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Abstract

A domination graph of a digraph D, $\text{dom}(D)$, is created using the vertex set of D, $V(D)$. There is an edge uv in $\text{dom}(D)$ whenever (u,z) or (v,z) is in the arc set of D, $A(D)$, for every other vertex $z \in V(D)$. For only some digraphs D has the structure of $\text{dom}(D)$ been characterized. Examples of this are tournaments and regular digraphs. The authors have characterizations for the structure of digraphs D for which $UG(D) = \text{dom}(D)$ or $UG(D) \cong \text{dom}(D)$. For example, when $UG(D) \cong \text{dom}(D)$, the only components of the complement of $UG(D)$ are complete graphs, paths and cycles. Here, we determine values of i and j for which $UG(D) \cong \text{dom}(D)$ and $UG^c(D) = C_4 \cup P_i \cup P_j$.

1 Introduction

Domination graphs were first introduced by Merz, Lundgren, Reid and Fisher [11] to describe the structure of the domination graphs and competition graphs of tournaments. Let D be a directed graph, or *digraph*, with nonempty vertex set $V(D)$ and arc set $A(D)$. The *domination graph of* D, $\text{dom}(D)$, is the graph created using the vertex set of D, $V(D)$. An edge uv is in $\text{dom}(D)$ if for every other vertex z in D, either (u,z) or (v,z) is in $A(D)$. The *competition graph of* D, $C(D)$, is created on the vertex set of D with an edge xy if there exists a third vertex $z \in V(D)$ such

[redacted]

that (x, z) and $(y, z) \in A(D)$. Given a *tournament* T, where there is exactly one arc between each pair of vertices, $\mathrm{dom}(T)$ is the complement of the competition graph of the tournament formed by reversing the arcs of T. Since $\mathrm{dom}(D)$ is a much sparser graph than the competition graph of a digraph, Merz et al. studied the domination graphs of tournaments to determine characteristics of the corresponding competition graphs. Such characteristics included the minimum and maximum number of edges in the competition graph of a tournament.

Since that time, further refinements have been made in the work on tournaments, including that done by Cho, Doherty, Kim and Lundgren ([1], [2]) and Merz et al. ([7], [8], [9], [10], [12]). For example, in [1], Cho et al. characterized the structure of the domination graphs of regular tournaments. Given the complexity of digraph structure, a complete characterization of domination graphs is probably an unreasonable expectation. Thus, classes of digraphs are studied. In our research, the underlying graph of a digraph is of particular interest. The *underlying graph of* D, $UG(D)$, is the graph obtained from D by removing the directions of the arcs. Previously, we have used underlying graphs to add to the knowledge base by characterizing digraphs D where $UG(D) = \mathrm{dom}(D)$ [4], and some digraphs where $UG(D) \cong \mathrm{dom}(D)$ ([3], [5], [6]). In this paper, we find values of i and j where $UG(D) \cong \mathrm{dom}(D)$ and $UG^c(D) = C_4 \cup P_i \cup P_j$.

In a digraph D, if $(u, v) \in A(D)$, then u is said to *dominate* v. When for every other vertex z in $V(D)$, either (u, z) or (v, z) is an arc in D, then u and v form a *dominating pair*. Thus, all edges in $\mathrm{dom}(D)$ are formed by dominating pairs of vertices. A digraph D is considered a *biorientation* of a graph G if for every edge $uv \in E(G)$, either (u, v) or (v, u) or both are arcs in D, and D contains no other arcs. If for edge uv in G, only one of arcs (u, v) or (v, u) is in D, then the arc is called an *orientation* of edge uv, or a *single arc*. We say edge uv in G is *bidirected* if it is replaced with arcs (u, v) and (v, u) in D. When all edges of G are bidirected edges in D, then D is a *complete biorientation* of G, also known as a *symmetric digraph*. Although bidirected edges are allowed in D, there are no directed loops.

When $UG(D) \cong \mathrm{dom}(D)$, there are often many edges. Let $UG^c(D)$ be the *complement of* $UG(D)$, where uv is an edge in $UG^c(D)$ if and only if it is not an edge in $UG(D)$. Similarly define the complement of $\mathrm{dom}(D)$, $\mathrm{dom}^c(D)$. If $UG(D) \cong \mathrm{dom}(D)$, then $UG^c(D) \cong \mathrm{dom}^c(D)$. The difference in the number of edges can be seen in Figure 1 at the beginning of Section 2 where $UG^c(D)$ is shown in part (a), and $UG(D)$ in part (b). It is quite apparent that $UG^c(D)$ has significantly fewer edges. Thus, it is easier to work with $UG^c(D)$ and $\mathrm{dom}^c(D)$.

To relate the results obtained from the complements to $UG(D)$ and $\mathrm{dom}(D)$, we use the concepts of the union and the join of graphs and digraphs. The *union* of two graphs, denoted $G \cup H$, is the graph on vertex set $V(G) \cup V(H)$ with edge set $E(G) \cup E(H)$. Similarly define the union of two digraphs. Figure 1(a) shows the union of C_4 and P_5. The *join* of graphs G and H, $G + H$, is the graph $G \cup H$ plus all edges between each vertex in G and each vertex in H. Similarly, the *join* of digraphs D_1 and D_2 is $D_1 \cup D_2$ plus all arcs (x, y) and (y, x) for each $x \in V(D_1)$, $y \in V(D_2)$. Figure 1(b) illustrates $C_4^c + P_5^c$.

The structure of $UG(D)$ is limited when $UG(D) \cong \mathrm{dom}(D)$. This is summed up in the following three results.

Theorem 1.1 *[5] If $D_1, \ldots, D_k$ are digraphs with $UG(D_i) \cong \mathrm{dom}(D_i)$ for $i = 1, \ldots, k$ and $D = D_1 + D_2 + \cdots + D_k$, then $UG(D) \cong \mathrm{dom}(D)$. Also*

1. $UG(D) = \sum_{i=1}^{k} UG(D_i)$;
2. $\mathrm{dom}(D) = \sum_{i=1}^{k} \mathrm{dom}(D_i)$;
3. $UG^c(D) = \bigcup_{i=1}^{k} UG^c(D_i)$;
4. $\mathrm{dom}^c(D) = \bigcup_{i=1}^{k} \mathrm{dom}^c(D_i)$.

Theorem 1.2 *[5] If $UG(D) \cong \mathrm{dom}(D)$, then each component of $UG^c(D)$ is either a complete graph, a path, or a cycle.*

Corollary 1.3 *[5] If $UG(D) \cong \mathrm{dom}(D)$, then D is the join of $D_1, D_2, \ldots, D_k$, where $UG(D_i)$ is isomorphic to an independent set, the complement of a path, or the complement of a cycle.*

Theorem 1.2 gives the three basic components that comprise $UG^c(D)$ for the digraphs in which we are interested. The structure of D and $UG(D)$ where $UG^c(D)$ is connected has been completely characterized [5], as have the cases where P_1, P_2 and C_4 are the components of $UG^c(D)$ [6], and $UG^c(D) = P_i \cup P_j$ [3]. In this paper, we find the values of i and j where $UG(D) \cong \mathrm{dom}(D)$, and $UG^c(D) \cong C_4 \cup P_i \cup P_j$. In the next section, we set up the preliminaries by discussing the general constructions, as well as previous results for $i, j = 1, 2$. In the final two sections, the case where $i = 1$ and $j \geq 3$ is examined as a special case and then the general case of $i \geq 2$, $j \geq 3$. The final theorem merges these cases to give combined results for $i, j \geq 1$.

2 The Preliminaries

To illustrate the basic ideas of tying together $UG^c(D)$ with $UG(D) \cong \mathrm{dom}(D)$, consider Figure 1. In part (a), $UG^c(D) = C_4 \cup P_5$. Then in part (b), $UG(D)$ is shown with the edges between all vertices in C_4^c and all vertices in P_5^c represented by a thick line. Consider what happens if directions are given to the edges of $UG(D)$. Even if all edges of $UG(D)$ are bioriented, some pairs of vertices will never dominate in D. Such pairs will never be adjacent in $\mathrm{dom}(D)$, so are always adjacent in $\mathrm{dom}^c(D)$. For example, consider pair x_1, x_3 in Figure 1(b). Neither x_1 nor x_3 is adjacent to vertex x_2, so cannot dominate x_2. Thus, $x_1 x_3$ is always an edge in $\mathrm{dom}^c(D)$. Similarly, an edge y_i, y_{i+2} will always be in $\mathrm{dom}^c(D)$, since neither vertex is adjacent to y_{i+1} in $UG(D)$. Figure 1(c) shows all edges that are always in $\mathrm{dom}^c(D)$ given $UG^c(D) = C_4 \cup P_5$. We call such an edge a *generated edge* or, collectively, the *generated subpaths* of $\mathrm{dom}^c(D)$.

The generated subpaths for the component P_n in $UG^c(D)$ are formally identified in the following lemma.

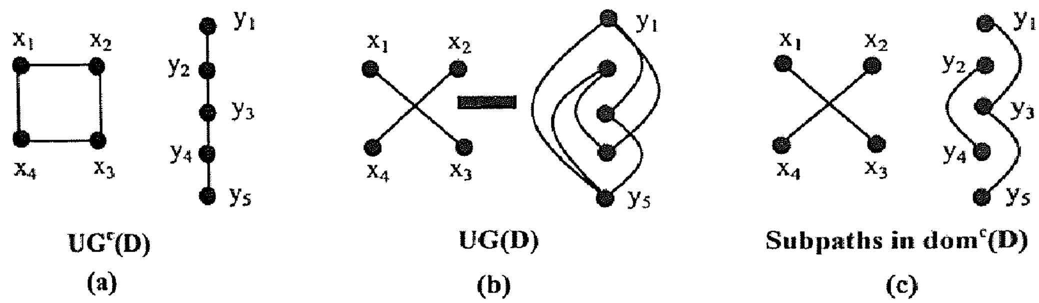


Figure 1: (a) $UG^c(D) = C_4 \cup P_5$. (b) $UG(D)$ with edges between all vertices of C_4^c and all vertices of P_5^c represented by the thick line. (c) All generated subpaths in $\mathrm{dom}^c(D)$.

Lemma 2.1 *[5] If $UG^c(D) = P_n = x_1, x_2, \ldots, x_n$ for $n \geq 3$, then*

1. *if n is odd, $x_1, x_3, \ldots, x_n$ and $x_2, x_4, \ldots, x_{n-1}$ are paths in $\mathrm{dom}^c(D)$, and*
2. *if n is even, $x_1, x_3, \ldots, x_{n-1}$ and $x_2, x_4, \ldots, x_n$ are paths in $\mathrm{dom}^c(D)$.*

Remark 2.2 *If uv is a generated edge in $\mathrm{dom}^c(D)$, then there exists a vertex z in $UG(D)$ such that uz and vz are not edges in $UG(D)$.*

This is true because if there were an edge, it could always be oriented toward z from u or v, creating D where u and v are a dominating pair. Since a generated edge is always in $\mathrm{dom}^c(D)$ for every biorientation of $UG(D)$, this cannot happen.

There can also be edges in $\mathrm{dom}^c(D)$ that are created. This requires a vertex x that either beats both u and v or is not adjacent to u and beats v. For example, orient edge x_3y_1 in Figure 1(b) from y_1 to x_3, making single arc (y_1, x_3) in D. Then neither x_3 nor y_2 dominates y_1, and edge x_3y_2 is "created" in $\mathrm{dom}^c(D)$. We call edge x_3y_2 and others like it a *created edge*.

Any vertex that two vertices do not dominate is referred to as a *source* of the edge between them in $\mathrm{dom}^c(D)$. For a generated edge, it is any vertex that is not adjacent to the pair. In Figure 1, x_2 and x_4 are sources for the pair x_1, x_3 in C_4. For a created edge, it is any vertex x as described in the preceding paragraph. From [6] we know the following.

Lemma 2.3 *[6] If $UG(D) \cong \mathrm{dom}(D)$ and every component of $UG^c(D)$ is isomorphic to K_1, K_2, or C_4, then no vertex of D is a source of more than one edge in $\mathrm{dom}^c(D)$.*

Lemma 2.4 *[6] If $UG(D) \cong \mathrm{dom}(D)$ and y is the source of two or more edges in $\mathrm{dom}^c(D)$, then the set of vertices which do not dominate y is contained in a component isomorphic to K_r, $r \geq 3$ in $UG^c(D)$.*

Since we do not have any components in $\mathrm{dom}^c(D)$ that contain K_r for $r \geq 3$, there can be no vertex that is the source of more than one edge in our constructions.

Corollary 2.5 *If $UG(D) \cong \mathrm{dom}(D)$ and $UG^c(D) = C_4 \cup P_i \cup P_j$, then any vertex is the source of at most one edge in $\mathrm{dom}^c(D)$.*

Now we look specifically at the vertices of C_4 and their role as sources.

Lemma 2.6 *Let $UG(D) \cong \mathrm{dom}(D)$ where every component of $UG^c(D)$ is isomorphic to K_1, K_2, or C_4. If x_1, x_2, x_3, x_4, x_1 forms C_4 in $UG^c(D)$, then x_i is the source of exactly one edge in $\mathrm{dom}^c(D)$: x_1 and x_3 are sources of x_2x_4; x_2 and x_4 are sources of x_1x_3.*

If $UG^c(D) \cong \mathrm{dom}^c(D)$, the generated edges must be supplemented with created edges. The following corollary to Lemma 2.6 shows that the vertices of C_4 cannot be sources for these created edges.

Corollary 2.7 *Let $UG(D) \cong \mathrm{dom}(D)$ and $C_4 = x_1, x_2, x_3, x_4, x_1$ be a component of $UG^c(D)$. Then x_1, x_2, x_3, and x_4 will not be sources for any created edges in $\mathrm{dom}^c(D)$. Furthermore, edges x_1x_3 and x_2x_4 are generated edges in $\mathrm{dom}^c(D)$.*

Since the vertices of C_4 cannot be sources, by Lemma 2.6 we conclude that the sources must come from the two paths that are the components of $UG^c(D)$. Let $P_i = y_1, \ldots, y_i$ and $P_j = z_1, \ldots, z_j$ be the two paths.

Lemma 2.8 *Let $K_1 = \{y\}$ be a component in $UG^c(D)$. Then vertex y in D can be the source of any created edge uv in $\mathrm{dom}^c(D)$ where $y \neq u, v$.*

Proof. If y is an isolated vertex in $UG^c(D)$, then y is adjacent to every other vertex in $UG(D)$. Therefore, if (y, u) and (y, v) are single arcs in D, y is a source of edge uv in $\mathrm{dom}^c(D)$ since u and v do not dominate y in D. Because there are no loops, y cannot be incident with any edge for which it is a source, so $y \neq u, v$. ■

Next, we find what vertices of paths can be sources, and what vertices are incident with the sourced edges in $\mathrm{dom}^c(D)$. In the following lemma, $N(y)$ is the *neighborhood of* y, which is the set of vertices adjacent to y.

Lemma 2.9 *[6] If $UG(D) \cong \mathrm{dom}(D)$ and $N(y) = \{x\}$ in $UG^c(D)$, then y is a source of at most one edge in $\mathrm{dom}^c(D)$, and this edge will be incident to x.*

Lemma 2.10 *[3] Let $UG(D) \cong \mathrm{dom}(D)$, $UG^c(D) = \bigcup_{i=1}^{k} P_{n_i}$, $n_i \geq 1$, and $P_{n_i} = x_{1i}, x_{2i}, \dots, x_{n_i i}$. If (u,v) in D is an orientation of edge uv in $UG(D)$, then $u = x_{1j}$ or $u = x_{n_j j}$ for some $1 \leq j \leq k$.*

Corollary 2.11 *If $UG(D) \cong \mathrm{dom}(D)$, each component of $UG^c(D)$ is either C_4 or a path, and y is the source of a created edge in $\mathrm{dom}^c(D)$, then y is an end vertex of a path.*

Lemma 2.12 *Let $UG(D) \cong \mathrm{dom}(D)$ where $P = x_1, \dots, x_i$ is a component of $UG^c(D)$ for $i \geq 3$. If x_1 or x_i is the source of a created edge in $dom^c(D)$, then that created edge is incident with x_2 or x_{i-1} respectively.*

Proof. Let (x_1, u) be a single arc in D. Then neither u nor x_2 dominates x_1, and ux_2 is an edge in $\mathrm{dom}^c(D)$. A similar argument holds for x_i and x_{i-1} when $i \geq 3$. ∎

The sources that can be used to create edges in $\mathrm{dom}^c(D)$ have now been identified, as well as the vertices with which those edges must be incident. Following is a lemma that shows the only possible vertex that can be the origin of more than one single arc in D is a vertex isomorphic to the component K_1 in $UG^c(D)$.

Lemma 2.13 *[6] Let $UG(D) \cong \mathrm{dom}(D)$ where every component of $UG^c(D)$ is isomorphic to K_1, K_2, or C_4. Let x be a vertex of D. If x is in a component isomorphic to C_4 in $UG^c(D)$, then x is the origin of no single arcs of D. If x is in a component isomorphic to K_2 in $UG^c(D)$, then x is the origin of at most one single arc of D. If x is in a component isomorphic to K_1 in $UG^c(D)$, then x is the origin of at most two single arcs of D.*

To complete the construction preliminaries, it is important to have an idea of where the created edges can be placed. Consider a path $P = y_1, y_2, \dots, y_{i-1}, y_i$ in $UG^c(D)$. We call y_1 and y_i the *outer vertices* of P. Vertices y_2 and y_{i-1} are called the *inner end vertices* (since these are the end vertices of generated subpaths). All other vertices of the paths are *inner vertices*. First, we determine under what conditions an edge may be incident with a vertex in $\mathrm{dom}^c(D)$.

Proposition 2.14 *Let $UG(D) \cong \mathrm{dom}(D)$ and $UG^c(D) = C_4 \cup P_i \cup P_j$. If $v \in V(D)$ is incident with exactly one generated edge in $\mathrm{dom}^c(D)$, then v is incident with at most one created edge.*

Proof. Vertices in $UG^c(D) = C_4 \cup P_i \cup P_j$ have degree of at most 2. Since $UG(D) \cong \mathrm{dom}(D)$, no vertex can be incident with more than two edges. ∎

Corollary 2.15 *Only a generated subpath P_1 in $\mathrm{dom}^c(D)$ can be incident with two created edges.*

Corollary 2.16 *Only specific vertices of components K_1, K_2, and P_3 in $UG^c(D)$ can be incident with two created edges in $\mathrm{dom}^c(D)$.*

One important aspect of joining subpaths to create a larger path is that we must be careful about what vertices are used to create an edge.

Lemma 2.17 *Let $UG(D) \cong \text{dom}(D)$ where every component of $UG^c(D)$ is isomorphic to a path or C_4. A created edge in $\text{dom}^c(D)$ between any two end vertices must have as a source a vertex $K_1 = \{y\}$ in $UG^c(D)$.*

Proof. The end vertices of any path can be sources for created edges in $\text{dom}^c(D)$. Any edge for which they are a source is incident with the corresponding inner end vertex in P_k, $k \geq 2$. Thus, end vertices cannot be used to create edges between end vertices. According to Lemma 2.8, $K_1 = \{y\}$ can create any edge, so is the only way that an edge can be created between two end vertices. ■

Corollary 2.18 *Let $UG(D) \cong \text{dom}(D)$ where every component of $UG^c(D)$ is isomorphic to a nontrivial path or C_4. Then there is no biorientation of D such that the end vertices of any path form a created edge in $\text{dom}^c(D)$.*

Another special case occurs when we try to create an edge between two inner end vertices to make a cycle. In order for this to happen, the path must have an odd number of vertices. Otherwise, the inner end vertices will not be on the same generated subpath and no cycle will be made. Figure 2 illustrates this concept. The figure is a mix of D and $\text{dom}^c(D)$. Single arc (v_1, v_6) causes neither v_2 nor v_6 to dominate v_1 in D. This creates edge v_2v_6 in $\text{dom}^c(D)$ between inner end vertices v_2 and v_6. The same edge can also be created with single arc (v_7, v_2). The generated subpaths in $\text{dom}^c(D)$, v_1, v_3, v_5, v_7 and v_2, v_4, v_6, are shown, as well as the created edge v_2v_6. After constructing this C_4, no other created edge can be adjacent to v_2 or v_6, as C_4 in $\text{dom}^c(D)$ cannot be adjacent to another edge. The following lemma formalizes this.

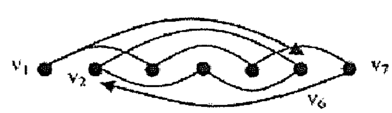


Figure 2: Generated subpaths v_1, v_3, v_5, v_7 and v_2, v_4, v_6 in $dom^c(D)$, created edge v_2v_6 in $dom^c(D)$, and the single arcs in D that create v_2v_6 in $dom^c(D)$.

Lemma 2.19 *Let $UG(D) \cong \text{dom}(D)$ where every component of $UG^c(D)$ is isomorphic to a path or C_4, and two inner end vertices of one subpath are joined by a created edge in $\text{dom}^c(D)$. Then the end vertices of the path from which the subpath was generated can be the source of no other edge.*

Proof. Let $u_1, u_2, \ldots, u_{i-1}, u_i$ be a path in $UG^c(D)$, where u_2 and u_{i-1} are inner end vertices of one subpath, and u_2u_{i-1} is an edge in $\text{dom}^c(D)$. From Lemma 2.12, we see that u_2 and u_{i-1} are incident with created edges from sources u_1 and u_i respectively. If u_1 or u_i was the source of a different edge, then it would be a pendant edge to the cycle $u_1, u_2, \ldots, u_{i-1}, u_i, u_1$ in $\mathrm{dom}^c(D)$, which indicates $\mathrm{dom}^c(D) \not\cong UG^c(D)$. ∎

To finalize the preliminaries, we take care of the cases $UG^c(D) = C_4 \cup P_i \cup P_j$ where $i, j = 1, 2$. These were special cases examined in [6]. The theorem from that paper is generalized, and we find that there is a biorientation for each of the four cases.

Theorem 2.20 *[6] Let G be a graph such that G is the union of r copies of K_1, s copies of K_2, and t copies of C_4 with $r + s + t \geq 1$. Then there exists a digraph D with* $\mathrm{dom}(D) cong UG(D) \cong G^c$ *if and only if $2t \leq r + s$, and if $s = 1$, then $r \geq 1$.*

Corollary 2.21 *Let $UG^c(D) = C_4 \cup P_i \cup P_j$ for $i, j = 1, 2$. Then there exists a biorientation of the edges of $UG(D)$ such that $UG(D) \cong \mathrm{dom}(D)$.*

3 Existence where $UG^c(D) = C_4 \cup P_1 \cup P_j$, $j \geq 3$

In every graph of $C_4 \cup P_1 \cup P_j$ when j is at least 3, there are $4 + (j - 1)$ edges. The generated edges in $\mathrm{dom}^c(D)$ total 2 from C_4 and $j - 2$ from the vertices of P_j. We must, therefore, be able to create 3 edges in $\mathrm{dom}^c(D)$ if we are to have $UG(D) \cong \mathrm{dom}(D)$.

Proposition 3.1 *Let $UG(D) \cong \mathrm{dom}(D)$, and $UG^c(D) = C_4 \cup P_1 \cup P_j$, where $P_1 = \{y\}$ and $P_j = z_1, \ldots, z_j$ for $j \geq 3$. Then y, z_1, and z_j must all be used as sources for distinct created edges in $\mathrm{dom}^c(D)$.*

Proof. There are j generated edges in $\mathrm{dom}^c(D)$. There are $j + 3$ edges in $UG^c(D)$. Since $UG^c(D) \cong \mathrm{dom}^c(D)$, three edges must be created in $\mathrm{dom}^c(D)$. Only y, z_1, and z_j can be sources of these edges, so each must be the source of exactly one created edge. ∎

To determine what must be done with these three edges, consider how the copy of C_4 must be designed in $\mathrm{dom}^c(D)$. The first question may be whether or not the vertices of C_4 in $UG^c(D)$ will be the vertices of C_4 in $\mathrm{dom}^c(D)$. The next proposition shows that this is not possible.

Proposition 3.2 *Let $UG(D) \cong \mathrm{dom}(D)$, and $UG^c(D) = C_4 \cup P_i \cup P_j$, $i \geq 1$, $j \geq 2$, where $C_4 = x_1, x_2, x_3, x_4, x_1$. Then at most one of the generated edges x_1x_3 or x_2x_4 is an edge of C_4 in $\mathrm{dom}^c(D)$.*

Proof. If both x_1x_3 and x_2x_4 are edges of C_4 in $\mathrm{dom}^c(D)$, then there must be two created edges joining them. From Lemma 2.12, we know that inner end vertices will be incident with the created edges for any source other than P_1. Thus, we must have two copies of P_1 to create the edges, which we do not. ∎

Given $P_j \geq 5$, it is natural to ask whether z_2z_{j-1} can be a created edge. Note that it is a generated edge when $j = 5$.

Lemma 3.3 *Let $UG(D) \cong \mathrm{dom}(D)$, and $UG^c(D) = C_4 \cup P_1 \cup P_j$, where $j \geq 5$ odd. Then $z_2 z_{j-1}$ is not a created edge in $\mathrm{dom}^c(D)$.*

Proof. Edge $z_2 z_{j-1}$ creates a cycle in $\mathrm{dom}^c(D)$. Since $UG^c(D) \cong \mathrm{dom}^c(D)$, the cycle is C_4, which indicates $j = 9$. By Lemma 2.19, z_1 and z_9 must both be the source of $z_2 z_{j-1}$. However, they must also source two distinct created edges. Thus, $z_2 z_{j-1}$ cannot be a created edge in $\mathrm{dom}^c(D)$. ■

Now we begin to determine the values for j that will yield $\mathrm{dom}^c(D) \cong C_4 \cup P_1 \cup P_j$. The method that is used throughout the remainder of the paper is to consider the values for j that will allow C_4 to be constructed out of 1, 2, or 3 of the generated subpaths in $\mathrm{dom}^c(D)$. We begin by determining the values of j where C_4 is created using one of the generated subpaths in $\mathrm{dom}^c(D)$.

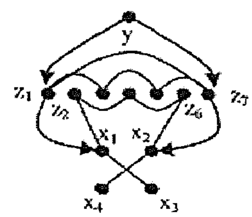


Figure 3: Example of a digraph and its associated $\mathrm{dom}^c(D)$ graph where $UG^c(D) = C_4 \cup P_1 \cup P_7$. Edges shown are in $\mathrm{dom}^c(D)$, while single arcs are in D. Bidirected edges of D are omitted.

Lemma 3.4 *If $UG(D) \cong \mathrm{dom}(D)$, $UG^c(D) = C_4 \cup P_1 \cup P_j$, $j \geq 3$, and C_4 in $\mathrm{dom}^c(D)$ is formed using only vertices from one generated subpath, then $j = 7, 8$. Furthermore, such D exists.*

Proof. In $UG^c(D)$, let $C_4 = x_1, x_2, x_3, x_4, x_1$, $P_1 = \{y\}$, and $P_j = z_1, \ldots, z_j$ for $j \geq 3$. Since one generated subpath is being used to form C_4, it must have four vertices. With $i = 1$, the subpath must be generated from the vertices of P_j, so $j = 7, 8, 9$. If $j = 9$, then the two inner end vertices would have to be joined to form $C_4 = z_2, z_4, z_6, z_8, z_2$, which violates Lemma 3.3. For $j = 7$, biorient all edges of $UG(D)$, except construct single arcs (y, z_1), (y, z_7), (z_1, x_1), and (z_7, x_2), creating edges $z_1 z_7$, $x_1 z_2$, and $x_2 z_6$ in $\mathrm{dom}^c(D)$. Thus, the components of $\mathrm{dom}^c(D)$ are $C_4 = z_1, z_3, z_5, z_7, z_1$, $P_1 = \{y\}$, and $P_7 = x_3, x_1, z_2, z_4, z_6, x_2, x_4$ (see Figure 3). For $j = 8$, biorient all edges of $UG(D)$, except construct single arcs (z_8, z_1), (z_1, x_1), (y, x_2), and (y, x_3), creating edges $z_1 z_7$, $x_1 z_2$, and $x_2 x_3$ in $\mathrm{dom}^c(D)$. Thus, the components of $\mathrm{dom}^c(D)$ are $C_4 = z_1, z_3, z_5, z_7, z_1$, $P_1 = \{y\}$, and $P_8 = x_4, x_2, x_3, x_1, z_2, z_4, z_6, z_8$. ■

The next lemma considers the possibility that C_4 is created using P_1 and P_3 or P_2 and P_2. The latter possibility is broken into two cases: $x_1 x_3$ is one P_2 or it is not.

Lemma 3.5 *If $UG(D) \cong \text{dom}(D)$, $UG^c(D) = C_4 \cup P_1 \cup P_j$, $j \geq 3$, and C_4 in $\text{dom}^c(D)$ is formed using only vertices from exactly two generated subpaths, then $j = 4, 5$. Furthermore, such D exists.*

Proof. In $UG^c(D)$, let $C_4 = x_1, x_2, x_3, x_4, x_1$, $P_1 = \{y\}$, and $P_j = z_1, \ldots, z_j$ for $j \geq 3$. We break this proof into two possibilities for the two subpaths. Either P_3 and P_1 are the two subpaths, or there are two P_2 that will be used. P_3 is a generated subpath when $j = 5, 6, 7$. P_1 must be y since there are no other K_1 generated. Vertices z_1 and z_j must be the sources of the two edges incident to y in C_4. By Lemma 2.12, these edges are also incident to z_2 and z_{j-1} respectively. So, z_2z_{j-1} cannot be a generated edge without forming a cycle with y. To form C_4, z_2 and z_{j-1} must be adjacent to a fourth vertex in a generated subpath of $\text{dom}^c(D)$ (Proposition 2.14). This implies that $P_3 = z_2, z_4, z_6$ is a generated subpath on $V(P_j)$, and $j = 7$ is the only possibility. However, generated edges x_1x_3 and x_2x_4 and subpath z_1, z_3, z_5, z_7 cannot be appended to form P_7. Therefore, $j \neq 7$.

Now consider that C_4 is created using two generated subpaths, P_2. Proposition 3.2 states that at most one of the edges x_1x_3 or x_2x_4 is an edge that can be used to create C_4. This indicates that at least one other generated subpath on two vertices must exist, so $j = 3, 4, 5$. Vertices z_1 and z_j must be used as sources, creating edges incident with z_2 and z_{j-1} respectively. While y may be used as a source for an edge in C_4, at least one of z_1 and z_j must be the source of an edge in C_4. If $j = 3$, then z_1 and z_3 create two edges incident with z_2, and C_4 cannot be created. If $j = 4$, biorient all edges of $UG(D)$, except construct single arcs (y, x_4), (y, z_4), (z_1, x_2), and (z_4, x_3), creating edges x_4z_4, x_2z_2, and x_3z_3 in $\text{dom}^c(D)$. So, $\text{dom}^c(D) \cong C_4 \cup P_1 \cup P_4$. If $j = 5$, biorient all edges of $UG(D)$, except construct single arcs (z_1, x_1), (z_5, x_3), (y, x_2), and (y, z_1), creating edges x_1z_2, x_3z_4, and x_2z_1 in $\text{dom}^c(D)$. So, $\text{dom}^c(D) \cong C_4 \cup P_1 \cup P_4$. Figure 4 shows this construction. ■

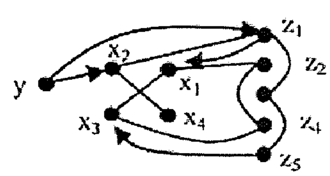


Figure 4: Example of a digraph and its associated $\text{dom}^c(D)$ graph where $UG^c(D) = C_4 \cup P_1 \cup P_5$. Edges shown are in $\text{dom}^c(D)$, while single arcs are in D. Bidirected edges of D are omitted.

While it is possible to continue looking at joining more and more subpaths, it is not necessary since there are only so many that can be used to create C_4. In the case where we have $UG^c(D) = C_4 \cup P_1 \cup P_j$, we can join at most two generated subpaths.

Lemma 3.6 *If $UG(D) \cong \text{dom}(D)$ and $UG^c(D) = C_4 \cup P_1 \cup P_j$ for $j \geq 3$, then there is no biorientation of the edges of $UG(D)$ such that the copy of C_4 in $\text{dom}^c(D)$ can be created using three or more generated subpaths.*

Proof. The only way to append three subpaths to form C_4 is to use one copy of P_2 and two copies of P_1. This can only occur when $j = 3$, creating $P_1 = y$ and $P_3 = z_1, z_2, z_3$. Both y and z_2 will be incident with two created edges. Sources z_1 and z_3 create two edges incident with z_2, only one of which can be incident with y. There is no source for another edge incident with y, so this construction is not possible. ∎

Combining Lemmas 3.4, 3.5, and 3.6, we have the following.

Theorem 3.7 *If $UG(D) \cong \mathrm{dom}(D)$ and $UG^c(D) = C_4 \cup P_1 \cup P_j$ for $j \geq 3$, then $j = 4, 5, 7, 8$. Furthermore, such D exists.*

4 Existence where $UG^c(D) = C_4 \cup P_i \cup P_j$, $i \geq 2$, $j \geq 3$

The existence of a biorientation for $UG(D)$ when $i = j = 2$ has been established in Corollary 2.21, so we begin the remaining cases with $i \geq 2$ and $j \geq 3$.

Lemma 4.1 *If $UG(D) \cong \mathrm{dom}(D)$, $UG^c(D) = C_4 \cup P_i \cup P_j$, $i, j \geq 2$, where $P_i = y_1, \ldots, y_i$ and $P_j = z_1, \ldots, z_j$, then y_1, y_i, z_1, and z_j must all be used as source vertices of distinct created edges in $\mathrm{dom}^c(D)$.*

Proof. There are $i + j - 2$ generated edges in $\mathrm{dom}^c(D)$. Since $UG^c(D)$ has $i + j + 2$ edges and $UG^c(D) \cong \mathrm{dom}^c(D)$, four edges must be created in $\mathrm{dom}^c(D)$. Corollary 2.11 states that y_1, y_i, z_1, and z_j are the only possible source vertices. So they must each source a distinct created edge in $\mathrm{dom}^c(D)$. ∎

The copies of P_i and P_j that are found in $\mathrm{dom}^c(D)$ are constructed by appending generated subpaths using created edges. Here we define $P_s \overset{A}{\cup} P_t$ as the graph obtained by creating an edge between an end vertex of P_s and an end vertex of P_t. When such an edge is created, we say that P_s has been *appended* to P_t. This operation is associative, but not necessarily commutative. For example, with $P_r \overset{A}{\cup} P_s \overset{A}{\cup} P_t$, we create a distinct edge between subpaths P_r and P_s, and another between P_s and P_t. This does not guarantee that $P_r \overset{A}{\cup} P_t \overset{A}{\cup} P_s$ is a possible construction.

As was done when $i = 1$, we look at constructing C_4 in $\mathrm{dom}^c(D)$ by appending 1, 2, and 3 generated subpaths. Since there is no P_1 as a component in $UG^c(D)$, there will be no constructions where any vertex is the origin of more than once single arc (see Lemma 2.13).

Lemma 4.2 *If $UG(D) \cong \mathrm{dom}(D)$, $UG^c(D) = C_4 \cup P_i \cup P_j$, $i \geq 2$, $j \geq 3$, where $P_i = y_1, \ldots, y_i$, $P_j = z_1, \ldots, z_j$, and C_4 is formed using vertices from exactly one generated subpath in $\mathrm{dom}^c(D)$, then $j = 8$.*

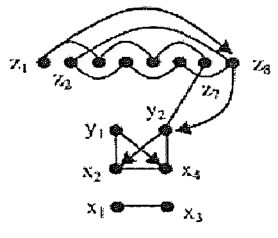
Proof. The generated subpath must have four vertices, and the possible values of j where this occurs are $j = 7, 8, 9$. There is no K_1 as a component in $UG^c(D)$. Thus, using Corollary 2.18 we see that this subpath cannot have two outer vertices, and $j \neq 7$. If the subpath uses two inner end vertices, then Lemma 2.19 shows that z_1 and z_j must be the sources for that edge. However, Lemma 4.1 indicates that z_1 and z_j must be the sources for distinct edges, so $j \neq 9$. Thus, $j = 8$. ■

Lemma 4.3 *If $UG(D) \cong \mathrm{dom}(D)$, $UG^c(D) = C_4 \cup P_i \cup P_j$, $i \geq 2$, $j \geq 3$, where $C_4 = x_1, x_2, x_3, x_4, x_1$, $P_i = y_1, \ldots, y_i$, $P_j = z_1, \ldots, z_j$, and C_4 in $\mathrm{dom}^c(D)$ is formed using vertices from exactly one generated subpath in P_j, then $i = 2, 4, 6, 7, 9, 12, 15$ and $j = 8$. Furthermore, such D exists.*

Proof. By Lemma 4.2 we know $j = 8$. Without loss of generality, we may take (z_1, z_8) to be a single arc in D for the remainder of the proof, thus creating edge z_2z_8 in $\mathrm{dom}^c(D)$ and $C_4 = z_2, z_4, z_6, z_8, z_2$. That leaves three sources, y_1, y_i, and z_j, and five generated subpaths with which to create P_i and P_8. These subpaths are x_1, x_3 and x_2, x_4, plus the two subpaths of P_i, and z_1, z_3, z_5, z_7. Note that z_7 is an inner end vertex and so will be incident with a created edge that has z_8 as the source.

We will consider the cases where i is even and i is odd.

1. $i = 2m$ is even. Two copies of P_m are generated on $V(P_i)$. We use these subpaths with x_1, x_3, x_2, x_4, and z_1, z_3, z_5, z_7 to create P_8 and P_{2m} in $\mathrm{dom}^c(D)$. Refer to P_8 and P_{2m} as paths S and T in no particular order. Each of the subpaths P_4, P_m and P_m have an inner end vertex, so each will be incident with a created edge. There are only three nonisomorphic ways to append the subpaths: $S = P_2 \overset{A}{\cup} P_4 \overset{A}{\cup} P_m \overset{A}{\cup} P_m$ and $T = P_2$; $S = P_2 \overset{A}{\cup} P_m \overset{A}{\cup} P_m$ and $T = P_2 \overset{A}{\cup} P_4$; and $S = P_2 \overset{A}{\cup} P_4 \overset{A}{\cup} P_m$ and $T = P_2 \overset{A}{\cup} P_m$.
   (a) $S = P_2 \overset{A}{\cup} P_4 \overset{A}{\cup} P_m \overset{A}{\cup} P_m$ and $T = P_2$. Since T is a path on 2 vertices, S must be the path on 8 vertices. So $2+4+2m = 8$, and $i = 2m = 2$. In addition to single arc (z_1, z_8), construct single arcs (z_8, y_2), (y_2, x_2), and (y_1, x_4) in D, and biorient all other edges of $UG(D)$. This creates edges y_2z_7, y_1x_2 and x_4y_2 in $\mathrm{dom}^c(D)$ so $\mathrm{dom}^c(D) \cong UG^c(D)$. This construction is shown in Figure 5(a).
   (b) $S = P_2 \overset{A}{\cup} P_m \overset{A}{\cup} P_m$ and $T = P_2 \overset{A}{\cup} P_4$. T is a path on 6 vertices, and S must be the path on 8 vertices, so $i = 6$. Similarly to 1(a), the construction of singe arcs (z_1, z_8), (z_8, x_4), (y_1, x_1), and (y_6, x_3) in D create the necessary edges in $\mathrm{dom}^c(D)$ so that $\mathrm{dom}^c(D) \cong UG^c(D)$. This construction is shown in Figure 5(b).
   (c) $S = P_2 \overset{A}{\cup} P_4 \overset{A}{\cup} P_m$ and $T = P_2 \overset{A}{\cup} P_m$. In this case, either S or T can be the path with 8 vertices. If S has 8 vertices, then $i = 4$. Similarly to 1(a), the construction of single arcs (z_1, z_8), (z_8, x_3), (y_1, x_1), and (y_4, x_2) in D create the necessary edges in $\mathrm{dom}^c(D)$ so that $\mathrm{dom}^c(D) \cong UG^c(D)$.
   On the other hand, if T has 8 vertices, then $i = 12$. Similarly to 1(a), the construction of single arcs (z_1, z_8), (z_8, y_1), (y_1, x_1), and (y_{12}, x_4) in D create the necessary edges in $\mathrm{dom}^c(D)$ so that $\mathrm{dom}^c(D) \cong UG^c(D)$.



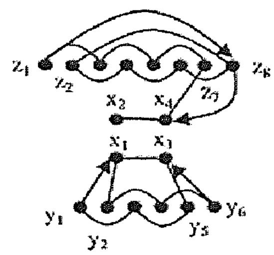


Figure 5: Examples of digraphs and their associated $\mathrm{dom}^c(D)$ graphs where C_4 is created using vertices of one subpath. Edges are shown for $\mathrm{dom}^c(D)$, and single arcs are shown for D. Bidirected arcs of D are not shown. (a) $UG^c(D) = C_4 \cup P_2 \cup P_8$. (b) $UG^c(D) = C_4 \cup P_6 \cup P_8$.

2. $i = 2m+1$ is odd. The two subpaths generated are an outer subpath on $m+1$ vertices and an inner subpath on m vertices. The inner subpath has two inner end vertices, which will be appended to two other subpaths, placing it in the middle of S or T. The subpath $P_4 = z_1, z_3, z_5, z_7$ has the only other inner end vertex, z_7. Thus, there are only four nonisomorphic ways to append the subpaths: $S = P_2 \overset{A}{\cup} P_m \overset{A}{\cup} P_4 \overset{A}{\cup} P_2$ and $T = P_{m+1}$; $S = P_2 \overset{A}{\cup} P_m \overset{A}{\cup} P_4 \overset{A}{\cup} P_{m+1}$ and $T = P_2$; $S = P_2 \overset{A}{\cup} P_m \overset{A}{\cup} P_{m+1}$ and $T = P_2 \overset{A}{\cup} P_4$; and $S = P_2 \overset{A}{\cup} P_m \overset{A}{\cup} P_2$ and $T = P_4 \overset{A}{\cup} P_{m+1}$. In the second and third case, S is odd, T is constant, and there is no way to construct P_8. The remaining two subcases are addressed below.

   (a) $S = P_2 \overset{A}{\cup} P_m \overset{A}{\cup} P_4 \overset{A}{\cup} P_2$ and $T = P_{m+1}$. Since m must be at least 1, S cannot be a path on 8 vertices, so $T = P_{m+1} = P_8$, and $i = 15$. Similarly to 1(a), the construction of single arcs (z_1, z_8), (z_8, x_3), (y_1, x_1), and (y_{15}, x_4) in D create the necessary edges in $\mathrm{dom}^c(D)$ so that $\mathrm{dom}^c(D) \cong UG^c(D)$.

   (b) $S = P_2 \overset{A}{\cup} P_m \overset{A}{\cup} P_2$ and $T = P_4 \overset{A}{\cup} P_{m+1}$. If S is a path on 8 vertices, then $m = 4$, and $i = 9$. Similarly to 1(a), the construction of single arcs (z_1, z_8), (z_8, y_1), (y_1, x_1), and (y_9, x_4) in D create the necessary edges in $\mathrm{dom}^c(D)$ so that $\mathrm{dom}^c(D) \cong UG^c(D)$.
   On the other hand, if T is a path on 8 vertices, then $m = 3$, and $i = 7$. Similarly to 1(a), the construction of single arcs (z_1, z_8), (z_8, y_1), (y_1, x_1), and (y_7, x_2) in D create the necessary edges in $\mathrm{dom}^c(D)$ so that $\mathrm{dom}^c(D) \cong UG^c(D)$.

Thus, if $j = 8$, then $i = 2, 4, 6, 7, 9, 12, 15$, and such D exists. ∎

Next, we look at the case where C_4 is created using two generated subpaths. The first lemma considers creating C_4 using generated subpaths P_1 and P_3, and the second lemma using P_2 and P_2.

Lemma 4.4 *If $UG(D) \cong \mathrm{dom}(D)$, $UG^c(D) = C_4 \cup P_i \cup P_j$, $i \geq 2$, $j \geq 3$, where $C_4 = x_1, x_2, x_3, x_4, x_1$, $P_i = y_1, \ldots, y_i$, $P_j = z_1, \ldots, z_j$, and C_4 is formed using vertices from exactly two generated subpaths P_1 and P_3 in $\mathrm{dom}^c(D)$, then $i = 2$ and $j = 6$. Furthermore, such D exists.*

Proof. Since P_1 is a vertex in C_4, there are two created edges incident with it. Because K_1 is not a component of $UG^c(D)$, P_1 must be a generated subpath of P_i or P_j. Without loss of generality, say P_i. Then $i = 2, 3$.

If $i = 2$, say that $P_1 = y_2$. Subpath P_3 must come from P_j and have exactly one inner end vertex to append to y_2. Thus, $j = 6$. Biorient all edges of $UG(D)$ except construct single arcs (z_1, y_1), (y_2, z_6), (z_6, x_2), and (y_1, x_4). The created edges form components $C_4 = y_1, z_2, z_4, z_6, y_1$, $P_2 = x_1, x_3$, and $P_6 = y_2, x_4, x_2, z_5, z_3, z_1$ in $\mathrm{dom}^c(D)$.

If $i = 3$ and $P_1 = y_2$, then y_1 and y_3 are sources of both edges incident with y_2. Thus, P_3 cannot contain another interior vertex since the created edges are each incident with only one. P_3 must be an outer subpath, so $j = 5$. However, if C_4 is constructed using z_1, z_3, z_5, then P_3 and P_5 cannot be created using the remaining subpaths, which are all even. ∎

Lemma 4.5 *If $UG(D) \cong \mathrm{dom}(D)$, $UG^c(D) = C_4 \cup P_i \cup P_j$, $i \geq 2$, $j \geq 3$, where $C_4 = x_1, x_2, x_3, x_4, x_1$, $P_i = y_1, \ldots, y_i$, $P_j = z_1, \ldots, z_j$, and C_4 is formed using vertices from exactly two generated subpaths P_2 and P_2 in $\mathrm{dom}^c(D)$, then $i = j = 4$ or $i = 5$ and $j = 2, 3, 4, 6, 9$. Furthermore, such D exists.*

Proof. Proposition 3.2 states that at most one of x_1x_3 or x_2x_4 will be an edge in C_4.

1. Suppose that x_1x_3 is an edge of C_4. The two created edges incident with x_1 and x_3 will be incident with two adjacent inner end vertices of P_i or P_j. Say it is P_i, making y_2, y_4 the generated subpath appended to x_1x_3 and $i = 5$. Without loss of generality, construct single arcs (y_1, x_1) and (y_5, x_3) in D so that $C_4 = x_1, y_2, y_4, x_3, x_1$ in $\mathrm{dom}^c(D)$. Now we find the values of j where such a D exists.

   Consider how P_5 can be made by appending the two generated subpaths on $V(P_j)$, x_2, x_4 and y_1, y_3, y_5. Only two created edges can be constructed using the remaining sources z_1 and z_j, and will be incident with z_2 and z_{j-1} respectively (Lemma 2.12). We look at the number of subpaths that are appended to construct P_5.

   Case 1: P_5 is a generated subpath. Then P_5 can have no inner end vertices, or it will be incident with a created edge. Thus, $j = 9$. Construct single arcs (z_1, y_1), (z_9, x_4), (y_1, x_1), and (y_5, x_3) in D and biorient all other edges of $UG(D)$. This creates the necessary edges in $\mathrm{dom}^c(D)$ so that $\mathrm{dom}^c(D) \cong UG^c(D)$.

   Case 2: P_5 is constructed using z_2 or z_{j-1}, but not both. So, z_2 and z_{j-1} are on different generated subpaths, and j is even. Say that z_2 on generated subpath

P_{z_2} is appended to another subpath P to create P_5. Now P does not contain z_{j-1}, so must either be x_2, x_4 or y_1, y_3, y_5. Thus, $|V(P_{z_2})| = 3$ or 2, resulting in $j = 6$ or $j = 4$. If $j = 4$, construct single arcs (z_1, y_1), (z_4, x_2), (y_1, x_1), and (y_5, x_3) in D and biorient all other edges of $UG(D)$. If $j = 6$, construct single arcs (z_1, x_2), (z_6, y_5), (y_1, x_1), and (y_5, x_3) in D and biorient all other edges of $UG(D)$. Both of these constructions create the necessary edges in $\text{dom}^c(D)$ so that $\text{dom}^c(D) \cong UG^c(D)$.

Case 3: P_5 is constructed using both z_2 and z_{j-1}. Either x_2, x_4 or y_1, y_3, y_5 but not both will be appended to the two inner end vertices (else there would be more than 5 vertices). The subpath that is not appended will be a path in $\text{dom}^c(D)$, so $j = 2, 3$. If $j = 3$, sources z_1 and z_3 create two edges incident with z_2. This forms a cycle in $\text{dom}^c(D)$, so $j \neq 3$. If $j = 2$, construct single arcs (z_1, y_1), (z_2, y_5), (y_1, x_1), and (y_5, x_3) in D and biorient all other edges of $UG(D)$. This creates the necessary edges in $\text{dom}^c(D)$ so that $\text{dom}^c(D) \cong UG^c(D)$.

2. Suppose that x_1x_3 is not an edge of C_4. The copy of C_4 cannot be created from one generated subpath in $\text{dom}^c(D)$ (Lemma 2.19). If C_4 were to be made from the two generated subpaths of one path, then that path is P_4. However, the only non-generated edge that can be created by either source z_1 or z_4 is z_2z_3, so C_4 cannot be constructed. Thus, C_4 must be constructed from a generated copy of P_2 in P_i and another in P_j. Since these paths must be appended using exactly two created edges, the two copies of P_2 together must have exactly two inner end vertices. This dictates that $i > 3$ or $j > 3$. Say that $i > 3$, so $i = 4, 5$ and $j = 3, 4, 5$ are the only values where P_2 is a generated subpath.

   We have already shown that D exists for $i = 5$ and $j = 2, 4, 6, 9$, so we need only consider $i = 4$ with $j = 3, 4$, and $i = 5$ with $j = 3, 5$. When $i = 4$, only one vertex of each generated subpath is an inner end vertex, so the subpath must be appended to another copy of P_2 with exactly one inner end vertex. Thus, $j \neq 3$ and $j = 4$. Construct single arcs (z_1, y_1), (z_4, x_3), (y_1, x_2), and (y_4, z_4) in D and biorient all other edges of $UG(D)$. This creates the necessary edges in $\text{dom}^c(D)$ so that $\text{dom}^c(D) \cong UG^c(D)$.

   When $i = 5$, the copy of P_2 is an inner subpath, so both vertices are inner end vertices. Thus, it can only be appended to a copy of P_2 where both vertices are outer vertices. This implies that $j = 3$. Construct single arcs (y_1, z_1), (y_5, z_3), (z_1, x_1), and (z_3, x_4) in D and biorient all other edges of $UG(D)$. This creates the necessary edges in $\text{dom}^c(D)$ so that $\text{dom}^c(D) \cong UG^c(D)$.

Given the restrictions of appending paths, these are the only possible constructions. So $i = j = 4$ or $i = 5$ and $j = 2, 3, 4, 6, 9$, and such D exists. ■

Finally, the case where C_4 is constructed using vertices from exactly three generated subpaths is examined.

Lemma 4.6 *If $UG(D) \cong \mathrm{dom}(D)$, $UG^c(D) = C_4 \cup P_i \cup P_j$, $i \geq 2$, $j \geq 3$, where $C_4 = x_1, x_2, x_3, x_4, x_1$, $P_i = y_1, \ldots, y_i$, $P_j = z_1, \ldots, z_j$, and C_4 is formed using vertices from exactly three generated subpaths in $\mathrm{dom}^c(D)$, then $i = 2$ and $j = 3$. Furthermore, such D exists.*

Proof. As seen in the proof of Lemma 3.6, this requires one copy of P_2 and two copies of P_1 as generated subpaths. If $i = 2$, there are 2 copies of P_1 generated. However, y_1y_2 will never be an edge in $\mathrm{dom}^c(D)$ (Corollary 2.18). Thus, only one of y_1 or y_2 may be a vertex in C_4. This implies the other copy of P_1 must come from $V(P_j)$. Since $j \geq 3$, only $j = 3$ produces a subpath with one vertex. Biorient all edges of $UG(D)$, except construct single arcs (z_1, y_1), (z_3, x_3), (y_1, z_3), and (y_2, x_1). This creates the necessary edges in $\mathrm{dom}^c(D)$ so that $\mathrm{dom}^c(D) \cong UG^c(D)$. If $i = 3$, then it has one subpath P_1, and the others must come from $V(P_j)$. Again, $j = 3$. However, all created edges are incident with y_2 and z_2, so there is no way to construct C_4. If $i \geq 4$, with $j \geq 3$, there are not two subpaths P_1. Thus, $i = 2$ and $j = 3$ is the only possibility. ■

As a grand finale, we bring together the results that appear at the end of Section 2, the end of Section 3, and the lemmas stated previously in this section.

Theorem 4.7 *If $UG(D) \cong \mathrm{dom}(D)$, and $UG^c(D) = C_4 \cup P_i \cup P_j$, then*

1. *$i = 1$ and $j = 1, 2, 4, 5, 7, 8$, or*
2. *$i = 2$ and $j = 2, 3, 5, 6, 8$, or*
3. *$i = 4$ and $j = 4, 5$, or*
4. *$i = 5$ and $j = 3, 6, 9$, or*
5. *$i = 8$ and $j = 4, 6, 7, 9, 12, 15$.*

Furthermore, in each case such a digraph exists.

Proof. Corollary 2.21 addresses existence for $i, j = 1, 2$. Further results for $i = 1$ were listed in Theorem 3.7. Except for $i = j = 2$, the results in parts (2)–(5) come directly from Lemmas 4.3, 4.4, 4.5, and 4.6. Although the creation of C_4 using four components was not specifically mentioned, it requires that $i = j = 2$, so that there are four copies of P_1 with which to construct C_4. This possibility is covered in Corollary 2.21. ■

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