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Digraphs with Isomorphic Underlying and Domination Graphs: 4-cycles and Pairs of Paths

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Abstract

A domination graph of a digraph $D$, $\text{dom}(D)$, is created using the vertex set of $D$, $V(D)$. There is an edge $uv$ in $\text{dom}(D)$ whenever $(u, z)$ or $(v, z)$ is in the arc set of $D$, $A(D)$, for every other vertex $z \in V(D)$. For only some digraphs $D$ has the structure of $\text{dom}(D)$ been characterized. Examples of this are tournaments and regular digraphs. The authors have characterizations for the structure of digraphs $D$ for which $UG(D) \cong \text{dom}(D)$ or $UG(D) \cong \text{dom}(D)$. For example, when $UG(D) \cong \text{dom}(D)$, the only components of the complement of $UG(D)$ are complete graphs, paths and cycles. Here, we determine values of $i$ and $j$ for which $UG(D) \cong \text{dom}(D)$ and $UG^\prime(D) = C_4 \cup P_i \cup P_j$.

1 Introduction

Domination graphs were first introduced by Merz, Lundgren, Reid and Fisher [11] to describe the structure of the domination graphs and competition graphs of tournaments. Let $D$ be a directed graph, or digraph, with nonempty vertex set $V(D)$ and arc set $A(D)$. The domination graph of $D$, $\text{dom}(D)$, is the graph created using the vertex set of $D$, $V(D)$. An edge $uv$ is in $\text{dom}(D)$ if for every other vertex $z$ in $D$, either $(u, z)$ or $(v, z)$ is in $A(D)$. The competition graph of $D$, $C(D)$, is created on the vertex set of $D$ with an edge $xy$ if there exists a third vertex $z \in V(D)$ such
that \((x, z)\) and \((y, z) \in A(D)\). Given a tournament \(T\), where there is exactly one arc between each pair of vertices, \(\text{dom}(T)\) is the complement of the competition graph of the tournament formed by reversing the arcs of \(T\). Since \(\text{dom}(D)\) is a much sparser graph than the competition graph of a digraph, Merz et al. studied the domination graphs of tournaments to determine characteristics of the corresponding competition graphs. Such characteristics included the minimum and maximum number of edges in the competition graph of a tournament.

Since that time, further refinements have been made in the work on tournaments, including that done by Cho, Doherty, Kim and Lundgren ([1], [2]) and Merz et al. ([7], [8], [9], [10], [12]). For example, in [1], Cho et al. characterized the structure of the domination graphs of regular tournaments. Given the complexity of digraph structure, a complete characterization of domination graphs is probably an unreasonable expectation. Thus, classes of digraphs are studied. In our research, the underlying graph of a digraph is of particular interest. The underlying graph of \(D\), \(\text{UG}(D)\), is the graph obtained from \(D\) by removing the directions of the arcs. Previously, we have used underlying graphs to add to the knowledge base by characterizing digraphs \(D\) where \(\text{UG}(D) = \text{dom}(D)\) [4], and some digraphs where \(\text{UG}(D) \cong \text{dom}(D)\) ([3], [5], [6]). In this paper, we find values of \(i\) and \(j\) where \(\text{UG}(D) \cong \text{dom}(D)\) and \(\text{UC}(D) = C_4 \cup P_i \cup P_j\).

In a digraph \(D\), if \((u, v) \in A(D)\), then \(u\) is said to dominate \(v\). When for every other vertex \(z\) in \(V(D)\), either \((u, z)\) or \((v, z)\) is an arc in \(D\), then \(u\) and \(v\) form a dominating pair. Thus, all edges in \(\text{dom}(D)\) are formed by dominating pairs of vertices. A digraph \(D\) is considered a biorientation of a graph \(G\) if for every edge \(uv \in E(G)\), either \((u, v)\) or \((v, u)\) or both are arcs in \(D\), and \(D\) contains no other arcs. If for edge \(uv\) in \(G\), only one of arcs \((u, v)\) or \((v, u)\) is in \(D\), then the arc is called an orientation of edge \(uv\), or a single arc. We say edge \(uv\) in \(G\) is bidirected if it is replaced with arcs \((u, v)\) and \((v, u)\) in \(D\). When all edges of \(G\) are bidirected edges in \(D\), then \(D\) is a complete biorientation of \(G\), also known as a symmetric digraph. Although bidirected edges are allowed in \(D\), there are no directed loops.

When \(\text{UG}(D) \cong \text{dom}(D)\), there are often many edges. Let \(\text{UG}^c(D)\) be the complement of \(\text{UG}(D)\), where \(uv\) is an edge in \(\text{UG}(D)\) if and only if it is not an edge in \(\text{UG}(D)\). Similarly define the complement of \(\text{dom}(D)\), \(\text{dom}^c(D)\). If \(\text{UG}(D) \cong \text{dom}(D)\), then \(\text{UG}^c(D) \cong \text{dom}^c(D)\). The difference in the number of edges can be seen in Figure 1 at the beginning of Section 2 where \(\text{UG}^c(D)\) is shown in part (a), and \(\text{UG}(D)\) in part (b). It is quite apparent that \(\text{UG}^c(D)\) has significantly fewer edges. Thus, it is easier to work with \(\text{UG}^c(D)\) and \(\text{dom}^c(D)\).

To relate the results obtained from the complements to \(\text{UG}(D)\) and \(\text{dom}(D)\), we use the concepts of the union and the join of graphs and digraphs. The union of two graphs, denoted \(G \cup H\), is the graph on vertex set \(V(G) \cup V(H)\) with edge set \(E(G) \cup E(H)\). Similarly define the union of two digraphs. Figure 1(a) shows the union of \(C_4\) and \(P_3\). The join of graphs \(G\) and \(H\), \(G + H\), is the graph \(G \cup H\) plus all edges between each vertex in \(G\) and each vertex in \(H\). Similarly, the join of digraphs \(D_1\) and \(D_2\) is \(D_1 \cup D_2\) plus all arcs \((x, y)\) and \((y, x)\) for each \(x \in V(D_1)\), \(y \in V(D_2)\). Figure 1(b) illustrates \(C_4^c + P_5^c\).
The structure of $UG(D)$ is limited when $UG(D) \cong \text{dom}(D)$. This is summed up in the following three results.

**Theorem 1.1** [5] If $D_1, \ldots, D_k$ are digraphs with $UG(D_i) \cong \text{dom}(D_i)$ for $i = 1, \ldots, k$ and $D = D_1 + D_2 + \cdots + D_k$, then $UG(D) \cong \text{dom}(D)$. Also

1. $UG(D) = \sum_{i=1}^{k} UG(D_i)$;
2. $\text{dom}(D) = \sum_{i=1}^{k} \text{dom}(D_i)$;
3. $UG^e(D) = \bigcup_{i=1}^{k} UG^e(D_i)$;
4. $\text{dom}^e(D) = \bigcup_{i=1}^{k} \text{dom}^e(D_i)$.

**Theorem 1.2** [5] If $UG(D) \cong \text{dom}(D)$, then each component of $UG^e(D)$ is either a complete graph, a path, or a cycle.

**Corollary 1.3** [5] If $UG(D) \cong \text{dom}(D)$, then $D$ is the join of $D_1, D_2, \ldots, D_k$, where $UG(D_i)$ is isomorphic to an independent set, the complement of a path, or the complement of a cycle.

Theorem 1.2 gives the three basic components that comprise $UG^e(D)$ for the digraphs in which we are interested. The structure of $D$ and $UG(D)$ where $UG^e(D)$ is connected has been completely characterized [5], as have the cases where $P_1$, $P_2$ and $C_4$ are the components of $UG^e(D)$ [6], and $UG^e(D) = P_1 \cup P_2$ [3]. In this paper, we find the values of $i$ and $j$ where $UG(D) \cong \text{dom}(D)$, and $UG^e(D) \cong C_4 \cup P_1 \cup P_2$. In the next section, we set up the preliminaries by discussing the general constructions, as well as previous results for $i, j = 1, 2$. In the final two sections, the case where $i = 1$ and $j = 3$ is examined as a special case and then the general case of $i \geq 2$, $j \geq 3$. The final theorem merges these cases to give combined results for $i, j \geq 1$.

## 2 The Preliminaries

To illustrate the basic ideas of tying together $UG^e(D)$ with $UG(D) \cong \text{dom}(D)$, consider Figure 1. In part (a), $UG^e(D) = C_3 \cup P_3$. Then in part (b), $UG(D)$ is shown with the edges between all vertices in $C_3$ and all vertices in $P_3$ represented by a thick line. Consider what happens if directions are given to the edges of $UG(D)$. Even if all edges of $UG(D)$ are bioriented, some pairs of vertices will never dominate in $D$. Such pairs will never be adjacent in $\text{dom}(D)$, so are always adjacent in $\text{dom}^e(D)$. For example, consider pair $x_1, x_3$ in Figure 1(b). Neither $x_1$ nor $x_3$ is adjacent to vertex $x_2$, so cannot dominate $x_2$. Thus, $x_1x_3$ is always an edge in $\text{dom}^e(D)$. Similarly, an edge $y_i, y_{i+2}$ will always be in $\text{dom}^e(D)$, since neither vertex is adjacent to $y_{i+1}$ in $UG(D)$. Figure 1(c) shows all edges that are always in $\text{dom}^e(D)$ given $UG^e(D) = C_3 \cup P_3$. We call such an edge a *generated edge* or, collectively, the *generated subpaths* of $\text{dom}^e(D)$.
The generated subpaths for the component \( P_n \) in \( UG^c(D) \) are formally identified in the following lemma.

\[ \begin{align*}
\text{UG}'(D) & \quad \text{UG}(D) \\
\text{Subpaths in dom'}(D)
\end{align*} \]

Figure 1: (a) \( UG^c(D) = C_4 \cup P_5 \). (b) \( UG(D) \) with edges between all vertices of \( C_4^c \) and all vertices of \( P_5^c \) represented by the thick line. (c) All generated subpaths in \( \text{dom}^c(D) \).

Lemma 2.1 [5] If \( UG^c(D) = P_n = x_1, x_2, \ldots, x_n \) for \( n \geq 3 \), then

1. if \( n \) is odd, \( x_1, x_3, \ldots, x_n \) and \( x_2, x_4, \ldots, x_{n-1} \) are paths in \( \text{dom}^c(D) \), and
2. if \( n \) is even, \( x_1, x_3, \ldots, x_{n-1} \) and \( x_2, x_4, \ldots, x_n \) are paths in \( \text{dom}^c(D) \).

Remark 2.2 If \( uv \) is a generated edge in \( \text{dom}^c(D) \), then there exists a vertex \( z \) in \( UG(D) \) such that \( uz \) and \( vz \) are not edges in \( UG(D) \).

This is true because if there were an edge, it could always be oriented toward \( z \) from \( u \) or \( v \), creating \( D \) where \( u \) and \( v \) are a dominating pair. Since a generated edge is always in \( \text{dom}^c(D) \) for every biorientation of \( UG(D) \), this cannot happen.

There can also be edges in \( \text{dom}^c(D) \) that are created. This requires a vertex \( x \) that either beats both \( u \) and \( v \) or is not adjacent to \( u \) and beats \( v \). For example, orient edge \( x_3y_1 \) in Figure 1(b) from \( y_1 \) to \( x_3 \), making single arc \( (y_1, x_3) \) in \( D \). Then neither \( x_4 \) nor \( y_2 \) dominates \( y_1 \), and edge \( x_3y_2 \) is “created” in \( \text{dom}^c(D) \). We call edge \( x_3y_2 \) and others like it a created edge.

Any vertex that two vertices do not dominate is referred to as a source of the edge between them in \( \text{dom}^c(D) \). For a generated edge, it is any vertex that is not adjacent to the pair. In Figure 1, \( x_2 \) and \( x_4 \) are sources for the pair \( x_1, x_3 \) in \( C_4 \). For a created edge, it is any vertex \( x \) as described in the preceding paragraph. From [6] we know the following.

Lemma 2.3 [6] If \( UG(D) \cong \text{dom}(D) \) and every component of \( UG^c(D) \) is isomorphic to \( K_1, K_2, \) or \( C_4 \), then no vertex of \( D \) is a source of more than one edge in \( \text{dom}^c(D) \).
Lemma 2.4 [6] If $UG(D) \cong \text{dom}(D)$ and $y$ is the source of two or more edges in $\text{dom}^c(D)$, then the set of vertices which do not dominate $y$ is contained in a component isomorphic to $K_r$, $r \geq 3$ in $UG^c(D)$.

Since we do not have any components in $\text{dom}^c(D)$ that contain $K_r$ for $r \geq 3$, there can be no vertex that is the source of more than one edge in our constructions.

Corollary 2.5 If $UG(D) \cong \text{dom}(D)$ and $UG^c(D) = C_4 \cup P_i \cup P_j$, then any vertex is the source of at most one edge in $\text{dom}^c(D)$.

Now we look specifically at the vertices of $C_4$ and their role as sources.

Lemma 2.6 Let $UG(D) \cong \text{dom}(D)$ where every component of $UG^c(D)$ is isomorphic to $K_1, K_2$, or $C_4$. If $x_1, x_2, x_3, x_4, x_1$ forms $C_4$ in $UG^c(D)$, then $x_i$ is the source of exactly one edge in $\text{dom}^c(D)$: $x_1$ and $x_3$ are sources of $x_2x_4$; $x_2$ and $x_4$ are sources of $x_1x_3$.

If $UG^c(D) \cong \text{dom}^c(D)$, the generated edges must be supplemented with created edges. The following corollary to Lemma 2.6 shows that the vertices of $C_4$ cannot be sources for these created edges.

Corollary 2.7 Let $UG(D) \cong \text{dom}(D)$ and $C_4 = x_1, x_2, x_3, x_4, x_1$ be a component of $UG^c(D)$. Then $x_1, x_2, x_3$, and $x_4$ will not be sources for any created edges in $\text{dom}^c(D)$. Furthermore, edges $x_1x_3$ and $x_2x_4$ are generated edges in $\text{dom}^c(D)$.

Since the vertices of $C_4$ cannot be sources, by Lemma 2.6 we conclude that the sources must come from the two paths that are the components of $UG^c(D)$. Let $P_i = y_1, \ldots, y_i$ and $P_j = z_1, \ldots, z_j$ be the two paths.

Lemma 2.8 Let $K_1 = \{y\}$ be a component in $UG^c(D)$. Then vertex $y$ in $D$ can be the source of any created edge $uv$ in $\text{dom}^c(D)$ where $y \neq u, v$.

Proof. If $y$ is an isolated vertex in $UG^c(D)$, then $y$ is adjacent to every other vertex in $UG(D)$. Therefore, if $(y, u)$ and $(y, v)$ are single arcs in $D$, $y$ is a source of edge $uv$ in $\text{dom}^c(D)$ since $u$ and $v$ do not dominate $y$ in $D$. Because there are no loops, $y$ cannot be incident with any edge for which it is a source, so $y \neq u, v$.

Next, we find what vertices of paths can be sources, and what vertices are incident with the sourced edges in $\text{dom}^c(D)$. In the following lemma, $N(y)$ is the neighborhood of $y$, which is the set of vertices adjacent to $y$.

Lemma 2.9 [6] If $UG(D) \cong \text{dom}(D)$ and $N(y) = \{x\}$ in $UG^c(D)$, then $y$ is a source of at most one edge in $\text{dom}^c(D)$, and this edge will be incident to $x$. 
Lemma 2.10 [5] Let $\text{UG}(D) \cong \text{dom}(D)$, $\text{UG}^e(D) = \bigcup_{i=1}^k P_{n_i}$, $n_i \geq 1$, and $P_{n_i} = x_{1i}, x_{2i}, \ldots, x_{n_ii}$. If $(u, v)$ in $D$ is an orientation of edge $uv$ in $\text{UG}(D)$, then $u = x_{1j}$ or $v = x_{n_ii}$ for some $1 \leq j \leq k$.

Corollary 2.11 If $\text{UG}(D) \cong \text{dom}(D)$, each component of $\text{UG}^e(D)$ is either $C_4$ or a path, and $y$ is the source of a created edge in $\text{dom}^e(D)$, then $y$ is an end vertex of a path.

Lemma 2.12 Let $\text{UG}(D) \cong \text{dom}(D)$ where $P = x_1, \ldots, x_i$ is a component of $\text{UG}^e(D)$ for $i \geq 3$. If $x_1$ or $x_i$ is the source of a created edge in $\text{dom}^e(D)$, then that created edge is incident with $x_2$ or $x_{i-1}$ respectively.

Proof. Let $(x_1, u)$ be a single arc in $D$. Then neither $u$ nor $x_2$ dominates $x_1$, and $ux_2$ is an edge in $\text{dom}^e(D)$. A similar argument holds for $x_i$ and $x_{i-1}$ when $i \geq 3$.

The sources that can be used to create edges in $\text{dom}^e(D)$ have now been identified, as well as the vertices with which those edges must be incident. Following is a lemma that shows the only possible vertex that can be the origin of more than one single arc in $D$ is a vertex isomorphic to the component $K_1$ in $\text{UG}^e(D)$.

Lemma 2.13 [6] Let $\text{UG}(D) \cong \text{dom}(D)$ where every component of $\text{UG}^e(D)$ is isomorphic to $K_1$, $K_2$, or $C_4$. Let $x$ be a vertex of $D$. If $x$ is in a component isomorphic to $C_4$ in $\text{UG}^e(D)$, then $x$ is the origin of no single arcs of $D$. If $x$ is in a component isomorphic to $K_2$ in $\text{UG}^e(D)$, then $x$ is the origin of at most one single arc of $D$. If $x$ is in a component isomorphic to $K_1$ in $\text{UG}^e(D)$, then $x$ is the origin of at most two single arcs of $D$.

To complete the construction preliminaries, it is important to have an idea of where the created edges can be placed. Consider a path $P = y_1, y_2, \ldots, y_{i-1}, y_i$ in $\text{UG}^e(D)$. We call $y_1$ and $y_i$ the outer vertices of $P$. Vertices $y_2$ and $y_{i-1}$ are called the inner end vertices (since these are the end vertices of generated subpaths). All other vertices of the paths are inner vertices. First, we determine under what conditions an edge may be incident with a vertex in $\text{dom}^e(D)$.

Proposition 2.14 Let $\text{UG}(D) \cong \text{dom}(D)$ and $\text{UG}^e(D) = C_4 \cup P_1 \cup P_3$. If $v \in V(D)$ is incident with exactly one generated edge in $\text{dom}^e(D)$, then $v$ is incident with at most one created edge.

Proof. Vertices in $\text{UG}^e(D) = C_4 \cup P_1 \cup P_3$ have degree of at most 2. Since $\text{UG}(D) \cong \text{dom}(D)$, no vertex can be incident with more than two edges.

Corollary 2.15 Only a generated subpath $P_1$ in $\text{dom}^e(D)$ can be incident with two created edges.

Corollary 2.16 Only specific vertices of components $K_1$, $K_2$, and $P_3$ in $\text{UG}^e(D)$ can be incident with two created edges in $\text{dom}^e(D)$.
One important aspect of joining subpaths to create a larger path is that we must be careful about what vertices are used to create an edge.

**Lemma 2.17** Let \( UG(D) \cong \text{dom}(D) \) where every component of \( UG^c(D) \) is isomorphic to a path or \( C_4 \). A created edge in \( \text{dom}^c(D) \) between any two end vertices must have as a source a vertex \( K_1 = \{y\} \) in \( UG^c(D) \).

**Proof.** The end vertices of any path can be sources for created edges in \( \text{dom}^c(D) \). Any edge for which they are a source is incident with the corresponding inner end vertex in \( P_k, k \geq 2 \). Thus, end vertices cannot be used to create edges between end vertices. According to Lemma 2.8, \( K_1 = \{y\} \) can create any edge, so is the only way that an edge can be created between two end vertices. ■

**Corollary 2.18** Let \( UG(D) \cong \text{dom}(D) \) where every component of \( UG^c(D) \) is isomorphic to a nontrivial path or \( C_4 \). Then there is no biorientation of \( D \) such that the end vertices of any path form a created edge in \( \text{dom}^c(D) \).

Another special case occurs when we try to create an edge between two inner end vertices to make a cycle. In order for this to happen, the path must have an odd number of vertices. Otherwise, the inner end vertices will not be on the same generated subpath and no cycle will be made. Figure 2 illustrates this concept. The figure is a mix of \( D \) and \( \text{dom}^c(D) \). Single arc \( (v_1, v_6) \) causes neither \( v_2 \) nor \( v_5 \) to dominate \( v_1 \) in \( D \). This creates edge \( v_2v_6 \) in \( \text{dom}^c(D) \) between inner end vertices \( v_2 \) and \( v_6 \). The same edge can also be created with single arc \( (v_7, v_2) \). The generated subpaths in \( \text{dom}^c(D) \), \( v_1, v_3, v_5, v_7 \) and \( v_2, v_4, v_6 \), are shown, as well as the created edge \( v_2v_6 \). After constructing this \( C_4 \), no other created edge can be adjacent to \( v_2 \) or \( v_6 \), as \( C_4 \) in \( \text{dom}^c(D) \) cannot be adjacent to another edge. The following lemma formalizes this.

![Figure 2: Generated subpaths \( v_1, v_3, v_5, v_7 \) and \( v_2, v_4, v_6 \) in \( \text{dom}^c(D) \), created edge \( v_2v_6 \) in \( \text{dom}^c(D) \), and the single arcs in \( D \) that create \( v_2v_6 \) in \( \text{dom}^c(D) \).](image)

**Lemma 2.19** Let \( UG(D) \cong \text{dom}(D) \) where every component of \( UG^c(D) \) is isomorphic to a path or \( C_4 \), and two inner end vertices of one subpath are joined by a created edge in \( \text{dom}^c(D) \). Then the end vertices of the path from which the subpath was generated can be the source of no other edge.

**Proof.** Let \( u_1, u_2, \ldots, u_{i-1}, u_i \) be a path in \( UG^c(D) \), where \( u_2 \) and \( u_{i-1} \) are inner end vertices of one subpath, and \( u_2u_{i-1} \) is an edge in \( \text{dom}^c(D) \). From Lemma 2.12, we see
that \( u_2 \) and \( u_{i-1} \) are incident with created edges from sources \( u_1 \) and \( u_i \) respectively. If \( u_1 \) or \( u_i \) was the source of a different edge, then it would be a pendant edge to the cycle \( u_1, u_2, \ldots, u_{i-1}, u_i, u_1 \) in \( \text{dom}(D) \), which indicates \( \text{dom}(D) \not\cong UG^c(D) \).

To finalize the preliminaries, we take care of the cases \( UG^c(D) = C_4 \cup P_1 \cup P_j \) where \( i, j = 1, 2 \). These were special cases examined in [6]. The theorem from that paper is generalized, and we find that there is a biorientation for each of the four cases.

**Theorem 2.20** [6] Let \( G \) be a graph such that \( G \) is the union of \( r \) copies of \( K_2 \), and \( s \) copies of \( C_4 \) with \( r + s + t \geq 1 \). Then there exists a digraph \( D \) with \( \text{dom}(D) \cong UG^c(D) \) if and only if \( 2t \leq r + s \), and if \( s = 1 \), then \( r \geq 1 \).

**Corollary 2.21** Let \( UG^c(D) = C_4 \cup P_1 \cup P_j \) for \( i, j = 1, 2 \). Then there exists a biorientation of the edges of \( UG(D) \) such that \( UG(D) \cong \text{dom}(D) \).

### 3 Existence where \( UG^c(D) = C_4 \cup P_1 \cup P_j, j \geq 3 \)

In every graph of \( C_4 \cup P_1 \cup P_j \) when \( j \) is at least 3, there are \( 4 + (j - 1) \) edges. The generated edges in \( \text{dom}^c(D) \) total 2 from \( C_4 \) and \( j - 2 \) from the vertices of \( P_j \). We must, therefore, be able to create 3 edges in \( \text{dom}^c(D) \) if we are to have \( UG(D) \cong \text{dom}(D) \).

**Proposition 3.1** Let \( UG(D) \cong \text{dom}(D) \), and \( UG^c(D) = C_4 \cup P_1 \cup P_j \), where \( P_1 = \{y\} \) and \( P_j = z_1, \ldots, z_j \) for \( j \geq 3 \). Then \( y, z_1 \), and \( z_j \) must all be used as sources for distinct created edges in \( \text{dom}^c(D) \).

**Proof.** There are \( j \) generated edges in \( \text{dom}^c(D) \). There are \( j + 3 \) edges in \( UG^c(D) \). Since \( UG^c(D) \cong \text{dom}^c(D) \), three edges must be created in \( \text{dom}^c(D) \). Only \( y \), \( z_1 \), and \( z_j \) can be sources of these edges, so each must be the source of exactly one created edge.

To determine what must be done with these three edges, consider how the copy of \( C_4 \) must be designed in \( \text{dom}^c(D) \). The first question may be whether or not the vertices of \( C_4 \) in \( UG^c(D) \) will be the vertices of \( C_4 \) in \( \text{dom}^c(D) \). The next proposition shows that this is not possible.

**Proposition 3.2** Let \( UG(D) \cong \text{dom}(D) \), and \( UG^c(D) = C_4 \cup P_1 \cup P_j \), \( i \geq 1, j \geq 2 \), where \( C_4 = x_1, x_2, x_3, x_4, x_1 \). Then at most one of the generated edges \( x_1x_3 \) or \( x_2x_4 \) is an edge of \( C_4 \) in \( \text{dom}^c(D) \).

**Proof.** If both \( x_1x_3 \) and \( x_2x_4 \) are edges of \( C_4 \) in \( \text{dom}^c(D) \), then there must be two created edges joining them. From Lemma 2.12, we know that inner end vertices will be incident with the created edges for any source other than \( P_1 \). Thus, we must have two copies of \( P_1 \) to create the edges, which we do not.

Given \( P_j \geq 5 \), it is natural to ask whether \( z_2z_{j-1} \) can be a created edge. Note that it is a generated edge when \( j = 5 \).
Lemma 3.3 Let $UG(D) \cong \text{dom}(D)$, and $UG^c(D) = C_4 \cup P_1 \cup P_j$, where $j \geq 5$ odd. Then $z_2 z_{j-1}$ is not a created edge in $\text{dom}^c(D)$.

Proof. Edge $z_2 z_{j-1}$ creates a cycle in $\text{dom}^c(D)$. Since $UG^c(D) \cong \text{dom}^c(D)$, the cycle is $C_4$, which indicates $j = 9$. By Lemma 2.19, $z_1$ and $z_9$ must both be the source of $z_2 z_{j-1}$. However, they must also source two distinct created edges. Thus, $z_2 z_{j-1}$ cannot be a created edge in $\text{dom}^c(D)$. \hfill \blacksquare

Now we begin to determine the values for $j$ that will yield $\text{dom}^c(D) \cong C_4 \cup P_1 \cup P_j$.

The method that is used throughout the remainder of the paper is to consider the values for $j$ that will allow $C_4$ to be constructed out of 1, 2, or 3 of the generated subpaths in $\text{dom}^c(D)$. We begin by determining the values of $j$ where $C_4$ is created using one of the generated subpaths in $\text{dom}^c(D)$.

Figure 3: Example of a digraph and its associated $\text{dom}^c(D)$ graph where $UG^c(D) = C_4 \cup P_1 \cup P_7$. Edges shown are in $\text{dom}^c(D)$, while single arcs are in $D$. Bidirected edges of $D$ are omitted.

Lemma 3.4 If $UG(D) \cong \text{dom}(D)$, $UG^c(D) = C_4 \cup P_1 \cup P_j$, $j \geq 3$, and $C_4$ in $\text{dom}^c(D)$ is formed using only vertices from one generated subpath, then $j = 7, 8$. Furthermore, such $D$ exists.

Proof. In $UG^c(D)$, let $C_4 = x_1, x_2, x_3, x_4, x_1, P_1 = \{y\}$, and $P_j = z_1, \ldots, z_j$ for $j \geq 3$. Since one generated subpath is being used to form $C_4$, it must have four vertices. With $i = 1$, the subpath must be generated from the vertices of $P_j$, so $j = 7, 8, 9$. If $j = 9$, then the two inner end vertices would have to be joined to form $C_4 = z_2, z_4, z_6, z_8, z_2$, which violates Lemma 3.3. For $j = 7$, biorient all edges of $UG(D)$, except construct single arcs $(y, z_1), (y, z_7), (z_1, x_1)$, and $(z_7, x_2)$, creating edges $z_1 z_7, x_1 z_2$, and $x_2 z_6$ in $\text{dom}^c(D)$. Thus, the components of $\text{dom}^c(D)$ are $C_4 = z_1, z_3, z_5, z_7, z_1, P_1 = \{y\}$, and $P_7 = x_3, x_1, z_2, z_4, z_6, x_2, x_4$ (see Figure 3). For $j = 8$, biorient all edges of $UG(D)$, except construct single arcs $(z_8, z_1), (z_1, x_1), (y, x_2)$, and $(y, z_3)$, creating edges $z_1 z_7, x_1 z_2$, and $x_2 z_3$ in $\text{dom}^c(D)$. Thus, the components of $\text{dom}^c(D)$ are $C_4 = z_1, z_3, z_5, z_7, z_1, P_1 = \{y\}$, and $P_8 = x_4, x_2, x_3, x_1, z_2, z_4, z_6, z_8$. \hfill \blacksquare

The next lemma considers the possibility that $C_4$ is created using $P_1$ and $P_8$ or $P_2$ and $P_3$. The latter possibility is broken into two cases: $x_1 z_3$ is one $P_2$ or it is not.
Lemma 3.5 If $UG(D) \not\equiv \text{dom}(D)$, $UG^c(D) = C_4 \cup P_1 \cup P_2$, $j \geq 3$, and $C_4 \in \text{dom}^c(D)$ is formed using only vertices from exactly two generated subpaths, then $j = 4, 5$. Furthermore, such $D$ exists.

Proof. In $UG^c(D)$, let $C_4 = x_1, x_2, x_3, x_4$, $P_1 = \{y\}$, and $P_2 = z_1, \ldots, z_j$ for $j \geq 3$. We break this proof into two possibilities for the two subpaths. Either $P_3$ and $P_1$ are the subpaths, or there are two $P_2$ that will be used. $P_3$ is a generated subpath when $j = 5, 6, 7$. $P_1$ must be $y$ since there are no other $K_1$ generated. Vertices $z_1$ and $z_j$ must be the sources of the two edges incident to $y$ in $C_4$. By Lemma 2.12, these edges are also incident to $z_2$ and $z_{j-1}$ respectively. So, $z_2z_{j-1}$ cannot be a generated edge without forming a cycle with $y$. To form $C_4$, $z_2$ and $z_{j-1}$ must be adjacent to a fourth vertex in a generated subpath of $\text{dom}^c(D)$ (Proposition 2.14). This implies that $P_3 = z_2, z_4, z_6$ is a generated subpath on $V(P_3)$ and $j = 7$ is the only possibility. However, generated edges $x_1x_3$ and $x_2x_4$ and subpath $z_1, z_3, z_5, z_7$ cannot be appended to form $P_7$. Therefore, $j \neq 7$.

Now consider that $C_4$ is created using two generated subpaths, $P_2$. Proposition 3.2 states that at most one of the edges $x_1x_3$ or $x_2x_4$ is an edge that can be used to create $C_4$. This indicates that at least one other generated subpath on two vertices must exist, so $j = 3, 4, 5$. Vertices $z_1$ and $z_j$ must be used as sources, creating edges incident with $z_2$ and $z_{j-1}$ respectively. While $y$ may be used as a source for an edge in $C_4$, at least one of $z_1$ and $z_j$ must be the source of an edge in $C_4$. If $j = 3$, then $z_1$ and $z_3$ create two edges incident with $z_2$, and $C_4$ cannot be created. If $j = 4$, biorient all edges of $UG(D)$, except construct single arcs $(y, x_4)$, $(y, z_4)$, $(z_1, x_2)$, and $(z_4, x_3)$, creating edges $x_3z_4$, $x_2z_3$, and $x_4z_3$ in $\text{dom}^c(D)$. So, $\text{dom}^c(D) \not\equiv C_4 \cup P_1 \cup P_2$. If $j = 5$, biorient all edges of $UG(D)$, except construct single arcs $(z_1, x_1)$, $(z_5, x_3)$, $(y, x_2)$, and $(y, z_1)$, creating edges $x_1z_2$, $x_3z_4$, and $x_2z_1$ in $\text{dom}^c(D)$. So, $\text{dom}^c(D) \not\equiv C_4 \cup P_1 \cup P_2$. Figure 4 shows this construction. 

![Figure 4: Example of a digraph and its associated $\text{dom}^c(D)$ graph where $UG^c(D) = C_4 \cup P_1 \cup P_2$. Edges shown are in $\text{dom}^c(D)$, while single arcs are in $D$. Bidirected edges of $D$ are omitted.](image)

While it is possible to continue looking at joining more and more subpaths, it is not necessary since there are only so many that can be used to create $C_4$. In the case where we have $UG^c(D) = C_4 \cup P_1 \cup P_2$, we can join at most two generated subpaths.

Lemma 3.6 If $UG(D) \not\equiv \text{dom}(D)$ and $UG^c(D) = C_4 \cup P_1 \cup P_2$ for $j \geq 3$, then there is no biorientation of the edges of $UG(D)$ such that the copy of $C_4$ in $\text{dom}^c(D)$ can be created using three or more generated subpaths.
Proof. The only way to append three subpaths to form $C_4$ is to use one copy of $P_2$ and two copies of $P_1$. This can only occur when $j = 3$, creating $P_i = y$ and $P_j = z_1, z_2, z_3$. Both $y$ and $z_2$ will be incident with two created edges. Sources $z_1$ and $z_3$ create two edges incident with $z_2$, only one of which can be incident with $y$. There is no source for another edge incident with $y$, so this construction is not possible.

Combining Lemmas 3.4, 3.5, and 3.6, we have the following.

**Theorem 3.7** If $UG(D) \cong dom(D)$ and $UG^c(D) = C_4 \cup P_i \cup P_j$ for $j \geq 3$, then $j = 4, 5, 7, 8$. Furthermore, such $D$ exists.

### 4 Existence where $UG^c(D) = C_4 \cup P_i \cup P_j$, $i \geq 2$, $j \geq 3$

The existence of a biorientation for $UG(D)$ when $i = j = 2$ has been established in Corollary 2.21, so we begin the remaining cases with $i \geq 2$ and $j \geq 3$.

**Lemma 4.1** If $UG(D) \cong dom(D)$, $UG^c(D) = C_4 \cup P_i \cup P_j$, $i, j \geq 2$, where $P_i = y_1, \ldots, y_i$ and $P_j = z_1, \ldots, z_j$, then $y_1, y_i, z_1, z_j$ must all be used as source vertices of distinct created edges in $dom^c(D)$.

Proof. There are $i + j - 2$ generated edges in $dom^c(D)$. Since $UG^c(D)$ has $i + j + 2$ edges and $UG^c(D) \cong dom^c(D)$, four edges must be created in $dom^c(D)$. Corollary 2.11 states that $y_1, y_i, z_1, z_j$ are the only possible source vertices. So they must each source a distinct created edge in $dom^c(D)$.

The copies of $P_i$ and $P_j$ that are found in $dom^c(D)$ are constructed by appending generated subpaths using created edges. Here we define $P_r \hat{\cup} P_s$ as the graph obtained by creating an edge between an end vertex of $P_r$ and an end vertex of $P_s$. When such an edge is created, we say that $P_r$ has been appended to $P_s$. This operation is associative, but not necessarily commutative. For example, with $P_r \hat{\cup} P_s \hat{\cup} P_t$, we create a distinct edge between subpaths $P_r$ and $P_s$, and another between $P_s$ and $P_t$. This does not guarantee that $P_r \hat{\cup} P_s \hat{\cup} P_t$ is a possible construction.

As was done when $i = 1$, we look at constructing $C_4$ in $dom^c(D)$ by appending $1, 2, 3$ generated subpaths. Since there is no $P_1$ as a component in $UG^c(D)$, there will be no constructions where any vertex is the origin of more than once single arc (see Lemma 2.13).

**Lemma 4.2** If $UG(D) \cong dom(D)$, $UG^c(D) = C_4 \cup P_i \cup P_j$, $i \geq 2$, $j \geq 3$, where $P_i = y_1, \ldots, y_i$, $P_j = z_1, \ldots, z_j$, and $C_4$ is formed using vertices from exactly one generated subpath in $dom^c(D)$, then $j = 8$.

Proof. The generated subpath must have four vertices, and the possible values of $j$ where this occurs are $j = 7, 8, 9$. There is no $K_1$ as a component in $UG^c(D)$. Thus,
using Corollary 2.18 we see that this subpath cannot have two outer vertices, and \( j \neq 7 \). If the subpath uses two inner end vertices, then Lemma 2.19 shows that \( z_1 \) and \( z_j \) must be the sources for that edge. However, Lemma 4.1 indicates that \( z_1 \) and \( z_j \) must be the sources for distinct edges, so \( j \neq 7 \). Thus, \( j = 8 \). ■

Lemma 4.3 If \( UG(D) \cong \text{dom}(D) \), \( UG^e(D) = C_4 \cup P_i \cup P_j \), \( i \geq 2 \), \( j \geq 3 \), where \( C_4 = x_1, x_2, x_3, x_4, x_1 \), \( P_i = y_1, \ldots, y_i \), \( P_j = z_1, \ldots, z_j \), and \( C_4 \) in \( \text{dom}^e(D) \) is formed using vertices from exactly one generated subpath in \( P_j \), then \( i = 2, 4, 6, 7, 9, 12, 15 \) and \( j = 8 \). Furthermore, such \( D \) exists.

Proof. By Lemma 4.2 we know \( j = 8 \). Without loss of generality, we may take \((z_1, z_8)\) to be a single arc in \( D \) for the remainder of the proof, thus creating edge \( z_2z_8 \) in \( \text{dom}^e(D) \) and \( C_4 = z_2, z_4, z_6, z_8, z_2 \). That leaves three sources, \( y_1, y_i, \) and \( z_j \), and five generated subpaths with which to create \( P_i \) and \( P_j \). These subpaths are \( x_1, x_3 \) and \( x_2, x_4 \), plus the two subpaths of \( P_i \), and \( z_1, z_3, z_5, z_7 \). Note that \( z_7 \) is an inner end vertex and so will be incident with a created edge that has \( z_8 \) as the source.

We will consider the cases where \( i \) is even and \( i \) is odd.

1. \( i = 2m \) is even. Two copies of \( P_m \) are generated on \( V(P_i) \). We use these subpaths with \( x_1, x_3, x_2, x_4, \) and \( z_1, z_3, z_5, z_7 \) to create \( P_3 \) and \( P_{2m} \) in \( \text{dom}^e(D) \).
   
   Refer to \( P_3 \) and \( P_{2m} \) as paths \( S \) and \( T \) in no particular order. Each of the subpaths \( P_4, P_m \) and \( P_m \) have an inner end vertex, so each will be incident with a created edge. There are only three nonisomorphic ways to append the subpaths: \( S = P_2^A P_3^A P_m^A P_m \) and \( T = P_2 \); \( S = P_2^A P_m^A P_m^A P_m \) and \( T = P_2^A P_4 \); and \( S = P_2^A P_4^A P_m \) and \( T = P_2 \).

   (a) \( S = P_2^A P_4^A P_m^A P_m \) and \( T = P_2 \). Since \( T \) is a path on 2 vertices, \( S \) must be the path on 8 vertices. So \( 2 + 4 + 2m = 8 \), and \( i = 2m = 2 \). In addition to single arc \((z_1, z_8)\), construct single arcs \((z_8, y_2)\), \((y_2, x_2)\), and \((y_1, x_4)\) in \( D \), and biorient all other edges of \( UG(D) \). This creates edges \( y_2z_7, y_1x_2 \) and \( x_4y_2 \) in \( \text{dom}^e(D) \) so \( \text{dom}^e(D) \cong UG^e(D) \). This construction is shown in Figure 5(a).

   (b) \( S = P_2^A P_m^A P_m^A P_m \) and \( T = P_2^A P_4 \). \( T \) is a path on 6 vertices, and \( S \) must be the path on 8 vertices, so \( i = 6 \). Similarly to 1(a), the construction of single arcs \((z_1, z_8)\), \((z_8, x_4)\), \((y_1, x_1)\), and \((y_2, x_2)\) in \( D \) create the necessary edges in \( \text{dom}^e(D) \) so that \( \text{dom}^e(D) \cong UG^e(D) \). This construction is shown in Figure 5(b).

   (c) \( S = P_2^A P_4^A P_m \) and \( T = P_2 \). In this case, either \( S \) or \( T \) can be the path with 8 vertices. If \( S \) has 8 vertices, then \( i = 4 \). Similarly to 1(a), the construction of single arcs \((z_1, z_8)\), \((z_8, z_3)\), \((y_1, x_1)\), and \((y_2, x_2)\) in \( D \) create the necessary edges in \( \text{dom}^e(D) \) so that \( \text{dom}^e(D) \cong UG^e(D) \).

   On the other hand, if \( T \) has 8 vertices, then \( i = 12 \). Similarly to 1(a), the construction of single arcs \((z_1, z_8), (z_8, y_1), (y_1, x_1), \) and \((y_2, x_4)\) in \( D \) create the necessary edges in \( \text{dom}^e(D) \) so that \( \text{dom}^e(D) \cong UG^e(D) \).
2. \( i = 2m + 1 \) is odd. The two subpaths generated are an outer subpath on \( m + 1 \) vertices and an inner subpath on \( m \) vertices. The inner subpath has two inner end vertices, which will be appended to two other subpaths, placing it in the middle of \( S \) or \( T \). The subpath \( P_4 = z_1, z_3, z_5, z_7 \) has the only other inner end vertex, \( z_7 \). Thus, there are only four nonisomorphic ways to append the subpaths: \( S = P_3 \cup P_m \cup P_2 \cup P_2 \) and \( T = P_3 \cup P_m \cup P_2 \cup P_2 \) and \( T = P_3 \); \( S = P_3 \cup P_m \cup P_m \cup P_2 \) and \( T = P_2 \cup P_4 \); and \( S = P_2 \cup P_m \cup P_2 \) and \( T = P_3 \cup P_{m+1} \). In the second and third case, \( S \) is odd, \( T \) is constant, and there is no way to construct \( P_5 \). The remaining two subcases are addressed below.

(a) \( S = P_3 \cup P_m \cup P_2 \cup P_2 \) and \( T = P_m+1 \). Since \( m \) must be at least 1, \( S \) cannot be a path on 8 vertices, so \( T = P_m+1 = P_8 \), and \( i = 15 \). Similarly to 1(a), the construction of single arcs \((z_1, z_8)\), \((z_8, x_3)\), \((y_1, x_1)\), and \((y_15, x_4)\) in \( D \) create the necessary edges in \( \text{dom}^c(D) \) so that \( \text{dom}^c(D) \cong UG^c(D) \).

(b) \( S = P_2 \cup P_m \cup P_2 \) and \( T = P_3 \cup P_{m+1} \). If \( S \) is a path on 8 vertices, then \( m = 4 \), and \( i = 9 \). Similarly to 1(a), the construction of single arcs \((z_1, z_8)\), \((z_8, y_1)\), \((y_1, x_1)\), and \((y_9, x_4)\) in \( D \) create the necessary edges in \( \text{dom}^c(D) \) so that \( \text{dom}^c(D) \cong UG^c(D) \).

On the other hand, if \( T \) is a path on 8 vertices, then \( m = 3 \), and \( i = 7 \). Similarly to 1(a), the construction of single arcs \((z_1, z_8)\), \((z_8, y_1)\), \((y_1, x_1)\), and \((y_7, x_2)\) in \( D \) create the necessary edges in \( \text{dom}^c(D) \) so that \( \text{dom}^c(D) \cong UG^c(D) \).

Thus, if \( j = 8 \), then \( i = 2, 4, 6, 7, 9, 12, 15 \), and such \( D \) exists.

Next, we look at the case where \( C_4 \) is created using two generated subpaths. The first lemma considers creating \( C_4 \) using generated subpaths \( P_1 \) and \( P_3 \), and the second lemma using \( P_2 \) and \( P_2 \).
Lemma 4.4 If \( UC(D) \cong \text{dom}(D) \), \( UG^c(D) = C_4 \cup P_i \cup P_j \), \( i \geq 2 \), \( j \geq 3 \), where \( C_4 = x_1, x_2, x_3, x_4, x_5, P_i = y_1, \ldots, y_i \), \( P_j = z_1, \ldots, z_j \), and \( C_4 \) is formed using vertices from exactly two generated subpaths \( P_1 \) and \( P_3 \) in \( \text{dom}^c(D) \), then \( i = 2 \) and \( j = 6 \). Furthermore, such \( D \) exists.

Proof. Since \( P_1 \) is a vertex in \( C_4 \), there are two created edges incident with it. Because \( K_1 \) is not a component of \( UG^c(D) \), \( P_1 \) must be a generated subpath of \( P_i \) or \( P_j \). Without loss of generality, say \( P_i \). Then \( i = 2 \), 3.

If \( i = 2 \), say that \( P_1 = y_2 \). Subpath \( P_3 \) must come from \( P_j \) and have exactly one inner end vertex to append to \( y_2 \). Thus, \( j = 6 \). Biorient all edges of \( UG(D) \) except construct single arcs \((z_1, y_1), (y_2, z_6), (z_6, x_2)\), and \((y_1, x_4)\). The created edges form components \( C_4 = y_1, z_2, z_4, z_5, y_1, P_2 = x_1, x_3, \) and \( P_6 = y_2, x_4, x_2, z_6, z_3, z_1 \) in \( \text{dom}^c(D) \).

If \( i = 3 \) and \( P_1 = y_3 \), then \( y_1 \) and \( y_3 \) are sources of both edges incident with \( y_3 \). Thus, \( P_3 \) cannot contain another interior vertex since the created edges are each incident with only one. \( P_2 \) must be an outer subpath, so \( j = 5 \). However, if \( C_4 \) is constructed using \( z_1, z_3, z_5 \), then \( P_3 \) and \( P_5 \) cannot be created using the remaining subpaths, which are all even. ■

Lemma 4.5 If \( UC(D) \cong \text{dom}(D) \), \( UG^c(D) = C_4 \cup P_i \cup P_j \), \( i \geq 2 \), \( j \geq 3 \), where \( C_4 = x_1, x_2, x_3, x_4, x_5, P_i = y_1, \ldots, y_i \), \( P_j = z_1, \ldots, z_j \), and \( C_4 \) is formed using vertices from exactly two generated subpaths \( P_2 \) and \( P_4 \) in \( \text{dom}^c(D) \), then \( i = j = 4 \) or \( i = 5 \) and \( j = 2, 3, 4, 6, 9 \). Furthermore, such \( D \) exists.

Proof. Proposition 3.2 states that at most one of \( x_1x_3 \) or \( x_2x_4 \) will be an edge in \( C_4 \).

1. Suppose that \( x_1x_3 \) is an edge of \( C_4 \). The two created edges incident with \( x_1 \) and \( x_3 \) will be incident with two adjacent inner end vertices of \( P_i \) or \( P_j \). Say it is \( P_2 \), making \( y_2, y_4 \) the generated subpath appended to \( x_1x_3 \) and \( i = 5 \). Without loss of generality, construct single arcs \((y_1, x_1)\) and \((y_5, x_3)\) in \( D \) so that \( C_5 = x_1, y_2, y_4, x_3, x_1 \) in \( \text{dom}^c(D) \). Now we find the values of \( j \) where such a \( D \) exists.

Consider how \( P_5 \) can be made by appending the two generated subpaths on \( V(P_j) \), \( x_2, x_4 \) and \( y_1, y_3, y_6 \). Only two created edges can be constructed using the remaining sources \( z_1 \) and \( z_j \), and will be incident with \( z_2 \) and \( x_{j-1} \) respectively (Lemma 2.12). We look at the number of subpaths that are appended to construct \( P_5 \).

Case 1: \( P_5 \) is a generated subpath. Then \( P_5 \) can have no inner end vertices, or it will be incident with a created edge. Thus, \( j = 9 \). Construct single arcs \((z_1, y_1), (z_9, x_4), (y_1, x_1)\), and \((y_6, x_3)\) in \( D \) and biorient all other edges of \( UG(D) \). This creates the necessary edges in \( \text{dom}^c(D) \) so that \( \text{dom}^c(D) \cong UG^c(D) \).

Case 2: \( P_5 \) is constructed using \( z_2 \) or \( x_{j-1} \), but not both. So, \( z_2 \) and \( x_{j-1} \) are on different generated subpaths, and \( j \) is even. Say that \( z_2 \) on generated subpath
P_{z_1} is appended to another subpath P to create P_5. Now P does not contain
z_{j-1}, so must either be x_2, x_4 or y_1, y_3, y_5. Thus, |V(P_{z_2})| = 3 or 2,
resulting in j = 6 or j = 4. If j = 4, construct single arcs (z_1, y_1), (z_4, x_2),
(y_1, x_1), and (y_5, x_3) in D and biorient all other edges of UG(D). If j = 6,
construct single arcs (z_1, x_2), (z_6, y_5), (y_1, x_1), and (y_5, x_3) in D and biorient all other edges of
UG(D). Both of these constructions create the necessary edges in dom_C(D) so
that dom_C(D) \cong UG_C(D).

Case 3: P_5 is constructed using both z_2 and z_{j-1}. Either x_2, x_4 or y_1, y_3, y_5
but not both will be appended to the two inner end vertices (else there would be
more than 5 vertices). The subpath that is not appended will be a path in
dom_C(D), so j = 2, 3. If j = 3, sources z_1 and z_3 create two edges incident
with z_2. This forms a cycle in dom_C(D), so j \neq 3. If j = 2, construct single
arcs (z_1, y_1), (z_2, y_5), (y_1, x_1), and (y_5, x_3) in D and biorient all other edges of
UG(D). This creates the necessary edges in dom_C(D) so that dom_C(D) \cong
UG_C(D).

2. Suppose that x_1x_3 is not an edge of C_4. The copy of C_4 cannot be created from
one generated subpath in dom_C(D) (Lemma 2.19). If C_4 were to be made from
the two generated subpaths of one path, then that path is P_4. However, the
only non-generated edge that can be created by either source z_1 or z_4 is z_2z_3,
so C_4 cannot be constructed. Thus, C_4 must be constructed from a generated
copy of P_2 in P_i and another in P_j. Since these paths must be appended using
exactly two created edges, the two copies of P_2 together must have exactly two
inner end vertices. This dictates that i > 3 or j > 3. Say that i > 3, so i = 4, 5
and j = 3, 4, 5 are the only values where P_2 is a generated subpath.

We have already shown that D exists for i = 5 and j = 2, 4, 6, 9, so we need
only consider i = 4 with j = 3, 4, and i = 5 with j = 3, 5. When i = 4, only
one vertex of each generated subpath is an inner end vertex, so the subpath
must be appended to another copy of P_2 with exactly one inner end vertex.
Thus, j \neq 3 and j = 4. Construct single arcs (z_1, y_1), (z_4, x_2), (y_1, x_1), and
(y_4, z_3) in D and biorient all other edges of UG(D). This creates the necessary
edges in dom_C(D) so that dom_C(D) \cong UG_C(D).

When i = 5, the copy of P_2 is an inner subpath, so both vertices are inner end
vertices. Thus, it can only be appended to a copy of P_2 where both vertices are
outer vertices. This implies that j = 3. Construct single arcs (y_1, z_1), (y_5, z_3),
(z_1, x_1), and (z_3, x_3) in D and biorient all other edges of UG(D). This creates
the necessary edges in dom_C(D) so that dom_C(D) \cong UG_C(D).

Given the restrictions of appending paths, these are the only possible construc-
tions. So i = j = 4 or i = 5 and j = 2, 3, 4, 6, 9, and such D exists.

Finally, the case where C_4 is constructed using vertices from exactly three gen-
erated subpaths is examined.
Lemma 4.6 If \( UG(D) \cong \text{dom}(D) \), \( UG^c(D) = C_4 \cup P_i \cup P_j \), \( i \geq 2 \), \( j \geq 3 \), where \( C_4 = x_1, x_2, x_3, x_4, x_1 \), \( P_i = y_1, \ldots, y_i \), \( P_j = z_1, \ldots, z_j \), and \( C_4 \) is formed using vertices from exactly three generated subpaths in \( \text{dom}^c(D) \), then \( i = 2 \) and \( j = 3 \). Furthermore, such \( D \) exists.

Proof. As seen in the proof of Lemma 3.6, this requires one copy of \( P_2 \) and two copies of \( P_1 \) as generated subpaths. If \( i = 2 \), there are 2 copies of \( P_1 \) generated. However, \( y_1, y_2 \) will never be an edge in \( \text{dom}^c(D) \) (Corollary 2.18). Thus, only one of \( y_1 \) or \( y_2 \) may be a vertex in \( C_4 \). This implies the other copy of \( P_1 \) must come from \( V(P_j) \). Since \( j \geq 3 \), only \( j = 3 \) produces a subpath with one vertex. Biorient all edges of \( UG(D) \), except construct single arcs \((z_1, y_1)\), \((z_3, x_3)\), \((y_1, z_3)\), and \((y_2, z_1)\). This creates the necessary edges in \( \text{dom}^c(D) \) so that \( \text{dom}^c(D) \cong UG^c(D) \). If \( i = 3 \), then it has one subpath \( P_1 \), and the others must come from \( V(P_j) \). Again, \( j = 3 \). However, all created edges are incident with \( y_2 \) and \( z_2 \), so there is no way to construct \( C_4 \). If \( i \geq 4 \), with \( j \geq 3 \), there are not two subpaths \( P_1 \). Thus, \( i = 2 \) and \( j = 3 \) is the only possibility. \( \blacksquare \)

As a grand finale, we bring together the results that appear at the end of Section 2, the end of Section 3, and the lemmas stated previously in this section.

Theorem 4.7 If \( UG(D) \cong \text{dom}(D) \), and \( UG^c(D) = C_4 \cup P_i \cup P_j \), then

1. \( i = 1 \) and \( j = 1, 2, 4, 5, 7, 8 \), or
2. \( i = 2 \) and \( j = 2, 3, 5, 6, 8 \), or
3. \( i = 4 \) and \( j = 4, 5 \), or
4. \( i = 5 \) and \( j = 3, 6, 9 \), or
5. \( i = 8 \) and \( j = 4, 6, 7, 9, 12, 15 \).

Furthermore, in each case such a digraph exists.

Proof. Corollary 2.21 addresses existence for \( i, j = 1, 2 \). Further results for \( i = 1 \) were listed in Theorem 3.7. Except for \( i = j = 2 \), the results in parts (2)-(5) come directly from Lemmas 4.3, 4.4, 4.5, and 4.6. Although the creation of \( C_4 \) using four components was not specifically mentioned, it requires that \( i = j = 2 \), so that there are four copies of \( P_1 \) with which to construct \( C_4 \). This possibility is covered in Corollary 2.21. \( \blacksquare \)

References


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