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Kim A. S. Factor
Marquette University, kim.factor@marquette.edu

Rebecca M. Kohler
Marquette University

Jason M. Derby
GE Healthcare

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Local Out-Tournaments with Upset Tournament Strong Components I: Full and Equal $\{0,1\}$-Matrix Ranks

Kim A. S. Factor and Rebecca M. Kohler
Marquette University
P.O. Box 1881, Milwaukee, WI 53201-1881
kim.factor@marquette.edu

Jason M. Darby
GE Healthcare
Waukesha, WI 53188

Abstract

A digraph D is a local out-tournament if the outset of every vertex is a tournament. Here, we use local out-tournaments, whose strong components are upset tournaments, to explore the corresponding ranks of the adjacency matrices. Of specific interest is the out-tournament whose adjacency matrix has boolean, nonnegative integer, term, and real rank all equal to the number of vertices, n. Corresponding results for biclique covers and partitions of the digraph are provided.

1 Introduction

The topics of local tournaments, $\{0,1\}$-matrix ranks, upset tournaments, and digraph biclique cover and partition numbers have been the foundation of many papers in the area of graph theory. Work in the area of local tournaments originates with Bang-Jensen [1]. Further work includes Bang-Jensen et al. [3], Bang-Jensen, Hell, and Huang ([4], [16]), and Huang [17], with the introduction of local in- and out-tournament digraphs by Bang-Jensen et al. [5].

Biclique cover and partition numbers of bipartite graphs and digraphs, as well as the related matrix ranks of the corresponding adjacency matrices, have been popular research topics during the past twenty-five years. As the answer to the interesting question of what digraphs have adjacency matrices with equal semiring ranks remains elusive, many have partially answered the question by considering certain classes of digraphs. The following list represents only a portion of the research that has been generated by this interest. See Brualdi et al. [7], Barefoot et al. [6], deCaen [9], Doherty et al. [10], Gregory et al. [11], Hefner (Factor) et al. ([12], [13], [14],[15]), Lundgren and Siewert ([18], [19], [20]), Maybee and Pullman [21], Monson et al. [22], Orlin [23], and Shader [25].

We further this research by bringing together concepts from these areas, and beginning the exploration of matrix ranks of the adjacency matrices of local out-tournaments. This is done through the use of upset tournaments that serve as the building blocks of the local out-tournaments. In this paper, we are interested in isolating the digraph structures that have adjacency matrices with full real rank, which is equal to the boolean, nonnegative integer, and term ranks.

The structure of the local out-tournament is determined in the first part of this paper following the definitions and preliminary results. Additionally, upset tournaments are defined and then used as the strong components of local out-tournaments. The resulting adjacency matrices are examined to determine which of these digraphs have corresponding adjacency matrices, A, where $r(A) = r_B(A) = r_{Z^+}(A) = r_t(A) = n$. Similar results follow for the associated biclique cover and partition numbers of the out-tournaments. Finally, open questions are discussed.

2 Terminology and Preliminaries

Many notational conventions are adopted from Bang-Jensen and Gutin [2]. A *digraph* $D = (V, A)$, where $V(D)$ is the nonempty *vertex set* of D and $A(D)$ is the *arc set* of D. For any arc $(u, v) \in A(D)$, we say that u *dominates* (or *beats*) v, and write $u \rightarrow v$. The *outset* of a vertex v, $O^+(v)$, is the set of all vertices that v dominates, and $|O^+(v)| = d^+(v)$. Similarly, the *inset* of a vertex v, $O^-(v)$, is the set of all vertices that dominate v, and $|O^-(v)| = d^-(v)$. In this paper, all digraphs are considered to be loopless. If we condense D by replacing each strong component with a vertex, the *strong component digraph*, $SC(D)$, is obtained. A digraph D is *connected* if its underlying graph is connected.

A *tournament* is a digraph where each pair of vertices defines exactly one arc. A *local out-tournament* (respectively, *local in-tournament*) is a digraph where the outset of every vertex is a tournament (respectively, the inset of every vertex). For ease in notation, these digraphs will often be referred to as *out-tournaments* and *in-tournaments*. A *local tournament* is a digraph where both the inset and outset of every vertex is a tournament. Local tournaments are also referred to more generally as *locally semicomplete digraphs*. To use the language of the majority of the research done on biclique covers and partitions and the associated matrix ranks, the authors will use the more specialized terms of local, in- and out-tournaments. Related to the results on in- and out-tournaments is out-branching and in-branching. A subdigraph T of D is an *out-branching* if T is a spanning, oriented tree of D and T has only one vertex v of in-degree zero. An *in-branching* is defined analogously with only one vertex of outdegree zero.

The relationship of domination is an important one in defining the structure of the out-tournament. Therefore, it is necessary to use notation that models certain nuances in the domination relationships. Let D_1 and D_2 be vertex disjoint digraphs. The notation $D_1 \Longrightarrow D_2$ means that there is no arc from $V(D_2)$ to $V(D_1)$. If every vertex in D_1 dominates every vertex in D_2, then we use $D_1 \rightarrow D_2$. Since we will be using tournaments as strong components, it will be the case that if arcs go one direction

between the strong components, then there will not be any going in the other direction. Therefore, we need to use $D_1 \mapsto D_2$, which means that $V(D_1)$ dominates $V(D_2)$ and there is no arc from $V(D_2)$ to $V(D_1)$.

Specifically in this paper, we will be constructing out-tournaments using upset tournaments as strong components. An *upset tournament* is a tournament on $n \geq 3$ vertices with score-list $\{1, 1, 2, 3, \ldots, n-3, n-2, n-2\}$. The *score-list* of a tournament is the multiset of the outdegrees of its vertices.

The *adjacency matrix* of a digraph D on n vertices is the $n \times n$ matrix $A = [a_{ij}]$ where $a_{ij} = 1$ if (v_i, v_j) is an arc in D, and equals 0 otherwise. Ranks corresponding to the $\{0,1\}$-matrix are the real rank, $r(A)$, the boolean rank, $r_B(A)$, and the nonnegative integer rank, $r_{Z^+}(A)$. The *boolean rank* of an $m \times n$ $\{0,1\}$-matrix is the smallest k for which there exist an $m \times k$ $\{0,1\}$-matrix B and a $k \times n$ $\{0,1\}$-matrix C such that $A = BC$ when boolean arithmetic is used $(1 + 1 = 1)$. Similarly, the *nonnegative integer rank* is the smallest k for which there exist $m \times k$ and $k \times n$ matrices B and C respectively such that $A = BC$, where the entries of B and C are nonnegative integers. If A is a $\{0,1\}$-matrix, then both B and C are $\{0,1\}$-matrices. The relationship between the boolean and nonnegative integer ranks is $r_B(A) \leq r_{Z^+}(A)$ for any $\{0,1\}$-matrix A. Since real rank can be defined similarly to nonnegative integer rank, only over all the real numbers, we have $r(A) \leq r_{Z^+}(A)$. There is no standard relationship between $r(A)$ and $r_B(A)$. Finally, the *term rank* of a matrix, $r_t(A)$, is the smallest number of rows and columns containing all of the nonzero entries of A. When A is a $\{0,1\}-$matrix, $r_B(A) \leq r_{Z^+}(A) \leq r_t(A)$. The relationship between the real rank and nonnegative integer rank also gives us $r(A) \leq r_{Z^+}(A) \leq r_t(A)$.

The real, nonnegative integer, and term ranks for the matrices of $n-$*tournaments*, tournaments on n vertices, were bounded by deCaen [9].

Theorem 2.1 [9] *If A is an n-tournament matrix, then $r(A) \geq n-1$.*

Corollary 2.2 [9] *If A is an n-tournament matrix, then $(n-1) \leq r(A) \leq r_{Z^+}(A) \leq r_t(A) \leq n$.*

These results indicate that if any tournament has equal ranks, then the ranks must equal $n-1$ or n. In general, $r_B(A)$ is very difficult to obtain. Thus, when looking for matrices with equal ranks, knowing the bounds on the remaining three ranks forces the search for tournament matrices where $r_B(A) = n-1$ or $r_B(A) = n$. Additionally, we know that $r_B(A) \leq r_t(A)$, so the term rank serves as an upper bound on the boolean rank in general $\{0,1\}-$matrices.

In this paper, we use the fact that it is known which upset tournaments have $r_B(A) = r_{Z^+}(A)$, and use it to help us characterize a class of local out-tournaments where all ranks equal n.

Gregory et al. [11] linked the boolean and nonnegative integer ranks of $\{0,1\}$-matrices to biclique cover and partition numbers of bipartite graphs. The *biclique cover number* of a graph G, $bc(G)$, is the smallest number of complete bipartite subgraphs that cover the edges of G. The *biclique partition number* of a graph G, $bp(G)$, is defined similarly using a partitioning of the edges of G. By labeling the rows of the adjacency matrix of a digraph D with a set of numbers and the columns with a disjoint set of numbers, the adjacency matrix of D also represents the adjacency matrix of a bipartite graph B. Using this common matrix, the following result is obtained.

Lemma 2.3 [11] *If D is a digraph, then $r_B(A) = bc(D)$ and $r_{Z^+}(A) = bp(D)$.*

The bicliques of B correspond to directed bicliques of D. In this paper, we use this relationship to extend the results obtained for the matrix ranks to include the biclique cover and partition numbers of the out-tournaments.

3 Local Out-Tournaments and Upset Tournaments

3.1 Out-Tournaments

Before examining the $\{0,1\}$-matrix ranks of the local out-tournaments, it is important to understand the structure of the digraphs. It is this that will determine which out-tournaments have adjacency matrices with full and equal ranks.

Bang-Jensen [1] shows that local tournaments have a structure that resembles that of tournaments. If D is a local tournament, then every strong component is a tournament. In addition, if two strong components are adjacent in D, then one completely dominates the other. For an out-tournament, however, not all of this structure is necessary. Since only the outset of each vertex need be a tournament, the constraints on the structure of the inset are relaxed. Thus, every strong component of an out-tournament is not necessarily a tournament, and complete domination is not required.

In-tournament digraphs were examined in depth by Bang-Jensen et al. [5], and much of the underlying structure identified. The following lemma and theorem are results for in-tournaments that are of specific interest in this paper in defining the structure of the out-tournaments. The corollaries following each result are the out-tournament equivalent, and come from the out-tournament being the converse of the in-tournament.

Lemma 3.1 [5] *Every connected in-tournament has an out-branching.*

Corollary 3.2 *Every connected out-tournament has an in-branching.*

Theorem 3.3 [5] *Let D be an in-tournament.*

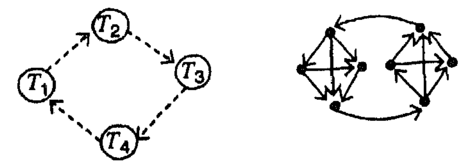
(a) Let A and B be distinct strong components of D. If a vertex $a \in A$ dominates some vertex in B, then $a \mapsto B$. Furthermore, $A \cap O^-(b)$ induces a tournament for each $b \in B$.

(b) If D is connected, then $SC(D)$ has an out-branching. Furthermore, if R is the root and A is any other component, there is a path from R to A containing all the components that can reach A.

Corollary 3.4 *Let D be an out-tournament.*

(a) Let A and B be distinct strong components of D. If a vertex $b \in B$ is dominated by some vertex in A, then $A \mapsto b$. Furthermore, $B \cap O^+(a)$ induces a tournament for each $a \in A$.

(b) If D is connected, then $SC(D)$ has an in-branching. Furthermore, if S is the vertex with out-degree of zero and A is any other component, there is a path from A to S containing all components that can reach S.



(a) (b)

Figure 1: (a) shows an out-tournament where tournaments form one strong component. (b) shows an out-tournament composed of two transitive tournaments.

In general, when constructing an out-tournament, it is not true that each strong component is a tournament. Figure 1(a) illustrates the possibility that an out-tournament might have a strong component that is comprised of separate tournaments. Figure 1(b) shows two transitive tournaments that form a strong component where the resulting digraph is an out-tournament. The strong component digraph for each of the out-tournaments in Figure 1 condenses down to one vertex. Thus, we can have an out-tournament where tournaments are some or all of the strong components, or out tournaments where none of the components are tournaments. Since the structure of out-tournaments has not been completely characterized, but has been described for the strong component structure, we focus on out-tournaments whose strong components are all tournaments.

3.2 Upset Tournaments

When implementing a structure where tournaments are the strong components, it is helpful for our purpose to use tournaments for which information exists as to the boolean, nonnegative integer, and term ranks of the tournament matrices. For this paper, we restrict our exploration to out-tournaments whose strong components are upset tournaments. To this end, we first describe the standard form that is used to represent the upset matrices, then verify that they are, indeed, strong tournaments.

Figure 2 shows an upset tournament in standard form by representing its upset path. The arcs (v_1, v_2) and (v_{n-1}, v_n) are in every upset path. All other arcs are directed in the opposite direction. The arc (v_i, v_j) can only be on the upset path when $i < j$. Vertices are presented in the order $v_1, v_2, \dots, v_n$.

As stated by Poet and Shader [24], every upset tournament is isomorphic to exactly

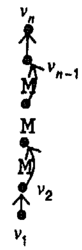


Figure 2: Upset tournament in standard form – all other arcs are directed down.

one upset tournament in standard form. Additionally, this results in having a unique path from v_1 to v_n.

Lemma 3.5 [24] *Let T be an upset tournament in standard form. Then T has a unique path from vertex v_1 to vertex v_n, and this path consists of the upset arcs of T.*

A result of Lemma 3.5 is that we know that an upset tournament is strongly connected.

Proposition 3.6 *If T is an upset tournament, then T is strongly connected.*

Proof. Let T be an upset tournament in standard form. By Lemma 3.5, there is a unique path from v_1 to v_n. Vertex v_n dominates all vertices except vertex v_{n-1}, and reaches v_{n-1} using arc (v_n, v_1) and the upset path. If v is a vertex on the upset path other than v_n, then v reaches v_{n-1} and v_n. It reaches all other vertices through v_n. If v is a vertex that is not on the upset path, then v dominates v_1, and reaches all vertices on the upset path through v_1. It then reaches all vertices not on the upset path from v_n. Thus, T is strongly connected. □

Because the upset tournaments are strong, they can be used as the strong components of a local out-tournament D. Corollary 3.4 guides the placement of arcs between upset tournaments T_i and T_j. Additionally, the structure of $SC(D)$ is acyclic, but the underlying graph is not necessarily a tree. The second part of the following lemma addresses the structure when two upset tournaments are dominated by a third upset tournament.

Lemma 3.7 *Let D be a local out-tournament with strong components T_i, T_j, and T_k where $v_i \in V(T_i)$, $v_j \in V(T_j)$, and $v_k \in V(T_k)$.*

(a) If T_i and T_j are upset tournaments where (v_i, v_j) is an arc in D, then $T_i \longmapsto v_j$.

(b) If T_i, T_j, and T_k are upset tournaments where $T_i \Longrightarrow T_j$ and $T_i \Longrightarrow T_k$, then $T_j \Longrightarrow T_k$ or $T_k \Longrightarrow T_j$.

Proof. Since upset tournaments are strong, part (a) follows directly from part (a) of Corollary 3.4. For (b), we can use part (b) from Corollary 3.4, but will prove it from the definition to support the further understanding of the tournament structure. Given that upset tournaments T_i, T_j and T_k are strong, if $T_i \Longrightarrow T_j$ and $T_i \Longrightarrow T_k$, then there exists $v_j \in V(T_j)$ and $v_k \in V(T_k)$ such that $T_i \longmapsto v_j$ and $T_i \longmapsto v_k$. By definition of an out-tournament, both v_j and v_k are in a tournament, so they must be adjacent. Thus, $v_j \rightarrow v_k$ or $v_k \rightarrow v_j$. From part (a), we extend this to $T_j \longmapsto v_k$ or $T_k \longmapsto v_j$. So, $T_j \Longrightarrow T_k$ or $T_k \Longrightarrow T_j$. □

4 Matrices and Matrix Ranks

Now that the structure of the out-tournaments with upset tournament strong components has been described, we direct our attention to finding which of these digraphs have adjacency matrices with $r(A) = r_t(A) = r_{Z^+}(A) = r_B(A) = n$. To do so, we use results on the matrix ranks of upset tournament matrices.

First, consider the basic structure of the adjacency matrix A of out-tournament D with strong components T_i. Let A_i be the adjacency matrices of the upset tournaments T_i. Thus $SC(D)$ has vertices T_i. We will carefully order the vertices of $SC(D)$ based upon the following proposition.

Proposition 4.1 [2] *Every acyclic digraph has an acyclic ordering of its vertices.*

Since $SC(D)$ is only guaranteed an in-branching, there may be more than one vertex in $SC(D)$ with indegree of zero. Thus, we cannot state that there is a path including every vertex. However, Proposition 4.1 states that there is an acyclic ordering of the T_i. We will assume this ordering of the T_i. This gives the adjacency matrix structure for $SC(D)$ shown in Figure 3. The ordering places each component along the diagonal.

$$\begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & \text{O} & \\ & \text{0's} & & T_k \end{bmatrix}$$

Figure 3: General adjacency matrix structure of SC(D), where D is a local tournament.

Keeping the same acyclic labeling of the T_i above, we obtain the adjacency matrix structure for the adjacency matrix of D shown in Figure 4. The two structures are the same only because the T_i are strong components in D. Note that the upper triangle regions of both matrices are not labeled with values. That is because these values can vary, while the values shown are set.

Consider how the structure of D dictates the placement of 1's in the upper triangular region of A. According to part (a) of Lemma 3.7, if (v_i, v_j) is an arc in D, then all of

$$\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & 0 & \\ 0's & & & A_k \end{bmatrix}$$

Figure 4: Adjacency matrix structure of an out-tournament where A_i are the adjacency matrices of the strong components.

the vertices in T_i dominate v_j. This translates into a column of 1's to the right of A_i and above A_j. There is a 1 in every row of A_i in the column corresponding to vertex v_j.

In the matrix, it may become necessary to discuss particular rows, columns, and regions. To help in the identification process, the following notation will be used. Let n_i be the number of vertices in T_i. So, $\sum_{i=1}^{k} n_i = n$. Further, let the vertices of T_i be labeled $v_{i1}, v_{i2}, ..., v_{in_i}$. If this is extended to the labeling of columns and rows in A, then vertex v_{jm} would be represented by column and row $n_1 + ... + n_{j-1} + m$.

To further identify the structure of these matrices, consider part (b) of Lemma 3.7 in conjunction with the acyclic labeling that has been adopted. With the acyclic labeling, if T_i and T_j are adjacent, then $T_i \Longrightarrow T_j$ if and only if $i < j$. Additionally, if $T_i \Longrightarrow T_j$ and $T_i \Longrightarrow T_k$, we will have $T_j \Longrightarrow T_k$ whenever $j < k$. Since the digraphs are isomorphic within labeling, we will assume the alpha ordering of $i < j < k$ for these indices. In A, if there is a submatrix of 1's in the rows of A_i that includes some columns of A_j and A_k, then there will be a submatrix of 1's in the rows of A_j in the same columns of A_k.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Figure 5: Adjacency matrix of an out-tournament with three upset tournament components, each on three vertices.

For an example, consider the out-tournament D consisting of upset tournament components $T_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$, $T_2 = \{(v_4, v_5), (v_5, v_6), (v_6, v_4)\}$ and $T_3 = \{(v_7, v_8), (v_8, v_9), (v_9, v_7)\}$. If $T_1 \longmapsto \{v_4, v_5\}$ and $T_1 \longmapsto \{v_8, v_9\}$,

then $T_2 \longmapsto \{v_8, v_9\}$. All of T_2 must dominate v_8 and v_9 in order to satisfy Lemma 3.7. The vertices in T_2 could also dominate more than $\{v_8, v_9\}$. It is the minimum set that must be dominated. The adjacency matrix $A(D)$ is shown in Figure 5.

In an upset tournament, every vertex has an outdegree greater than zero. So, every row in the adjacency matrix of an upset tournament contains a 1. Visually, a directed biclique of a digraph is a submatrix which forms a block of 1's in the digraph's adjacency matrix. In a biclique partition, these submatrices must be disjoint. In a biclique cover, they may overlap. Given the structure of the adjacency matrices here, every biclique in A_i can be expanded to cover any 1's to the right of A_i in a biclique cover. This relationship is important in determining what upset tournaments can be used as strong components in out-tournaments where $bc(D) = bp(D) = n$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$B_1 = \{v_1, v_4, v_5, v_6\} \to \{v_2\}$
$B_2 = \{v_2, v_5, v_6\} \to \{v_3\}$
$B_3 = \{v_3, v_6\} \to \{v_1, v_4\}$
$B_4 = \{v_4\} \to \{v_1, v_2, v_5\}$
$B_5 = \{v_5\} \to \{v_1, v_2, v_3, v_6\}$

Figure 6: Adjacency matrix of an upset tournament on 6 vertices, and a minimum biclique cover.

Consider the matrix in Figure 6 representing an upset tournament with vertices $v_1, ..., v_6$. A minimum biclique cover is given, where B_i are the bicliques. Each of the B_i can be expanded to cover any 1's in a column to the right or the left of this submatrix. Figure 7 shows the same bicliques expanded to cover 1's representing the vertices in the original matrix dominating vertices v_8 and v_9.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$B_1' = \{v_1, v_4, v_5, v_6\} \to \{v_2, v_8, v_9\}$
$B_2' = \{v_2, v_5, v_6\} \to \{v_3, v_8, v_9\}$
$B_3' = \{v_3, v_6\} \to \{v_1, v_4, v_8, v_9\}$
$B_4' = \{v_4\} \to \{v_1, v_2, v_5, v_8, v_9\}$
$B_5' = \{v_5\} \to \{v_1, v_2, v_3, v_6, v_8, v_9\}$

Figure 7: Submatrix where all of the vertices of the upset tournament dominate vertices v_8 and v_9. The expanded biclique cover is given.

Lemma 4.2 *Let D be an out-tournament with k upset tournament strong compo-*

nents, T_i. Then $bc(D) \leq \sum_{i=1}^{k} bc(T_i)$.

Proof. Let $B_i = X_i \to Y_i$ be any maximal biclique in a minimum biclique covering of T_i. Suppose that T_i is not the terminal vertex in $SC(D)$. Then there exist arcs from T_i to at least one other tournament component T_j. Let $Z_j \subseteq V(T_j)$ be the set of vertices dominated by T_i, and $B_i' = X_i \to (Y_i \cup Z_j)$. The collection of all B_i in the biclique covering cover all arcs in T_i by definition, so the collection of all B_i' also cover those arcs. Every vertex in T_i dominates Z_j. Since T_i is strong, each vertex has outdegree greater than zero, and so must be contained in some X_i of B_i. Thus, every arc from X_i to Z_j is in B_i', so every arc from T_i to T_j is covered. Taking every B_i' for every T_i in D, we obtain a cover for D using only the number of bicliques used to cover each of the individual upset tournaments. Therefore, $bc(D) \leq \sum_{i=1}^{k} bc(T_i)$. □

Corollary 4.3 *Let A be the adjacency matrix of an out-tournament with k upset tournament strong components, where A_i is the adjacency matrix of strong component T_i. Then $r_B(A) \leq \sum_{i=1}^{k} r_B(A_i)$.*

Thus to find the matrices with full and equal ranks, the submatrices, A_i, must have $r_B(A_i) = n_i$. So we look for upset tournaments where $bc(T_i) = n_i$. Since $bc(T_i) \leq bp(T_i)$, the upset tournaments must have $bc(T_i) = bp(T_i) = n_i$.

Theorem 4.4 [18] *Let T be an upset tournament in standard form on $n \geq 6$ vertices. Then $bc(T) = n$ if and only if the upset path does not contain any arcs of the form (v_i, v_{i+1}) for $3 \leq i \leq n-3$.*

When we have $n_i \geq 6$, Theorem 4.4 gives us the structure that must be used for the upset tournament strong components of the out-tournament. What about upset tournaments on 3, 4 or 5 vertices? To answer this question, we use results from Gregory, et al. [11]. A set S of independent 1's of a $\{0,1\}$-matrix is said to be *isolated* if no two 1's are in a 2×2 submatrix of 1's.

Lemma 4.5 [11] *If the adjacency matrix A of a digraph D has an isolated set of r 1's, then $r_B(A) = bc(D) \geq r$.*

We use this result in the proof of the following lemma where we establish the boolean and nonnegative integer ranks of upset tournaments on 3, 4 or 5 vertices.

Lemma 4.6 *If T is an upset tournament on $n = 3, 4,$ or 5 vertices with adjacency matrix A, then $bc(T) = bp(T) = n$, and $r_B(A) = r_{Z^+}(A) = n$.*

Proof. When $n = 3$, there is exactly one upset tournament on n vertices in standard form, and it has upset path (v_1, v_2), (v_2, v_3). Entries a_{12}, a_{23} and a_{31} of the adjacency matrix are isolated 1's. When $n = 4$, there is exactly one upset tournament on n vertices in standard form, and it has upset path (v_1, v_2), (v_2, v_3) and (v_3, v_4). Entries a_{12}, a_{23}, a_{34} and a_{41} of the adjacency matrix are isolated 1's. When $n = 5$, there are two distinct upset tournaments on n vertices in standard form. One has upset path (v_1, v_2), (v_2, v_4) and (v_4, v_5), and isolated 1's a_{12}, a_{24}, a_{31}, a_{45}, and a_{53}. The other has upset path (v_1, v_2), (v_2, v_3), (v_3, v_4) and (v_4, v_5), and isolated 1's a_{12}, a_{23}, a_{34}, a_{45}, and a_{51}. By Lemma 4.5, all of the above upset tournaments have $r_B(A) = bc(T) \geq n$. Since $r_{Z^+}(A) = bp(T) \geq r_B(A) = bc(T)$, we have $bc(T) = bp(T) = n$, and $r_B(A) = r_{Z^+}(A) = n$. □

Next, the real rank must be considered in the final characterization of the out-tournaments. The following theorem relates real rank to nonnegative integer rank in upset tournaments.

Theorem 4.7 [25] *Let A be an adjacency matrix corresponding to an upset tournament. Then $r(A) = r_{Z^+}(A)$.*

This translates to $r(A_i) = r_{Z^+}(A_i)$ for the upset tournament strong components. Since it is possible for the real rank to be less than both the boolean and nonnegative integer ranks in general, it remains to show that $r(A) = n$ in the matrices we have discussed where $r_B(A) = r_{Z^+}(A) = n$. That will be done in the proof of the following theorem, which characterizes the out-tournaments with upset tournament strong components with full and equal ranks.

Theorem 4.8 *Let D be an out-tournament with k upset tournament strong components, T_i, and adjacency matrix A. For each T_i, either T_i is on 3, 4 or 5 vertices or it does not contain any arcs of the form (v_j, v_{j+1}) for $3 \leq j \leq n_i - 3$ for $n_i \geq 6$ if and only if $r_B(A) = r_{Z^+}(A) = r_t(A) = r(A) = n$.*

Proof. $(\Rightarrow)$ If $n = 3, 4$ or 5, the calculated real rank of the adjacency matrices for any upset tournament in standard form is 3, 4 and 5 respectively. This combined with Lemma 4.6 gives us $r_B(A) = r_{Z^+}(A) = r(A) = n$. If $n \geq 6$ and there are no arcs of the form (v_j, v_{j+1}) for $3 \leq j \leq n_i - 3$, $1 \leq i \leq k$, we know from Theorem 4.4 that $r_B(A_i) = r_{Z^+}(A_i) = n_i$. Also, from Corollary 4.3, $r_B(A) \leq \sum_{i=1}^{k} r_B(A_i) = n$. To show that $r_B(A) = n$, we will show that $bc(D) = n$. Consider minimum biclique covers of T_i and T_j. Because $V(T_i) \cap V(T_j) = \emptyset$, no fewer bicliques can be used to cover $A(T_i)$ and $A(T_j)$ if arcs are created from T_i to T_j. So, $bc(D) \geq \sum_{i=1}^{k} bc(T_i)$. In the proof of Lemma 4.2, we know that $bc(D) \leq \sum_{i=1}^{k} bc(T_i)$. Therefore, $bc(D) = \sum_{i=1}^{k} bc(T_i) = n$, so $r_B(A) = n$. Since $r_B(A) \leq r_{Z^+}(A)$, we have $r_{Z^+}(A) = n$. Finally, we examine the real rank of A. The rows of each A_i are linearly independent, so there is no linear combination of these rows that equals the zero vector. For $r(A)$ to be less than n, there must be a linear combination of the rows of A that equal the zero vector. The rows of A_1 cannot be used, as there is no linear combination of these rows that will give the first n_1 entries of the zero vector, and all entries below A_1 in A are 0's. Thus, we can only use rows below A_1. For similar reasons, we cannot use the rows of A_2. Following this reasoning inductively, we find that there is no linear combination of the rows of A that equal the zero vector, and so $r(A) = n$. In all cases, since $r(A) \leq r_t(A)$, we have $r_t(A) = n$.

($\Leftarrow$) $r_B(A) = n$ implies that $\sum_{i=1}^{k} r_B(A_i) = n$. So, each A_i must have full boolean rank, $r_B(A_i) = n_i$. Since $r_B(A) \le r_{Z^+}(A)$, $r_{Z^+}(A_i) = n_i$ for each $i = 1, ..., k$. Thus, $r_B(A_i) = r_{Z^+}(A_i) = n_i$ for $1 \le i \le k$. This only occurs when T_i is on 3, 4, or 5 vertices, or by Theorem 4.4 when $n \ge 6$ and there are no arcs of the form (v_j, v_{j+1}) for $3 \le j \le n_i - 3$. $\square$

Although this paper concentrates on local out-tournaments, the same results hold for local in-tournaments.

5 Miles to Go

A characteristic of upset tournaments that makes them interesting is that it is also known when $r_B(A) = r_{Z^+}(A) = n - 1$. If we use these upset tournaments as strong components in an out-tournament D, the singular matrices A_i make for a variety of rank values.

$$\begin{bmatrix}
0&1&0&0&0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0\\
1&0&0&1&0&0&1&0&0&0&0&0\\
1&1&0&0&1&0&1&0&0&0&0&0\\
1&1&1&0&0&1&1&0&0&0&0&0\\
1&1&1&1&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&0&1&0&0\\
0&0&0&0&0&0&1&1&0&0&1&0\\
0&0&0&0&0&0&1&1&1&0&0&1\\
0&0&0&0&0&0&1&1&1&1&0&0
\end{bmatrix}$$

Figure 8: Adjacency matrix of an out-tournament where T_i are upset tournaments on 6 vertices, and $r_B(A) \neq r_{Z^+}(A)$.

To illustrate this, consider the matrix in Figure 8. The upset tournament components T_1 and T_2 have six vertices each with upset paths isomorphic to (v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5). Each has $bc(T_i) = bp(T_i) = n - 1$. All vertices of T_1 dominate the first labeled vertex of T_2 to form a local out-tournament. As shown in Section 4, the bicliques of T_1 can be expanded to cover the arcs from T_1 to T_2. So, $r_B(A) = (n_1 - 1) + (n_2 - 1) = 10$. However, the partitions cannot be expanded in this way. Nor can the partitions in T_2 be expanded upward to cover the 1's. While this in itself is not enough to show that $r_{Z^+} > 10$, it must be at least 11 since $r(A) = 11$.

What if, instead of dominating the first labeled vertex of T_2, the vertices of T_1 dominate the second? Figure 9 shows this slightly different adjacency matrix. Here, we have the same boolean rank as in Figure 8, but a biclique in the partition cover of T_2 can be extended to cover the column of 1's above it. Therefore, $r_B(A) = r_{Z^+}(A) = 10$. As a bonus, $r(A) = 10$ as well. This shows that for a local out-

$$\begin{bmatrix}
0&1&0&0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&1&0&0&0&0\\
1&0&0&1&0&0&0&1&0&0&0&0\\
1&1&0&0&1&0&0&1&0&0&0&0\\
1&1&1&0&0&1&0&1&0&0&0&0\\
1&1&1&1&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&0&1&0&0\\
0&0&0&0&0&0&1&1&0&0&1&0\\
0&0&0&0&0&0&1&1&1&0&0&1\\
0&0&0&0&0&0&1&1&1&1&0&0
\end{bmatrix}$$

Figure 9: Adjacency matrix of an out-tournament where T_i are upset tournaments on 6 vertices, and $r_B(A) = r_{Z^+}(A)$.

tournament, $r(A) < n - 1$ is possible, unlike the case for tournaments.

So the question now becomes, how can local out-tournaments with upset tournament strong components be constructed where $r_B(A) = r_{Z^+}(A) < n$ and $r_B(A) = r_{Z^+}(A) = r_t(A) = r(A) < n$? Additionally, what local out-tournaments have adjacency matrices with equality for some subsets of these ranks, and what are the subsets?

Naturally, the ranks of the adjacency matrices of local tournaments and local out-tournaments with a variety of strong tournaments as components can be explored. Hopefully, a characterization as to the local, local out- and local in-tournaments whose adjacency matrices have equal $\{0, 1\}$ $-$matrix ranks can be obtained. This paper provides the first inroad to that characterization. It has also been an opportunity to bring together two different areas of research within graph theory.

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