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Digraphs with isomorphic underlying and domination graphs: Pairs of paths

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Abstract

A domination graph of a digraph D, $dom(D)$, is created using the vertex set of D and edge $uv \in E(dom(D))$ whenever $(u, z) \in A(D)$ or $(v, z) \in A(D)$ for any other vertex $z \in V(D)$. Here, we consider directed graphs whose underlying graphs are isomorphic to their domination graphs. Specifically, digraphs are completely characterized where $UG^c(D)$ is the union of two disjoint paths.

1 Introduction

Domination graphs were first introduced by Merz, Lundgren, Reid and Fisher [10] to describe the structure of the domination graphs and competition graphs of tournaments. Since that time, further refinements have been made in the work on tournaments, including that done by Cho, Doherty, Kim and Lundgren ([1], [2]) and Merz et al. ([6], [7], [8], [9], [10], [11]). However, the characterization of the structure of the domination graph of arbitrary digraphs has remained elusive. The authors have added to the knowledge in this area by characterizing digraphs D where the underlying graph of D is equal to its domination graph [3], and have characterized some digraphs where the graphs are isomorphic ([4], [5]). We add to that body of knowledge in this paper by characterizing digraphs whose underlying and domination graphs are isomorphic, $UG(D) \cong dom(D)$, and $UG^c(D)$ is the graph of two disjoint paths.

Let D be a directed graph, or *digraph*, with nonempty vertex set $V(D)$ and arc set $A(D)$. If $(u, v) \in A(D)$, then u is said to *dominate* v. Further, u is the *origin* of the arc (u, v), and v is the *terminating vertex*. When for every other vertex z in $V(D)$, either (u, z) or (v, z) is an arc in D, then u and v form a

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dominating pair. The *domination graph of* D, *dom* (D), is an undirected graph with the vertex set $V(D)$, where there is an edge between every dominating pair. A digraph D is considered a *biorientation* of a graph G if for every edge $uv \in E(G)$, either (u,v) or (v,u) or both are arcs in D, and D contains no other arcs. The *underlying graph of* D, $UG(D)$, is the graph for which D is a biorientation. If for edge uv in G, only one of edges (u,v) or (v,u) is in D, then the arc is called an *orientation* of edge uv. When all edges of G are bidirected edges in D, then D is a *complete biorientation* of G, also known as a *symmetric digraph.* Although bidirected edges are allowed in D, there are no directed loops.

When the underlying graph of D is isomorphic to the domination graph of D, it is its nature to have many edges. Thus, in most cases, it is easier to obtain results regarding $UG(D)$ and $dom(D)$ by observing patterns in $UG^c(D)$ and $dom^c(D)$, which are sparse graphs. To relate the results obtained from the complements to $UG(D)$ and $dom(D)$, we use the concepts of the union and the join of graphs and digraphs. The *union* of two graphs or digraphs is the graph or digraph formed by the union of their vertices as well as their sets of edges or arcs. The *join* of two graphs G and H, $G+H$, is the graph that consists of $G \cup H$ and all edges joining the vertices in G with the vertices in H. We extend this definition to directed graphs as follows. The *join* of D_1 and D_2 consists of $D_1 \cup D_2$ together with all bidirectional edges between every vertex of D_1 and every vertex of D_2.

We know that the structure of $UG(D)$ is limited to a small number of constructs. It can be summed up by the following three results.

Theorem 1.1 *[4] If $D_1,\ldots,D_k$ are directed graphs such that $UG(D_i) \cong dom(D_i)$ for $i=1,\ldots,k$ and $D = D_1 + D_2 + \ldots + D_k$, then $UG(D) \cong dom(D)$. Also*

1. $UG(D) = \sum_{i=1}^{k} UG(D_i)$
2. $dom(D) = \sum_{i=1}^{k} dom(D_i)$
3. $UG^c(D) = \bigcup_{i=1}^{k} UG^c(D_i)$
4. $dom^c(D) = \bigcup_{i=1}^{k} dom^c(D_i)$

Theorem 1.2 *[4] If $UG(D)$ is isomorphic to $dom(D)$, then $UG^c(D)$ is comprised of one or more connected components, each either a complete graph, a path, or a cycle.*

Corollary 1.3 *[4] If $UG(D)$ is isomorphic to $dom(D)$, then D is the join of $D_1, D_2, \ldots, D_k$, where $UG(D_i)$ is isomorphic to an independent set, the complement of a path, or the complement of a cycle.*

Theorem 1.2 gives three basic components that comprise the complement of the underlying graph in which we are interested. The structure of D and $UG(D)$ where $UG^c(D)$ is one component has been completely characterized [4], as has the case where P_1, P_2 and C_4 are the components [5]. In this paper, we further the research by characterizing the underlying graphs and the directed graphs where $UG(D) \cong dom(D)$, and $UG^c(D) = P_i \cup P_j$. In the next section, we determine for what values of j $UG^c(D) = P_i \cup P_j$ exists for the special cases of $i = 1, 2$. We then use the information from Section 2 to formulate the characterizations of the digraphs D that can be formed for the associated underlying graphs. Finally, we conclude the characterizations of $UG^c(D)$ and D where $i, j \geq 3$ in Sections 4 and 5. Some of the proofs required are quite long and interrupt the flow of information, so have been placed in their own section at the end of the paper.

2 Structure of $UG^c(D) = P_i \cup P_j$, $i = 1, 2$

Of immediate consequence when determining the structure of $UG^c(D)$ for any i and j is the edges that are formed in $dom^c(D)$ regardless of the structure of D. The following lemma lists the paths that are always part of $dom^c(D)$ when P_n for $n \geq 3$ is a component of $UG^c(D)$. These paths are used extensively in this paper, and will be referred to as the *generated subpaths in* $dom^c(D)$.

Lemma 2.1 *[4] If $UG^c(D) = P_n = x_1, x_2, \ldots, x_n$ for $n \geq 3$, then*

1. *if n is odd, $x_1, x_3, \ldots, x_n$ and $x_2, x_4, \ldots, x_{n-1}$ are paths in $dom^c(D)$, and*
2. *if n is even, $x_1, x_3, \ldots, x_{n-1}$ and $x_2, x_4, \ldots, x_n$ are paths in $dom^c(D)$.*

Further, we know that a biorientation of $UG(D)$ exists for each of the P_n, $n \geq 3$, where $UG(D) \cong dom(D)$. This is stated in the following lemma.

Lemma 2.2 *[4] Let D be a directed graph on $n \geq 3$ vertices and $UG^c(D) = P_n = x_1, \ldots, x_n$. Then $dom^c(D) \cong P_n$ if and only if for every edge $uv \in E(UG(D))$, (u,v) and (v,u) are arcs in D except for the following:*

1. *if n is odd, exactly one of the following is an orientation of the associated edge(s) in $UG(D)$:*
   (a) (x_1, x_n),
   (b) (x_n, x_1),
   (c) (x_1, x_n) and (x_n, x_{n-3}), or
   (d) (x_n, x_1) and (x_1, x_4), and
2. *if n is even, exactly one of the following is an orientation of the associated edge(s) in $UG(D)$:*
   (a) (x_1, x_{n-1}),
   (b) (x_n, x_2),
   (c) (x_1, x_{n-1}) and (x_n, x_2),

(d) (x_n, x_2) and (x_1, x_4), or
(e) (x_1, x_{n-1}) and (x_n, x_{n-3}).

Of particular note, the oriented edges (x_n, x_{n-3}) and (x_1, x_4) of the preceding lemma form edges in $dom^c(D)$ that are in the generated subpaths of Lemma 2.1. These serve a special purpose when we characterize D, so the following two corollaries are listed here for later use. The first follows from construction of $dom^c(D)$.

Corollary 2.3 *If $UG^c(D) = P_n$, $n \geq 4$, oriented edges (x_1, x_4) and (x_n, x_{n-3}) produce edges x_2x_4 and $x_{n-1}x_{n-3}$ in $dom^c(D)$.*

Further, this guarantees that when we use oriented edges (x_n, x_{n-3}) and (x_1, x_4), there will be no new edges appearing in $dom^c(D)$.

Corollary 2.4 *If $UG^c(D) = P_n$, $n \geq 4$, oriented edges (x_1, x_4) and (x_n, x_{n-3}) create no additional edges in $dom^c(D)$.*

To show that $UG^c(D) = P_1 \cup P_j$ can exist for all $j \geq 1$ such that $UG(D) \cong dom(D)$, we need to show that if the component P_1 is added to the graph $UG^c(D) = P_j$, an underlying graph will be created where it is possible to still create a digraph D preserving isomorphism. We can do that using the orientations given in Lemma 2.2.

Theorem 2.5 *Let $UG^c(D) = P_1 \cup P_j$. For all $j \geq 1$, there exists a biorientation of the edges of $UG(D)$ such that $UG(D) \cong dom(D)$.*

Proof. Let $j = 1$. Then $UG(D)$ equals the edge uv. The vertices u and v dominate for either orientation of the edge uv or the bidirection of the edge. Thus, $UG(D) \cong dom(D)$.

Let $j = 2$. Then $UG(D) = uv_1 \cup uv_2$. The orientation $(u, v_1) \cup (u, v_2)$ produces edges uv_1 and uv_2 in $dom(D)$. Thus, $UG(D) \cong dom(D)$.

Let $j \geq 3$. If $D_1 = P_1$ and $D_2 = P_j$, where $UG^c(D) = D_1 \cup D_2$, then by Theorem 1.1, $UG(D) \cong dom(D)$ where $D = D_1 + D_2$. ■

Now we turn our attention to characterizing the j for which $UG(D) \cong dom(D)$ and $UG^c(D) = P_2 \cup P_j$. To do so, there must be more understanding of the orientation of edges to form D and the affect this has on $dom^c(D)$. We work with $dom^c(D)$ because when $UG(D) \cong dom(D)$, $UG^c(D) \cong dom^c(D)$, and it is easier to work with the fewer edges in the complements.

Given any edge uv in $dom^c(D)$, we know that vertices u and v cannot form a dominating pair in D. Therefore, there must be at least one vertex z in D such that neither (u, z) nor (v, z) is an arc. We will call z a *source* of edge uv in $dom^c(D)$. Note that an edge in $dom^c(D)$ may have multiple sources.

The next few results eliminate certain vertices as candidates for sources, and restrict the number of edges for which a vertex may be a source. In our construction of a digraph where $UG^c(D)$ is the union of two paths, it is natural to ask whether it is possible for a vertex to be the source of more than one edge in $dom^c(D)$.

Lemma 2.6 *[5] If $UG(D) \cong dom(D)$ and y is the source of two or more edges in $dom^c(D)$, then the set of vertices which do not dominate y is contained in a component isomorphic to K_r, $r \geq 3$ in $UG^c(D)$.*

Since we have no components isomorphic to K_r, $r \geq 3$ in $UG^c(D)$, we have no vertices that are the source of more than one edge in $dom^c(D)$. There are two ways in which a vertex z may be a source of edge uv in $dom^c(D)$. The first is if it is not adjacent to vertices u and v in $UG(D)$. The second is if we create the source z by making it the origin of the oriented edge (z, u) if z is not adjacent to v, or (z, v) if z is not adjacent to u, or both if z is adjacent to both u and v. We can obtain the list of vertices that are candidates for becoming the origin of an oriented edge. The following lemma is used as the foundation for the choices.

Lemma 2.7 *[4] Let D be a digraph on n vertices, and (u, v) in D be the orientation of edge uv in $UG(D)$, where $\deg(u) = k$ in $UG(D)$. If $k < n - 2$, then K_3 is a subgraph of $dom^c(D)$.*

The preceding lemma thus leads to the following set of vertices that may serve as the origin for any oriented edge in D when $UG(D) \cong dom(D)$ and $UG^c(D)$ is comprised of disjoint paths.

Lemma 2.8 *Let D be any digraph such that $UG(D) \cong dom(D)$ and $UG^c(D)$ is comprised of components $\bigcup_{i=1}^{k} P_{n_i}$ where $P_{n_i} = x_{1i}, x_{2i}, ..., x_{n_i i}$ and $n_i \geq 1$ is the number of vertices for path P_{n_i}. If $UG(D) \cong dom(D)$ and (u, v) in D is an orientation of edge uv in $UG(D)$, then $u = x_{1j}$ or $u = x_{n_j j}$ for some j, $1 \leq j \leq k$.*

Proof. Consider $UG^c(D) = \bigcup_{i=1}^{k} P_{n_i}$ where $P_{n_i} = x_{1i}, x_{2i}, ..., x_{n_i i}$ and $n_i \geq 1$, and (u, v) is an orientation of edge uv. Let $n = \sum_{i=1}^{k} n_i$. According to Lemma 2.7, if $\deg(u) < n - 2$ in $UG(D)$, then K_3 is a subgraph of $dom^c(D)$. Thus, $\deg(u) \geq n - 2$. This indicates that in $UG^c(D)$, $\deg(u) \leq 1$. So, u must be K_1, or the end vertex of a path. Therefore, we obtain the list of vertices, which are the first and last vertices of each P_{n_i}. ■

Lemma 2.6 states that a vertex can be the source for at most one edge in $dom^c(D)$. Now we ask whether a vertex z may be the source of one edge uv if z is adjacent to both u and v in $UG(D)$. The answer is given in the next lemma where we find that if z is the origin of one oriented edge in D, it cannot be the origin of another oriented edge in D when the paths have at least three vertices. The results are generalized to k paths.

Lemma 2.9 *Let D be any digraph such that $UG(D) \cong dom(D)$ and $UG^c(D)$ is comprised of components $\bigcup_{i=1}^{k} P_{n_i}$ where $P_{n_i} = x_{1i}, x_{2i}, ..., x_{n_i i}$ and $n_i \geq 2$ is the number of vertices for path P_{n_i}. If (z, u) in D is an orientation of edge uz in $UG(D)$, then there is no vertex v such that (z, v) in D is an orientation of edge vz in $UG(D)$.*

Proof. Suppose that there are orientations (z, u) and (z, v) in D. By Lemma 2.8, z must be one of the end vertices of the path, and there exists a vertex z_k that is not adjacent to z in $UG(D)$. So, z_k, u and v do not dominate z and form K_3 in $dom^c(D)$, contradicting $UG(D) \cong dom(D)$. Thus, (z, u) or (z, v) may be an orientation of an edge in D, but not both. ■

Now we turn our attention to the structure of the paths themselves in $dom^c(D)$. The generated subpaths in $dom^c(D)$ are only in constructions for P_j when $j \geq 3$. However Lemma 2.1 does give an indication of the length of the paths formed automatically in $dom^c(D)$. As j becomes larger, the generated subpaths in $dom^c(D)$ on the vertices $v_1, ..., v_j$ become longer than P_2. Thus, it is necessary to know if $P_2 = u_1u_2$ in $UG^c(D)$ can also form $P_2 = u_1u_2$ in $dom^c(D)$. If not, there are only a few possible values for j so that $UG^c(D) = P_2 \cup P_j$ can yield an isomorphic $dom^c(D)$. The following lemma states that $P_2 = u_1u_2$ in $UG^c(D)$ is not possible.

Lemma 2.10 *Let $UG(D) \cong dom(D)$, and $UG^c(D) = P_2 \cup P_j$ for $j \geq 3$, where $P_2 = u_1u_2$ and $P_j = v_1, ..., v_j$. Then u_1u_2 is not an edge in $dom^c(D)$.*

Proof. Suppose that u_1u_2 is an edge in $dom^c(D)$. Then some vertex z must be a source of that edge. Vertices u_1 and u_2 cannot be the source of an edge with which they are incident. Thus, $z = v_k$ for some $k = 1, ..., j$. Since v_k is adjacent to both u_1 and u_2, the oriented edges (v_k, u_1) and (v_k, u_2) must both be in D. But Lemma 2.9 states that this cannot be. Therefore, u_1u_2 is not an edge in $dom^c(D)$. ■

Corollary 2.11 *Let $UG(D) \cong dom(D)$, and $UG^c(D) = P_2 \cup P_j$ for $j \geq 3$, where $P_2 = u_1u_2$ and $P_j = v_1, ..., v_j$. Then P_2 in $dom^c(D)$ is either equal to u_iv_k for some $i = 1, 2$ and some $k = 1, ..., j$ or v_iv_k for some $1 \leq i < k \leq j$.*

Now we can formulate the structure of $UG^c(D)$ given the preceding results. When it is shown that $UG^c(D) \cong dom^c(D)$, we generally skip directly to the consequence of $UG(D) \cong dom(D)$. Figure 1 shows the construction for $j = 4$ given in the proof for Theorem 2.12. Bidirectional edges are not shown. The dashed lines represent the edges in $UG^c(D)$, so are not bidirected edges in D. In the figure, $P_2 = v_2v_4$ in $dom^c(D)$.

Theorem 2.12 *Let $UG^c = P_2 \cup P_j$. There exists a biorientation D of the edges of $UG(D)$ such that $UG(D) \cong dom(D)$ if and only if $j = 1, 2, 3, 4, 5$.*

Proof. ($\Longrightarrow$) Theorem 2.5 shows the case where $j = 1$. For $j \geq 2$, according to Corollary 2.11, we must construct P_2 in $dom^c(D)$ using u_iv_k or v_kv_l. First consider $P_2 = u_iv_k$ in $dom^c(D)$. Here, v_k must be the generated subpath P_1 so that $u_iv_k = P_2$ in $dom^c(D)$. Therefore, $j \leq 3$. Next consider $P_2 = v_kv_l$ in $dom^c(D)$. The edge v_kv_l must be the generated subpath P_2, so Lemma 2.1 gives us $j \leq 5$.

($\Longleftarrow$) The case where $j = 1$ is shown in Theorem 2.5.

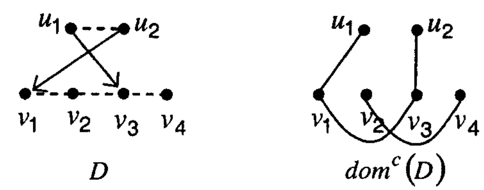

Figure 1: A digraph D where $UG^c(D) = P_2 \cup P_4$ and the associated graph $dom^c(D)$. Dashed lines represent $UG^c(D)$, and bioriented edges of D are omitted.

For $j = 2$, let (u_1, v_1) and (u_2, v_2) be oriented edges of $UG(D)$. Vertex u_1 is the source of edge u_2v_1 in $dom^c(D)$, and vertex u_2 is the source of edge u_1v_2. No other edges are formed, so $UG(D) \cong dom(D)$.

For $j = 3$, let (v_1, u_1) and (u_1, v_3) be oriented edges of $UG(D)$, and bidirect all other edges of $UG(D)$. Here, v_2 and v_1, v_3 are generated subpaths in $dom^c(D)$. Additionally, vertex v_1 is the source of edge u_1v_2 in $dom^c(D)$, and vertex u_1 is the source of edge u_2v_3. Thus, $dom^c(D) = u_1v_2 \cup u_2, v_3, v_1$, and $UG(D) \cong dom(D)$.

Using similar orientations for $j = 4$ and $j = 5$, (u_1, v_3) with (u_2, v_1) and (u_1, v_5) with (u_2, v_1) respectively, we find that these biorientations result in $UG(D) \cong dom(D)$. ■

3 Characterization of D where $UG^c(D) = P_i \cup P_j$, $i = 1, 2$

As might be expected, the characterization of all digraphs that can be formed using the underlying graphs specified in the previous section, is a somewhat tedious process. We will place the longer proofs into the final section of the paper so that the flow of the results are not interrupted by lengthy construction proofs.

To begin, we consider $i = 1$. The following lemma provides all of the additional support necessary before characterizing the digraphs D where $UG(D) \cong dom(D)$ and $UG^c(D) = P_1 \cup P_j$.

Lemma 3.1 *Let $UG^c(D) = P_1 \cup P_j$, where $P_1 = u$, $P_j = v_1, ..., v_j$ for $j \neq 2$, and $UG(D) \cong dom(D)$. Then $u = P_1$ in $dom^c(D)$.*

Proof. If $j = 1$, then the edge uv_1 in $UG(D)$ can be either of the two orientations or the biorientation in D. Thus, u and v_1 form a dominating pair, and are nonadjacent in $dom^c(D)$.

If $j = 3$, then v_1v_3 is a generated subpath in $dom^c(D)$. Thus, only u or v_2 can possibly equal P_1 in $dom^c(D)$. If v_2 is P_1, then either uv_1 or uv_3 is an edge in $dom^c(D)$. Say that uv_1 is an edge. Then there is a source vertex z in D such that neither u nor v_1 dominates z. By Lemma 2.9, z cannot be adjacent to both u and v_1. Vertex u is adjacent to all other vertices in $UG(D)$, so $z = v_2$. This implies that (v_2, u) must be an oriented edge so that v_1 and u do not dominate v_2. However, v_2 cannot be the origin of an oriented edge according to Lemma 2.8. Thus, uv_1 is not an edge in $dom^c(D)$. With a similar argument, we see that uv_3 is not an edge in $dom^c(D)$. Thus, $u = P_1$ in $dom^c(D)$ is the only possibility. It can be realized by applying the assignment of oriented edges associated with P_3 outlined in Lemma 2.2, and bidirecting the edges uv_i for $i = 1, 2, 3$.

If $j \geq 4$, then P_1 is not a generated subpath in $dom^c(D)$, so the only possibility is vertex u. The graph $dom^c(D)$ that is isomorphic to $UG^c(D)$ can be realized by applying the assignment of oriented edges associated with P_j outlined in Lemma 2.2, and biorienting the edges uv_i for $i = 1, ..., j$. ■

This leads to the following characterization of digraphs where $UG(D) \cong dom(D)$ and $UG^c(D) = P_1 \cup P_j$.

Theorem 3.2 *Let $UG^c(D) = P_1 \cup P_j$, where $P_1 = u$ and $P_j = v_1, ..., v_j$. $UG(D) \cong dom(D)$ if and only if D is of the form:*

1. *If $j = 1$, then D is an orientation of the edge uv_1 or the biorientation of that edge.*
2. *If $j = 2$, then (v_i, u) is an orientation for $i = 1$ or 2, and $(v_{i'}, u)$ is not an orientation for $i = 1$ or 2 and i' the remaining value, or (u, v_1) and (u, v_j) are orientations.*
3. *If $j \geq 3$ and v_pv_q is the edge in $dom^c(D)$ connecting the generated subpaths, then D is the digraph where all edges of $UG(D)$ are bidirected in D except for one of the following:*
   - *(a) the only oriented edges are as described in Lemma 2.2, or*
   - *(b) (u, v_p) and (u, v_q) are orientations of edges uv_p and uv_q respectively, or*
   - *(c) the edges are oriented as described in Lemma 2.2, and u is the origin of at most two oriented edges (u, v_k) and (u, v_l), $k < l$, where*
     - *i. u is the origin of only one oriented edge, (u, v_k), of edge uv_k for $k = 1, ..., j$, or*
     - *ii. u is the origin of two oriented edges where $k = 1, ..., j-2$ and $l = k + 2$, or $k = p$ and $l = q$.*

The proof of Theorem 3.2 can be found in the final section of this paper.

When $i = 1$, it is possible for a single vertex, namely u, to be the origin of two oriented edges. However, once $i, j \geq 2$, that possibility is eliminated, as was outlined in Lemmas 2.8 and 2.9. Although there are similarities in $UG^c(D) = P_2 \cup P_j$ for $j = 2, 3, 4, 5$, the differences are enough that we list the results

separately. First, we use the following lemma and its corollary to establish source vertices for $UG^c(D) = u_1u_2 \cup v_1v_2$. The set of vertices adjacent to a vertex v is the *neighborhood* of v, $n(v)$.

Lemma 3.3 *[5] If $UG(D) \cong dom(D)$ and $n(y) = \{x\}$ in $UG^c(D)$, then y is a source of at most one edge in $dom^c(D)$, and this edge will be incident to x.*

Corollary 3.4 *If $UG(D) \cong dom(D)$ and $UG^c(D) = u_1u_2 \cup v_1v_2$, then each vertex in D can be the source of at most one edge in $dom^c(D)$. Furthermore, u_1 may only be the source of an edge incident with u_2, u_2 the source of an edge incident with u_1, and similarly for v_1 and v_2.*

Since there are only two edges in $UG^c(D)$, we desire only two edges in $dom^c(D)$. Therefore, we pick only two of the vertices in $u_1u_2 \cup v_1v_2$ to be the origin of the oriented edges in D. Although it might seem possible to orient all four edges in $UG(D)$ in such a way that only two edges are formed in $dom^c(D)$, this cannot be done. Following, all digraphs D where $UG^c(D) = P_2 \cup P_2$ and $UG(D) \cong dom(D)$ are characterized.

Theorem 3.5 *Let $UG^c(D) = P_2 \cup P_2$. Further, let u be a vertex of one of the paths and let u' be the other vertex of that path. Let v be a vertex of the other path and v' its second vertex. $UG(D) \cong dom(D)$ if and only if (u, v) and (u', v') are oriented edges, and all other edges of $UG(D)$ are bidirected in D.*

Proof. ($\Longrightarrow$) Let u be u_1 or u_2 and v be v_1 or v_2. Further, suppose that (u, v) is an oriented edge in D. This creates edge $u'v$ in $dom^c(D)$. Since $UG(D) \cong dom(D)$, $dom^c(D)$ must contain only one more edge, uv'. According to Corollary 3.4, only vertex u' or vertex v may be the source of this edge. Therefore, (u', v') or (v, u) are the possible oriented edges in D that will create the edge in $dom^c(D) = P_2 \cup P_2$. Since (u, v) is an oriented edge, (v, u) is not a viable choice. Thus, (u', v') must be an oriented edge in D. Suppose that there are other oriented edges. No additional edges can be formed in $dom^c(D)$. The only arc that has not been discussed earlier is (v', u'), but it would bidirect edge $u'v'$, which is oriented in creating $dom^c(D)$. Thus, there are no other oriented edges possible in D. Since u is any of the four vertices in $UG(D)$, this holds for all cases

($\Longleftarrow$) If (u, v) and (u', v') are oriented edges, then $u'v$ and uv' are edges in $dom^c(D)$. The number of vertices in each path is less than 3, so there are no generated subpaths in $dom^c(D)$. Thus, $dom^c(D) = P_2 \cup P_2$, so $UG^c(D) \cong dom^c(D)$, and $UG(D) \cong dom(D)$. ■

Now we characterize D where $UG^c(D) = P_2 \cup P_3$. We begin by determining what vertices cannot be the sources of edges in $dom^c(D)$ outside of the edges in the generated subpaths.

Lemma 3.6 *Let $UG^c(D) = u_1u_2 \cup P_3$ where $P_3 = v_1, v_2, v_3$. If $UG(D) \cong dom(D)$, then (v_1, v_3) and (v_3, v_1) are both arcs in D.*

Proof. If (v_1, v_3) or (v_3, v_1) is an oriented edge, then by Lemma 2.2, vertices v_1, v_2 and v_3 form P_3 in $dom^c(D)$. Since u_1 and u_2 cannot form P_2 in $dom^c(D)$, according to Lemma 2.10, there is no way to create P_2 in $dom^c(D)$, and $UG^c(D) \ncong dom^c(D)$. Thus, both (v_1, v_3) and (v_3, v_1) must be arcs in D. ■

One characteristic that begins to appear now and will follow the constructions through all of the pairs of paths, concerns multiple sources for an edge. If more than one vertex in these digraphs can be the source of the same edge in $dom^c(D)$, then we can use any combination of the oriented edges in D that create the edge without creating new edges in $dom^c(D)$. However, we must be careful that each vertex is the source of at most one edge.

It is now possible to characterize all digraphs where $UG^c(D) = P_2 \cup P_3$. Figure 2 shows a possible construct using the vertex labeling convention adopted in the theorem and its proof. The figure shows a digraph where the oriented edges are formed using part (1) of Theorem 3.7. The random choice of u and v allows the characterization of all digraphs without listing each isomorphic labeling.

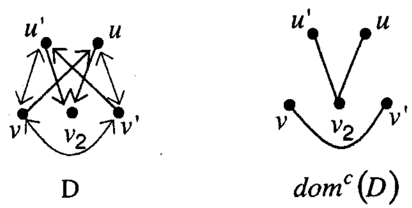

Figure 2: D shows a maximum number of oriented edges when $UG^c(D) = P_2 \cup P_3$. Vertex labeling is arbitrary.

Theorem 3.7 *Let $UG^c(D) = P_2 \cup P_3$. Further, let u be a vertex of P_2 and let u' be the other vertex. Let v be an end vertex of P_3 and v' be the other end vertex. $UG(D) \cong dom(D)$ if and only if every edge of $UG(D)$ is bidirected in D except for the following.*

1. (a) (u', v_2), (v, u) or (v', u) or any combination of these are oriented edges of $UG(D)$ in D, and

(b) (u, v_2), (v, u') or (v', u') or any combination of these are oriented edges of $UG(D)$ in D such that u, u', v and v' are the origin of at most one oriented edge, or

2. (u, v) is an oriented edge of $UG(D)$ in D, and (u', v_2) or (v', u) or both are oriented edges of $UG(D)$ in D.

The proof of Theorem 3.7 can be found in the final section of this paper.

When $UG^c(D) = P_2 \cup P_4$, we have the first instance where there are two nontrivial generated subpaths in $dom^c(D)$. Either of these paths will be the one to form P_2 in $dom^c(D)$. Again, there is much symmetry here, so the labeling we choose gives us all possible digraphs. Figure 3 represents just one of the selections that the labeling can produce, and aids in the understanding of the proof to the theorem.

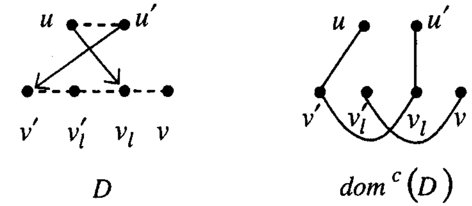

Figure 3: Example of labeling used where $UG^c(D) = P_2 \cup P_4$.

Theorem 3.8 *Let $UG^c(D) = P_2 \cup P_4$. Further, let $P_2 = u, u'$ and $P_4 = v, v_l, v'_l, v'$ for arbitrary selections of end vertices u, u', v, and v' in $UG^c(D)$. $UG(D) \cong dom(D)$ if and only if every edge of $UG(D)$ is bidirected in D except for the following.*

1. (u', v') is an oriented edge of $UG(D)$ in D, and
2. (u, v_l) or (v, u') or both are oriented edges of $UG(D)$ in D, and
3. (v, v') or (v', v) or both are arcs in D such that v and v' each are the origin of at most one oriented edge.

The proof for Theorem 3.8 can be found in the final section of this paper.

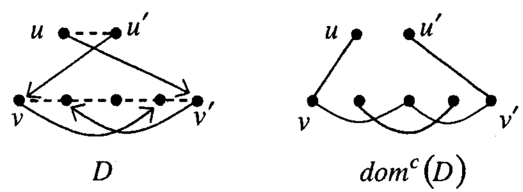

Figure 4: D shows a maximum number of oriented edges when $UG^c(D) = P_2 \cup P_5$. Dashed edges are $UG^c(D)$, and bidirectional edges are omitted for simplicity.

For the final characterization in this section, we have $dom^c(D)$ generated with very little choice of what vertices form P_2. The generated subpaths are

P_2 and P_3. Since $u_1u_2 \neq P_2$ in $dom^c(D)$, only v_2v_4 can fill that function. Figure 4 shows a digraph with the maximum number of oriented edges where $UG^c(D) = P_2 \cup P_5$ and $UG(D) \cong dom(D)$.

Theorem 3.9 *Let $UG^c(D) = P_2 \cup P_5$. Further, let $P_2 = u, u'$ and $P_5 = v, v_l, v_3, v_l', v'$ for arbitrary selections of end vertices u, u', v, and v' in $UG^c(D)$. $UG(D) \cong dom(D)$ if and only if every edge of $UG(D)$ is bidirected in D except for the following.*

1. *(u', v) and (u, v') are both oriented edges of $UG(D)$ in D, and*
2. *(v, v_l') or (v', v_l) or both or neither are oriented edges of $UG(D)$ in D.*

Proof. Paths v, v_3, v' and v_l, v_l' are generated subpaths in $dom^c(D)$.

($\Longrightarrow$) Since uu' cannot be an edge in $dom^c(D)$, $P_2 = v_lv_l'$. Thus, $P_5 = u, v, v_3, v', u'$ in $dom^c(D)$. Edges uv and $u'v'$ need to have a source in D. Since v is an end vertex, there is no v_k that can be used as a source of the edge for reasons explained in the proof of Theorem 3.8. Therefore, (u', v) is the only oriented edge that will form uv in $dom^c(D)$, so it must be in every biorientation of $UG(D)$. For similar reasons, (u, v') is the only oriented edge generating $u'v'$ in $dom^c(D)$, so must be in every biorientation of $UG(D)$, proving part (1).

Corollary 2.4 allows that we may use oriented edges (v, v_l') and (v', v_l) without creating new edges in $dom^c(D)$. Since these oriented edges are not necessary for the production of an additional edge in $dom^c(D)$, if they are used, then they can appear in a biorientation singly or together, proving part (2). Since u and v are arbitrary selections of the end vertices, we obtain all possible biorientations.

($\Longleftarrow$) Vertices u' and u are sources of edges uv and $u'v'$ respectively in $dom^c(D)$ when (u', v) and (u, v') are oriented edges of $UG(D)$ in D. Vertices v and v' are both sources of edge v_lv_l' when (v, v_l') and/or (v', v_l) are oriented edges in D, and v_3 is the only source of that edge otherwise. Thus, $dom^c(D) = v_l, v_l' \cup u, v, v_3, v', u'$, and $UG(D) \cong dom(D)$. ■

4 Structure of $UG^c(D) = P_i \cup P_j$ for $i, j \geq 3$

If we were interested in seeing only what pairs of paths can comprise $UG^c(D)$ so that $UG(D) \cong dom(D)$, the answer would be simple.

Theorem 4.1 *Let $UG^c(D) = P_i \cup P_j$ where $i, j \geq 3$. There exists a biorientation of the edges of $UG(D)$ such that $UG(D) \cong dom(D)$ for every value of $i, j \geq 3$.*

Proof. This follows directly from Theorem 1.1 and Lemma 2.2. ■

However, we are interested in much more than just existence. The main goal is to characterize all digraphs where $UG^c(D) = P_i \cup P_j$ and $UG(D) \cong dom(D)$. Therefore, we must also consider the structure of $UG^c(D)$ when

paths in $dom^c(D)$ are formed using vertices from both $V(P_i)$ and from $V(P_j)$. We therefore continue the discussion of the underlying graph by expanding upon Lemma 2.8. Because only the end vertices of the paths in $UG^c(D)$ may be used as origins of oriented edges in D, certain edges cannot occur in $dom^c(D)$. The following lemma details the edges that will never appear in that graph.

Lemma 4.2 *Let $P_i = u_1, ..., u_i$ and $P_j = v_1, ..., v_j$ be paths that are components of $UG^c(D)$ for $i, j \geq 3$. If $UG \cong dom(D)$, then u_1v_1, u_1v_j, u_iv_1 and u_iv_j are not edges in $dom^c(D)$.*

Proof. Lemma 2.8 states that only u_1, u_i, v_1 or v_j can be the origin of an oriented edge in D. A vertex can also not be the source of more than one edge or an edge with which it is incident in $dom^c(D)$. Thus, to form u_1v_1 in $dom^c(D)$, either u_i or v_j must be the source. Both u_i and v_j are adjacent to u_1 and v_1, so the oriented edges (u_i, u_1) and (u_i, v_1) or (v_j, u_1) and (v_j, v_1) need to be in D for either of the two vertices to be the source of the edge u_1v_1. This contradicts Lemma 2.9, so u_1v_1 does not have any possible source and cannot be produced in $dom^c(D)$ when $UG(D) \cong dom(D)$. Similar arguments hold for the other three edges between the end vertices of the paths. ■

The previous lemma has important consequences for $UG^c(D)$ when both i and j are odd. Recall that when i is odd, $u_1, u_3, ..., u_i$ is a generated subpath in $dom^c(D)$. When we have two such paths, they can never be connected to form a larger path in $dom^c(D)$, since we cannot form edges between the end vertices.

Corollary 4.3 *Let $P_i = u_1, ..., u_i$ and $P_j = v_1, ..., v_j$ be paths that are components of $UG^c(D)$ for odd $i, j \geq 3$. Further, let $U_1 = u_1, u_3, ..., u_i$, $U_2 = u_2, u_4, ..., u_{i-1}$, $V_1 = v_1, v_3, ..., v_j$ and $V_2 = v_2, v_4, ...v_{j-1}$ be the generated subpaths in $dom^c(D)$, where U_1V_1 denotes the path $u_1, u_3, ..., u_i, v_1, v_3, ..., v_j$. If $UG(D) \cong dom(D)$, then U_1V_1 is not a path in $dom^c(D)$.*

Now we have the information necessary to further characterize $UG^c(D)$ where we expect $dom^c(D)$ to be formed using vertices from both $V(P_i)$ and $V(P_j)$.

Theorem 4.4 *Let $P_i = u_1, ..., u_i$ and $P_j = v_1, ..., v_j$ be paths that are components of $UG^c(D)$ for $3 \leq i \leq j$. There exists a biorientation D of the edges of $UG(D)$ such that $UG(D) \cong dom(D)$ and u_kv_l is an edge in $dom^c(D)$ for some k and l if and only if*

1. *$j = i$,*
2. *$j = i + 1$,*
3. *$j = 2i - 1$,*
4. *$j = 2i$, or*
5. *$j = 2i + 1$.*

The proof of Theorem 4.4 can be found in the final section of this paper.

5 Characterization of D where $UG^c(D) = P_i \cup P_j$ for $i, j \geq 3$

In building digraphs as biorientations of their underlying graphs, Theorems 4.1 and 4.4 separate the characterization into two parts. The first is where P_i and P_j are created in $dom^c(D)$ using $V(P_i)$ and $V(P_j)$ respectively from $UG^c(D)$. The second is where generated subpaths from $V(P_i)$ and $V(P_j)$ are connected to form the paths. Now, to characterize all D where P_i and P_j are formed in $dom^c(D)$ using Lemma 2.2, we need the following result.

Lemma 5.1 *Let $P_i = u_1, ..., u_i$ and $P_j = v_1, ..., v_j$ be paths that are components of $UG^c(D)$ for $i, j \geq 3$. Also, let $u = u_1$ or u_i and $v = v_1$ or v_j. Any oriented edge (u, v_l) or (v, u_k) for $1 \leq k \leq i$ and $1 \leq l \leq j$, creates an edge in $dom^c(D)$ between $V(P_i)$ and $V(P_j)$.*

Proof. If $u = u_1$ or u_i, then u_2v_l or $u_{i-1}v_l$ respectively is an edge in $dom^c(D)$. The same argument holds for v. ∎

This result makes it clear that in the case where Lemma 2.2 is used to produce paths P_i and P_j in $dom^c(D)$, all edges in $UG(D)$ between the vertices of P_i^c and P_j^c must be bidirected.

Theorem 5.2 *Let $UG^c(D) = P_i \cup P_j$ for $i, j \geq 3$ where $P_i = u_1, ..., u_i$ and $P_j = v_1, ..., v_j$. $UG(D) \cong dom(D)$ and u_kv_l is not an edge in $dom^c(D)$ for any $1 \leq k \leq i$, $1 \leq l \leq j$, if and only if the edges of $UG(P_i^c)$ and $UG(P_j^c)$ are bioriented as stated in Lemma 2.2 and all other edges are bidirected to form D.*

Proof. ($\Longrightarrow$) Since $UG(D) \cong dom(D)$ and u_kv_l is not an edge in $dom^c(D)$, Lemma 5.1 indicates that no oriented edge may exist between $u_1, ..., u_i$ and $v_1, ..., v_j$. So all edges between $V(P_i)$ and $V(P_j)$ in D are bidirected. Thus, the paths must be formed in $dom^c(D)$ as outlined in Lemma 2.2. This produces P_i on the vertices $u_1, ..., u_i$ and P_j on the vertices $v_1, ..., v_j$ in $dom^c(D)$.

($\Longleftarrow$) Given a biorientation of the edges in $UG(P_i^c)$ and $UG(P_j^c)$ pursuant to Lemma 2.2, we know that P_i and P_j are formed on vertices $u_1, ..., u_i$ and $v_1, ..., v_j$ respectively in $dom^c(D)$. Since all other edges are bidirected, all other pairs of vertices dominate, and no additional edges are created in $dom^c(D)$. Thus, $dom^c(D) = P_i \cup P_j$ so that $UG(D) \cong dom(D)$, and u_kv_l is not an edge for any $1 \leq k \leq i$, $1 \leq l \leq j$. ∎

Now we turn to the more interesting characterization. That where $dom^c(D)$ contains at least one edge between $V(P_i)$ and $V(P_j)$. We know that only four vertices may be the origin of oriented edges in D, and each is the source of at most one edge in $dom^c(D)$. In order to construct $dom^c(D)$ so that it is isomorphic to $UG^c(D)$ where vertices from both $V(P_i)$ and $V(P_j)$ are used in each path, it is helpful to detail what edges are created in $dom^c(D)$ given an oriented edge.

Lemma 5.3 *Let $P_i = u_1, ..., u_i$ and $P_j = v_1, ..., v_j$ be components of $UG^c(D)$ where $i, j \geq 3$.*

1. *If (u_1, v_k) for $1 \leq k \leq j$ is an oriented edge in D, then u_2v_k is an edge in $dom^c(D)$.*
2. *If (u_i, v_k) for $1 \leq k \leq j$ is an oriented edge in D, then $u_{i-1}v_k$ is an edge in $dom^c(D)$.*
3. *If (v_1, u_k) for $1 \leq k \leq i$ is an oriented edge in D, then u_kv_2 is an edge in $dom^c(D)$.*
4. *If (v_j, u_k) for $1 \leq k \leq j$ is an oriented edge in D, then u_kv_{j-1} is an edge in $dom^c(D)$.*

Proof. In each case, the edge created in $dom^c(D)$ is between the one vertex not adjacent to the origin of the oriented edge and the vertex dominated by that same vertex. They do not dominate, and thus form an edge in $dom^c(D)$. ∎

A consequence of this lemma is that some edges between the paths U_kV_l can be formed in $dom^c(D)$ in two ways, or one way, or cannot be formed. An example of edges that cannot exist was given in Lemma 4.2. The previous lemma only addresses edges formed between the two sets of vertices. Sources for edges within each set were given in Lemma 2.2. The following corollary restates the results in Lemma 5.3 in terms of the number of ways in which edge in $dom^c(D)$ can be formed using oriented edges in D. This is important information for proving the final characterizations.

Corollary 5.4 *Let $P_i = u_1, ..., u_i$ and $P_j = v_1, ..., v_j$ be components of $UG^c(D)$ where $i, j \geq 3$. If (u_k, v_l) is an edge in $dom^c(D)$ where $k = 2$ or $i - 1$ and $l = 2$ or $j - 1$, then there are two possible sources for the edge. All other edges of the form (u_s, v_t)in $dom^c(D)$ have at most one source.*

The structure of D supersedes the possible labelings of the vertices, so the focus is on the relationships of the oriented edges to each other. In this way, the isomorphic labelings are incorporated into the final results. Of course, to be able to indicate which vertices are involved with the oriented edges in D and the edges in $dom^c(D)$, some labeling convention must be adopted. Thus, we will let u and u' be the end vertices of one path, with v and v' the end vertices of the other path. At times, we will need to discuss the vertices that are adjacent to u, u', v, and/or v' in $UG^c(D)$. Thus, if $u = u_1$ or u_i, then $u_k = u_2$ or u_{i-1} respectively. Likewise, if $v = v_1$ or v_j, then $v_l = v_2$ or v_{j-1} respectively. Similar labelings will be utilized for u'_k and v'_l.

From Theorem 4.4, the cases we need to consider in our characterization of D where $UG^c(D) = P_i \cup P_j$, are 1) $j = i$, 2) $j = i + 1$, 3) $j = 2i - 1$, 4) $j = 2i$, and 5) $j = 2i + 1$. First, we consider the case where $i = j$ is odd. The only oriented edges will be between the end vertices of the original two paths. Therefore, there are two oriented edge formations possible. These are shown in Figure 5.

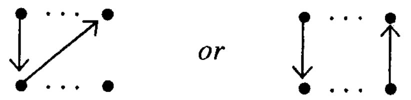

Figure 5: The only two possible oriented edge formations between $V(P_i)$ and $V(P_j)$ when $i = j \geq 3$ are odd.

Theorem 5.5 *Let $UG^c(D) = P_i \cup P_j$, where $i = j \geq 3$ are odd. Further, let $P_i = u, u_k, \ldots, u'_k, u'$ and $P_j = v, v_l, \ldots, v'_l, v'$ in $UG^c(D)$. $UG(D) \cong dom(D)$ where there is an edge between $V(P_i)$ and $V(P_j)$ in $dom^c(D)$ if and only if every edge of $UG(D)$ is bidirected in D except for the following.*

1. *(u, v) is an oriented edge of $UG(D)$ in D, and*
2. *exactly one of (v, u') or (v', u') is an oriented edge of $UG(D)$ in D, and*
3. *for $i, j \geq 5$, oriented edges may be formed as stated in Corollary 2.3 such that u, u', v, and v' are each the origin of at most one oriented edge.*

Proof. ($\Longrightarrow$) Pursuant to Lemma 2.1, there is one odd generated subpath and one even generated subpath on each of $V(P_i)$ and $V(P_j)$ in $dom^c(D)$. To create two paths of length i, each odd path must have an edge to an even path. Since edges must exist between $V(P_i)$ and $V(P_j)$ in $dom^c(D)$, the odd subpath generated on $V(P_i)$ must have an edge to the even subpath generated on $V(P_j)$, and similarly for their counterparts. Say that uv_k is any such an edge in $dom^c(D)$, forming path $u', \ldots, u, v_k, \ldots, v'_k$. Then (u, v) must be an oriented edge in D. This results in uv_l never being an edge in $dom^c(D)$ when $UG(D) \cong dom(D)$. To create the other path in $dom^c(D)$, we must form edge uv'_l, $u'v_l$, or $u'v'_l$. Edges uv'_l and $u'v_l$ are formed by oriented edges (v', u) and (v, u') in D respectively. These correspond to isomorphic digraphs. Thus, we only need list (v, u'). This relationship is shown in the first digraph of Figure 5. Finally, if edge $u'v'_l$ is in $dom^c(D)$, (v', u') must be an oriented edge of D. Thus, within isomorphic labeling, (u, v) must be an oriented edge along with one of (v, u') or (v', u') since $UG(D) \cong dom(D)$. Additionally, oriented edges listed in part (3) may be created as stated in Corollary 2.4 as long as no vertex is the origin of more than one oriented edge.

($\Longleftarrow$) If (u, v) and (v, u') are oriented edges in D, then edges u_kv and $u'v_l$ are formed in $dom^c(D)$, creating two paths with i vertices each. If (u, v) and (v', u') are oriented edges in D, then edges u_kv and $u'v'_l$ are formed in $dom^c(D)$, creating two paths with i vertices each. In both cases, $UG(D) \cong dom(D)$. Any directed edge in part (3) that does not create a vertex that is the origin of more than one oriented edge is allowed, with no additional edges formed. Therefore, $dom^c(D) = P_i \cup P_j$ where $i = j$ are odd, and $UG(D) \cong dom(D)$. ■

Next we examine the case where $i = j$ is even. To do so, it is easiest to separate possible orientations into classes determined by the structure of $dom^c(D)$. We will describe digraphs where the paths in $dom^c(D)$ have edges where 1) two end vertices are used in the two adjoining edges, 2) no end vertices are used in the two adjoining edges, and 3) one end vertex is used in the two edges.

In determining the digraphs that result in isomorphic underlying and domination graphs, we find that there is one formation that is not allowed in D when we look at the first case listed above. That where two end vertices are in the two edges of $dom^c(D)$. The following lemma shows that in this case in D, the oriented edges will always be from one set of vertices, $V(P_i)$, to the other set, $V(P_j)$, where the paths are arbitrarily labeled.

Lemma 5.6 *Let $UG^c(D) = P_i \cup P_j$ with $i = j$ even and $UG(D) \cong dom(D)$. Further, let x and w be end vertices in $UG^c(D)$. If xy and wz are edges between $V(P_i)$ and $V(P_j)$ in $dom^c(D)$, then x and w are both end vertices of P_i or both end vertices of P_j.*

Proof. Suppose that x and w are in separate vertex sets. Let $x = u$ in P_i and u' be the other end vertex. Also, let $v = w$ in P_j and v' be the other end vertex. Then $u, \ldots, u'_k$, $u_k, \ldots, u'$, $v, \ldots, v'_l$ and $v_l, \ldots, v'$ are the four generated subpaths in $dom^c(D)$. Since $xy = uy$ and $wz = vz$ are edges in $dom^c(D)$, $y \neq v'_l$ and $z \neq u'_k$, else paths $u'_k, \ldots u, v'_l, \ldots, v$ and $v'_l, \ldots, v, u'_k, \ldots, u$ are formed, and u and/or v appears in more than one path in $dom^c(D)$. Thus, $y = v_l$ and $z = u_k$. But then there is only one way to create edge uv_l in $dom^c(D)$, and that is with oriented edge (v, u) in D. Likewise, the one way to form edge u_kv in $dom^c(D)$ is with oriented edge (u, v) in D. They cannot be used together, as they form a bidirected edge. Thus, both x and w must be end vertices of either P_i or P_j. ■

With the preceding lemma, it is now possible to list the oriented edges that may occur in D when $i = j$ is even. Figure 6 illustrates the possible oriented edges that may occur in D given the number of end vertices in $UG^c(D)$ that are used to connect the paths in $dom^c(D)$.

Theorem 5.7 *Let $UG^c(D) = P_i \cup P_j$, where $i, j \geq 3$ are even. Further, let $P_i = u, u_k, \ldots, u'_k, u'$ and $P_j = v, v_l, \ldots, v'_l, v'$ in $UG^c(D)$. $UG(D) \cong dom(D)$ where there is an edge between $V(P_i)$ and $V(P_j)$ in $dom^c(D)$, if and only if every edge of $UG(D)$ is bidirected in D except for the following.*

1. (a) *(u, v) and (u', v') are both oriented edges in D, or*
   - i. *(u, v_l) or (v, u_k) or both are oriented edges in D, and*
   - ii. *(u', v'_l) or (v', u'_k) or both are oriented edges in D, or*
   - i. *(u, v) is an oriented edge in D, and*
   - ii. *(u', v_k) or (v, u'_k) or both are edges in D, and*

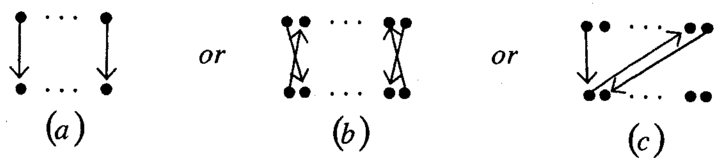

Figure 6: Digraphs where P_i and P_j are formed in $dom^c(D)$ using (a) two, (b) zero, and (c) one of the end vertices from $UG^c(D)$. All are shown with the maximum number of oriented edges between $V(P_i)$ and $V(P_j)$. Bidirected edges are omitted.

2. *for $i, j \geq 4$, oriented edges may be formed as stated in Corollary 2.3 such that u, u', v, and v' are each the origin of at most one oriented edge.*

The proof of Theorem 5.7 can be found in the last section of this paper.

To continue the characterization, we observe the case where $j = i+1$. The values for i and j alternate odd and even, which in practice does not make a difference with the results. However, it is important to understand when we are dealing with the even path and when the odd path is discussed. While we could split the results into odd and even, that is not necessary if we generalize the paths. So for this case, we will let $P_e = e, e_k, ..., e'_k, e'$ be the even path in $UG^c(D)$, and $P_o = o, o_l, ..., o'_l, o'$ be the odd path. Thus, one generated subpath in $dom^c(D)$ has end vertices e and e'. They can only be joined to an interior vertex, o_l or o'_l, of the other path to create the odd path in $dom^c(D)$. There is only one distinct way to do that. This narrows down the choices. Once the choice is made, as represented by the directed edge from the upper left to the lower left corner of every digraph in Figure 7, there is a variety of ways to produce the even path in $dom^c(D)$.

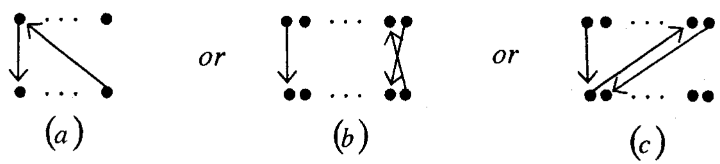

Figure 7: Digraphs where P_i and P_j are formed in $dom^c(D)$ when $j = i+1$ and there is an edge between $V(P_i)$ and $V(P_j)$. All are shown with the maximum number of oriented edges between $V(P_i)$ and $V(P_j)$. The even set of vertices is on the top, and bidirected edges are omitted.

Theorem 5.8 *Let $UG^c(D) = P_i \cup P_j$, where $i \geq 3$ and $j = i+1$. Further, let $P_e = e, e_k, ..., e'_k, e'$ be the even path in $UG^c(D)$, and $P_o = o, o_l, ..., o'_l, o'$ be the odd path. $UG(D) \cong dom(D)$ where there is an edge between $V(P_i)$ and $V(P_j)$ in $dom^c(D)$, if and only if every edge of $UG(D)$ is bidirected in D except for the following.*

1. *(e, o) is an oriented edge in D, and*
   (a) *(o', e) is an oriented edge in D, or*
   (b) *(e', o'_l) or (o', e'_k) or both are oriented edges in D, or*
   (c) *(e', o_l) or (o, e'_k) or both are oriented edges in D, and*
2. *oriented edges may be formed as stated in Corollary 2.3 such that e, e', o, and o' are each the origin of at most one oriented edge.*

Proof. Part (2) is valid by Corollary 2.4. The remaining arguments deal with part (1).

($\Longrightarrow$) If we let o be either of the end vertices of the odd path in $UG^c(D)$, then let $e_k o$ be the edge formed in $dom^c(D)$, we obtain an arbitrary labeling similar to that in the proof of Theorem 5.7. Path $P_1 = o', ..., o, e_k, ...e'$ is the odd path formed in $dom^c(D)$. The arguments here follow the same logic as the proofs in Theorems 5.5 and 5.7. However, once we have chosen o and e_k, there is a selection for how to form the remaining path in $dom^c(D)$. Figure 7(a) illustrates the option of having the one end vertex, e, that is not in P_1 as one of the vertices incident with the edge that connects the remaining two subpaths. The only way for this to occur is for (o', e) to be an oriented edge in D. The remaining two possible connecting edges, $e'_k o_l$ and $e'_k o'_l$, can be formed in two ways each, as listed in parts 1 (b) and 1 (c) of the theorem statement. These two options are not isomorphic, as the relationships of the oriented edges are different, as seen in Figure 7(b) and (c).

($\Longleftarrow$) By Lemma 5.3, if (e, o) is an arc, then edge $e_k o$ is in $dom^c(D)$, and each of parts (a) through (c) creates an edge connecting the remaining two subpaths in $dom^c(D)$. This results in $dom^c(D)$ consisting of two disjoint paths with i and $j = i+1$ vertices. Thus, $UG^c(D) \cong dom^c(D)$ and $UG(D) \cong dom(D)$. ■

All of the previous cases have dealt with paths P_i and P_j in $dom^c(D)$ that were created by connecting two subpaths for each. Now we turn our attention to those cases where P_i is one of the generated subpaths in $dom^c(D)$, and P_j is created by connecting the remaining three subpaths.

Remark 5.9 *If $j = 2i-1$, $2i$ or $2i+1$, where $UG(D) \cong dom(D)$, then P_i in $dom^c(D)$ is a generated subpath on $V(P_j)$.*

We begin now with the case where $j = 2i-1$. Figure 8 shows examples of part (1) in the following theorem. The digraphs are shown on the same set of vertices as their associated $dom^c(D)$ graphs. Edges shown are those for $dom^c(D)$, and bidirected edges of D are omitted for simplicity. Labeling on part (a) shows the labeling convention adopted.

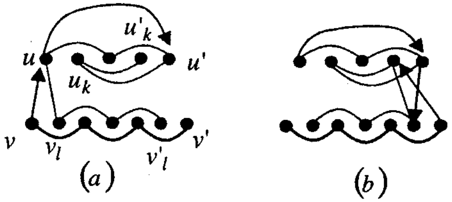

Figure 8: Examples of digraphs and their associated $dom^c(D)$ graphs where i is odd, $j = 2i - 1$, and P_i is a generated subpath on $V(P_j)$. Edges shown are in $dom^c(D)$, while arcs are in D. Bidirected edges are omitted.

Theorem 5.10 *Let $UG^c(D) = P_i \cup P_j$, where $i \geq 3$ and $j = 2i-1$. Further, let $P_i = u, u_k, ..., u'_k, u'$ and $P_j = v, v_l, ..., v'_l, v'$ in $UG^c(D)$. $UG(D) \cong dom(D)$ where there is an edge between $V(P_i)$ and $V(P_j)$ in $dom^c(D)$, if and only if every edge of $UG(D)$ is bidirected in D except for the following.*

1. *If edges are oriented in $V(P_i^c)$ as stated in Lemma 2.2, then*
   - *(a) (v, u) is an oriented edge in D, or*
   - *(b) if i is odd, then (u', v_l) or (v, u'_k) or both are oriented edges in D.*
2. *If edges are not oriented in $V(P_i^c)$ as stated in Lemma 2.2, then*
   - *(a) (v, u) is an oriented edge in D, and*
     - *i. (u, v'_l) or (v', u_k) or both are oriented edges in D, or*
     - *ii. if i is odd, then (u', v'_l) or (v', u'_k) or both are oriented edges in D, or*
     - *iii. if i is even, then (v', u') is an oriented edge in D, or*
   - *(b) if i is even, then*
     - *i. (u, v_l) or (v, u_k) or both are oriented edges in D, and*
     - *ii. (u', v'_l) or (v', u'_k) or both are oriented edges in D.*
3. *If i is odd or even, oriented edges stated in Corollary 2.3 may be used in addition to the required arcs in (1) and (2) as long as u, u', v, and v' are each the origin of at most one oriented edge in D.*

The proof of Theorem 5.10 can be found in the final section of this paper.

Continuing with the cases where P_i in $dom^c(D)$ is a generated subpath, we proceed to $j = 2i$. Unlike the cases where $j = 2i + 1$ or $2i - 1$, here P_j is on an even number of vertices. This allows the choice of which generated

subpath, V_1 or V_2, will be P_i in $dom^c(D)$. Generally, we will let $P_i = V_2 = v_l, ..., v'$. Figure 9 shows a digraph and its associated $dom^c(D)$ graph on the same set of vertices. It illustrates part 2(*b*)(*ii*) of the following theorem, and uses the minimum number of oriented edges. In the figure, $P_i = V_2$ and $P_j = U_1, v'_l, ..., v, U_2$, where $U_1 = u, u'_k$ and $U_2 = u_k, u'$.

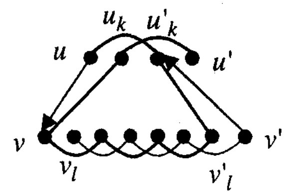

Figure 9: A digraph D where $j = 2i$ on the same set of vertices as $dom^c(D)$. Edges shown are those in $dom^c(D)$, while arcs are in D. Bioriented edges of D are omitted.

Theorem 5.11 *Let $UG^c(D) = P_i \cup P_j$, where $i \geq 3$ and $j = 2i$. Further, let $P_i = u, u_k, ..., u'_k, u'$ and $P_j = v, v_l, ..., v'_l, v'$ in $UG^c(D)$. $UG(D) \cong dom(D)$ where there is an edge between $V(P_i)$ and $V(P_j)$ in $dom^c(D)$, if and only if every edge of $UG(D)$ is bidirected in D except for the following.*

1. *If edges are oriented in $V(P_i^c)$ as stated in Lemma 2.2, then*
   - *(a) (v', u) is an oriented edge in D, or*
   - *(b) if i is odd, then $(u'v)$ is an oriented edge in D.*
2. *If edges are not oriented in $V(P_i^c)$ as stated in Lemma 2.2, and*
   - *(a) if i is odd, then (v', u) is an oriented edge in D, and*
     - *i. (u, v) is an oriented edge in D, or*
     - *ii. (u', v) is an oriented edge in D, or*
   - *(b) if i is even, then (u, v) is an oriented edge in D and*
     - *i. (v', u) is an oriented edge in D, or*
     - *ii. (u', v'_l) or (v', u'_k) or both are oriented edges in D.*
3. *If i is odd or even, oriented edges stated in Corollary 2.3 may be used in addition to the required arcs in (1) and (2) as long as u, u', v, and v' are each the origin of at most one oriented edge in D.*

The proof of Theorem 5.11 can be found in the final section of this paper.

To conclude the characterization of D where $UG^c(D) = P_i \cup P_j$ and $UG(D) \cong dom(D)$, the case where $j = 2i+1$ is examined. With j being odd, the choice for P_i in $dom^c(D)$ is set, and $V_1 = v, ..., v'$ must be connected to the two generated subpaths U_1 and U_2 to form P_j in $dom^c(D)$. There are very few nonisomorphic ways in which this can be done, so the final theorem has few options.

Theorem 5.12 *Let $UG^c(D) = P_i \cup P_j$, where $i \geq 3$ and $j = 2i+1$. Further, let $P_i = u, u_k, ..., u'_k, u'$ and $P_j = v, v_l, ..., v'_l, v'$ in $UG^c(D)$. $UG(D) \cong dom(D)$ where there is an edge between $V(P_i)$ and $V(P_j)$ in $dom^c(D)$, if and only if every edge of $UG(D)$ is bidirected in D except for the following.*

1. *If i is odd, then edges are oriented in $V(P_i^c)$ as stated in Lemma 2.2, and $(u'v)$ is an oriented edge in D.*
2. *If i is even, then (u, v) and (u', v') are oriented edges in D.*
3. *If i is odd or even, oriented edges stated in Corollary 2.3 may be used in addition to the required arcs in (1) and (2) as long as u, u', v, and v' are each the origin of at most one oriented edge in D.*

Proof. Let $V_1 = v, ..., v'$ and $V_2 = v_l, ..., v'_l$. The only choice for P_i in $dom^c(D)$ is $P_i = V_2$.

($\Longrightarrow$) $UG(D) \cong dom(D)$ and $P_i = V_2$ in $dom^c(D)$, so subpaths U_1, U_2 and V_1 must be connected to form P_j in $dom^c(D)$. V_1 has end vertices v and v', which can only form edges with interior vertices u_k and u'_k in $V(P_i)$. If i is odd, then $U_1 = u, ..., u'$ cannot connect to V_1 since u and u' are not interior vertices. Thus, U_1 must connect to $U_2 = u_k, ..., u'_k$, which must be connected to V_1. Thus, Lemma 2.2 must be used to connect U_1 and U_2, forming path $u, ..., u', u_k, ...u'_k$. The only edge that can connect this path to V_1 is $u'_k v$, where v is either end vertex of V_1. So oriented edge (u', v) in D is the only option when i is odd.

If i is even, then Lemma 2.2 creates path $u, ..., u'_k, u_k, ..., u'$ on $V(P_i)$, which cannot be connected to V_1 in $dom^c(D)$ since u and u' cannot form an edge with v or v'. Thus, U_1 and U_2 must each be connected to V_1, and Lemma 2.2 cannot be used. Only u_k and u'_k can be connected to v and v'. Choose u and v arbitrarily. Then $u_k v$ and $u'_k v'$ are the edges needed in $dom^c(D)$. Thus, edges (u, v) and (u', v') are oriented edges in D.

From Corollary 2.4, we are guaranteed that the arcs in part 3 will not alter the relationship $UG(D) \cong dom(D)$ as long as vertices u, u', v, and v' are each the origin of at most one oriented edge in D.

($\Longleftarrow$) In all constructions for parts (1) and (2), $P_i = V_2$. P_j is formed as follows, creating $UG(D) \cong dom(D)$ with an edge between $V(P_i)$ and $V(P_j)$. In part (1), $P_j = U_1 U_2 V_1$. In part (2), $P_j = U_1, v', ..., v, U_2$. In all cases, the oriented edges in Corollary 2.3 may be used and create no new edges. Since u, u', v, and v' are each the origin of at most one oriented edge, $UG(D) \cong dom^c(D)$. ∎

6 Proofs of Selected Results Omitted Earlier

Following is the proof for **Theorem 3.2.**

Proof. ($\Longleftarrow$) The case where $j = 1$ is obvious as the two vertices always dominate in D.

When $j = 2$, $E(UG(D)) = \{uv_1, uv_2\}$. If (v_1, u) is an orientation and (v_2, u) is not, then u and v_2 do not dominate, and form P_2 in $dom^c(D)$ with $v_1 = P_1$. Likewise, if (v_2, u) is an orientation and (v_1, u) is not, then u and v_1 form P_2 in $dom^c(D)$ with $v_2 = P_1$. If (u, v_1) and (u, v_2) are oriented edges, then $v_1 v_2$ forms P_2 in $dom^c(D)$ with $u = P_1$. Thus, $UG(D) \cong dom(D)$.

When $j \geq 3$, for part (a), Lemma 2.2 gives us constructions that result in the formation of P_j in $dom^c(D)$. Since u dominates all of the v_i, no relationships between the v_i are changed, and uv_i is an edge in $dom(D)$. Thus, $u = P_1$ and the v_i form P_j in $dom^c(D)$, giving $UG^c(D) \cong dom^c(D)$ and $UG(D) \cong dom(D)$.

For part (b), if (u, v_p) and (u, v_q) are the only orientations of the edges of $UG(D)$, then the only edges formed in $dom^c(D)$ without the source u are the generated subpaths. The vertex u is the source for edge $v_p v_q$ in $dom^c(D)$. There is no vertex in P_j that is not adjacent to u, so only the edge $v_p v_q$ is formed. Since $v_p v_q$ joins the two subpaths in $dom^c(D)$, forming P_j, and u is P_1 for reasons explained in part (a), $UG(D) \cong dom(D)$.

For part (c), let the edges between vertices v_i be oriented as in Lemma 2.2. First, consider the additional orientation (u, v_k) for $k = 1, ..., j$. The vertex u is not the source for any edge in $dom^c(D)$ since every vertex other than v_k dominate it. Thus, only the edges in $dom^c(D)$ formed by oriented edges specified in Lemmas 2.2 and 2.1 are created, and u is an isolated vertex. Therefore, $dom^c(D) = P_1 \cup P_j$ and $UG(D) \cong dom(D)$. Now consider that u is the origin of two oriented edges (u, v_k) and (u, v_l) where $k = 1, ..., j-2$ and $l = k+2$. Then u is a source of the edge $v_k v_{k+2}$ in $dom^c(D)$, for which vertex v_{k+1} is also a source since it is not adjacent to either vertex. Or, if $k = p$ and $l = q$, then u is a source for the edge $v_p v_q$ in $dom^c(D)$, for which a vertex v_i is also a source, as determined in Lemma 2.2. In either case, no new edges are formed in $dom^c(D)$, u is an isolated vertex in $dom^c(D)$, and the vertices v_i form the path P_j. Thus, $UG(D) \cong dom(D)$.

($\Longrightarrow$) The case where $j = 1$ is obvious as $UG(D) = K_2$ and the two vertices always dominate in D.

When $j = 2$ and $UG(D) \cong dom(D)$, then there must be a source to one edge in $dom^c(D)$. A vertex cannot be the source of any edge with which it is incident. Therefore, if it is possible, u must be the source for $v_1 v_2$, v_1 must be the source for uv_2 and v_2 must be the source for uv_1. For the first case, u must dominate both v_1 and v_2 in order to be the source. In the second case, v_1 and v_2 are not adjacent, so the orientation (v_1, u) of edge uv_1 makes uv_2 an edge in $dom^c(D)$ since neither u nor v_2 dominates v_1. However, (u, v_2) must be an oriented edge in D, so that uv_1 is an edge in $UG(D)$. A similar argument holds for the case where v_2 is the source for uv_1.

When $j \geq 3$, Lemma 2.2 shows a construction that creates an isomorphic copy of P_j in $dom^c(D)$ for $j \geq 3$. Lemma 3.1 guarantees that u must be equal to P_1 in $dom^c(D)$ since $UG(D) \cong dom(D)$. Therefore, no v_i will be the origin of an oriented edge with terminal vertex u, since edge uv_{i+1} or uv_{i-1} would be created in $dom^c(D)$. Thus, u will be the only possible origin for any additional oriented edges outside of those created by Lemma 2.2. The vertex u will either be a source vertex, or it will not be. If it is not, since u is adjacent to all vertices in $UG(D)$, either all edges uv_i for $i = 1, ..., j$ will be bidirected, proving part 3(a) above, or exactly one will have the orientation (u, v_i). If not, u would be a source of an edge in $dom^c(D)$. This proves part 3(c)(i) above.

If u is a source for an edge in $dom^c(D)$ and we do not use Lemma 2.2 to create path P_j in $dom^c(D)$, then u must be the source for edge v_pv_q so that P_j is created in $dom^c(D)$. Since u is adjacent to both vertices in $UG(D)$, both (u, v_p) and (u, v_q) must be oriented edges in D, proving part (2).

If u is a source for an edge in $dom^c(D)$ and we use Lemma 2.2 to create P_j in $dom^c(D)$, then u must be the source for an edge that is in P_j. Thus, u must be the source of an edge v_iv_{i+2}, for $i = 1, ..., j-2$, or v must be the source of edge v_pv_q. Since u is adjacent to all vertices in $UG(D)$, (u, v_i) and (u, v_{i+2}) or (u, v_p) and (u, v_q) must be oriented edges in D, proving part 3(c)(ii).

Lemma 2.6 guarantees that u cannot be the source of more than one edge in $dom^c(D)$, so it is the origin of at most two oriented edges in D. ∎

Following is the proof for **Theorem 3.7.**

Proof. ($\Longrightarrow$) By Lemma 2.10, when $UG(D) \cong dom(D)$, uu' is not an edge in $dom^c(D)$. Therefore, either vv', u_1v_2, or u_2v_2 must be P_2 in $dom^c(D)$. Since u can be either vertex u_1 or u_2, the second two choices reduce to uv_2. Edge vv' is a generated subpath in $dom^c(D)$, so if $vv' = P_2$ in $dom^c(D)$, edges uv_2 and $u'v_2$ must also be in $dom^c(D)$ forming P_3. From Lemma 2.8 we know that only vertices u, u', v, and v' may be the sources of additional edges in $dom^c(D)$. Edge uv_2 can be generated with oriented edges (u', v_2), (v, u) or (v', u). Likewise, edge $u'v_2$ can be generated with oriented edges (u, v_2), (v, u') or $(v'u')$. We may use any combination of the oriented edges to create each of the edges in $dom^c(D)$. So at least one from each group must be in D so that the associated edge is created in $dom^c(D)$. However, Lemma 2.9 restricts the number of edges we may orient. Therefore, only digraphs where u, u', v and v' are the origin of at most one oriented edge of the preceding form are possible when $UG(D) \cong dom(D)$. Additionally, according to Lemma 3.6, edge vv' must be bidirected in D. There are no other sources for the given edges in $dom^c(D)$, so all other edges in $UG(D)$ must be bidirected in D.

If $uv_2 = P_2$ in $dom^c(D)$, then $u'v$ or $u'v'$ must also be edges in $dom^c(D)$. Within isomorphic labeling, we will generate edge $u'v'$. This may be done only by using oriented edge (u, v'). Note that (v, u') will not produce the desired edge in this case, and there are no other vertices that may serve as the origin of an oriented edge since u' and v' cannot be the source of their own edge. To generate the edge uv_2, possible oriented edges are (u', v_2), (v, u) and (v', u). However, since (u, v') must be an oriented edge in every biorientation of $UG(D)$, and $UG(D) \cong dom(D)$, (v', u) can never be used. Thus, we must have (u, v')

and any selection of the other two directed edges.

($\Longleftarrow$) The edge vv' is generated as stated in Lemma 2.1. Vertices u', v and/or v' are sources for the edge uv_2 in $dom^c(D)$ when D has oriented edges (u', v_2), (v, u) and/or (v', u) respectively. No other edges in $dom^c(D)$ are generated by these directed edges. Likewise, vertices u', v and/or v' are sources for the edge $u'v_2$ in $dom^c(D)$ when D has oriented edges (u, v_2), (v, u') and/or (v', u') respectively. As long as each of u, u', v and v' are the origin of at most one oriented edge in D, any combination of these oriented edges with at least one from each group results in $dom^c(D) = vv' \cup u, v_2, u'$, and $UG(D) \cong dom(D)$.

If (u, v) is an oriented edge in D, then $u'v$ is an edge in $dom^c(D)$, and u', v, v' is a path on three vertices. When D has oriented edges (u', v_2) and/or (v', u), edge uv_2 is formed in $dom^c(D)$. Any digraph D with one or both of these oriented edges of $UG(D)$ will have the edge uv_2. Thus, $dom^c(D) = uv_2 \cup u', v, v'$, and $UG(D) \cong dom(D)$. ∎

Following is the proof for **Theorem 3.8.**

Proof. ($\Longrightarrow$) Paths v, v_l' and v', v_l are generated subpaths in $dom^c(D)$. Since uu' cannot be an edge in $dom^c(D)$, either vv_l' or $v'v_l$ form P_2 when $UG(D) \cong dom(D)$. Since the choice of v is arbitrary, say that $v, v_l' = P_2$ in $dom^c(D)$. Then $u, v', v_l, u' = P_4$ in $dom^c(D)$. Thus, edges uv' and $u'v_l$ must be formed. The vertex v' is an end vertex. The only vertex not adjacent to v' in $UG(D)$ cannot be used as the source of an edge in $dom^c(D)$. Therefore, the only directed edge that can be used to form uv' is (u', v'). So (u', v') must be an oriented edge in every biorientation of $UG(D)$, proving part (1).

To form edge $u'v_l$ in $dom^c(D)$, we are not restricted in the same way as that for edge uv'. Here, v_l is an interior vertex. Vertex v is not adjacent to v_l in $UG(D)$, and may be the origin of an oriented edge in D. Therefore, the oriented edges that can be used separately or together to form edge $u'v_l$ in $dom^c(D)$ are (u, v_l) and (v, u'), proving part (2).

Part (3) follows from Corollaries 2.3 and 2.4 as well as Lemma 2.9.

($\Longleftarrow$) Paths vv_l' and $v'v_l$ are generated subpaths in $dom^c(D)$. Vertex u' is the source for edge uv' in $dom^c(D)$ when (u', v') is an oriented edge in D. If (u, v_l) and/or (v, u') are oriented edges in D, edge $u'v_l$ is created in $dom^c(D)$. Vertex v is a source of edge $v'v_l$ in $dom^c(D)$ when (v, v') is an oriented edge in D, and $v'v_l$ also has the source v_l', as neither vertex is adjacent to v_l' in $UG(D)$. Likewise, v' is a source of edge vv_l' when (v', v) is an oriented edge in D, where v_l is always a source for vv_l'. Thus, if either or both of these edges is in D, no new edges appear in $dom^c(D)$. Since all other edges of $UG(D)$ are bidirected in D, there are no other edges formed in $dom^c(D)$. Thus, $dom^c(D) = v, v_l' \cup u, v', v_l, u'$, and $UG(D) \cong dom(D)$. ∎

Following is the proof for **Theorem 4.4.**

Proof. ($\Longrightarrow$) If i or j is even, then there are two generated subpaths, each of length $\frac{i}{2}$ or $\frac{j}{2}$ in $dom^c(D)$. If i or j is odd, then the two generated subpaths in $dom^c(D)$ are of length $\frac{i+1}{2}$ and $\frac{i-1}{2}$, or $\frac{j+1}{2}$ and $\frac{j-1}{2}$ respectively. When $UG(D) \cong dom(D)$ and we want edge u_kv_l to be in $dom^c(D)$, we must be able to connect each path formed with vertices $u_1, ..., u_i$ to a path formed with vertices $v_1, ..., v_j$.

First, we concentrate on each subpath on $V(P_i)$ being connected to a different subpath on $V(P_j)$. Consider i and j both even. Then we must have $\frac{i}{2} + \frac{j}{2} = i$ or j, so $j = i$.

Consider i and j both odd. From Corollary 4.3, we know that we cannot connect the two odd paths in $dom^c(D)$. Therefore, the odd subpath generated on vertices $u_1, ..., u_i$ must augment the shorter even subpath generated on vertices $v_1, ..., v_j$. The same holds true for the other odd subpath. Thus, the only time that $\frac{i+1}{2} + \frac{j-1}{2} = i$ and $\frac{i+1}{2} + \frac{i}{2} = j$ (or vice versa) is when $i = j$.

Consider one of i or j odd. Say that i is odd. Since $u_k v_l$ is in $dom^c(D)$, $u_1, u_3, ..., u_i$ must be connected to a path that is the same length as the path that $u_2, u_4, ..., u_{i-1}$ must be connected to, namely $v_1, v_3, ..., v_{j-1}$ and $v_2, v_4, ..., v_j$. Thus, $j = i + 1$.

Now we concentrate on the possibility that one subpath is not connected to another subpath. Since $i \leq j$, this subpath must form P_i in $dom^c(D)$. Thus, it is one of the generated subpaths formed on $V(P_j)$. The length of subpaths on $V(P_j)$ is $\frac{j}{2}$ if j is even, or $\frac{j-1}{2}$ and $\frac{j+1}{2}$ if j is odd. Thus, $j = 2i$, $2i + 1$, or $2i - 1$ respectively.

($\Longleftarrow$) Constructions that do not depend upon this theorem are given in Theorems 5.5, 5.7, 5.8, 5.10, 5.11, and 5.12, which take the values for i and j given in the statement of Theorem 4.4 and give biorientations of the underlying graph resulting in $UG(D) \cong dom(D)$ with an edge between $V(P_i)$ and $V(P_j)$.

Thus, there are biorientation of the edges of $UG(D)$ such that $UG(D) \cong dom(D)$ where there are edges between the u_i and the v_j. ■

Following is the proof for **Theorem 5.7.**

Proof. Part (2) is valid by Corollary 2.4. The remaining arguments deal with part (1).

($\Longrightarrow$) The four generated subpaths are of the same length in $dom^c(D)$. Since there is an edge between $V(P_i)$ and $V(P_j)$ in $dom^c(D)$, each of these paths on a subset of $V(P_i)$ must be connected to a path from $V(P_j)$ to form the isomorphic graph. We know that uv, uv', $u'v$, and $u'v'$ cannot be edges in $dom^c(D)$. Thus, only two, zero or one of u, u', v and v' can be incident with an edge connecting the two sets of vertices in $dom^c(D)$. We will separate the characterization into these three possibilities.

If two end vertices are used, let v be one of them. From Lemma 5.6, v' must be the other of the two vertices since they must be from the same set of vertices, $V(P_i)$ or $V(P_j)$. Any edge containing v or v' in $dom^c(D)$ will have u or u' as a source. Let u be either vertex, and (u, v) an oriented edge in D. Then (u', v') must be the other oriented edge so that two edges incident to v and v' are created in $dom^c(D)$. The arbitrary selection of u and v includes all possible labelings that produce this. Thus, edges $u_k v$ and $u'_k v'$ are in $dom^c(D)$, creating two paths that are disjoint, with i vertices each. Figure 6(a) shows the generic orientation of the only two edges that can accomplish this.

If there are no end vertices connecting P_i and P_j in $dom^c(D)$, then all of the

vertices u_k, u'_k, v_l, and v'_l must be incident with the two edges between vertex sets. Arbitrarily pick u_k to label one of the four possible interior vertices. Arbitrarily label one of the other interior vertices of the other vertex set as v_l. So, in $UG^c(D)$, we have paths $u, u_k, ..., u'_k, u'$ and $v, v_l, ..., v'_l, v'$. Say that $u_k v_l$ and $u'_k v'_l$ are the two edges in $dom^c(D)$. This selection includes all possible edges between the two sets where no end vertices are used. Two paths with i vertices each are created. To form these edges in $dom^c(D)$, Corollary 5.4 indicates that two oriented edges may be used, and we can use one or both of them in D. Oriented edges (u, v_l) or (v, u_k) or both may be used to create edge $u_k v_l$, while oriented edges (u', v'_l) or (v', u'_k) or both may be used to create edge $u'_k v'_l$. The possibility where all of these arcs are used is shown in Figure 6(b).

If there is one end vertex that is incident with an edge connecting $V(P_i)$ and $V(P_j)$ in $dom^c(D)$, let us call that vertex v. Choose u to be either end vertex of the other vertex set. Oriented edges (u, v) and (u', v) are the only ones that will produce an edge in $dom^c(D)$ that is incident with v. The arbitrary nature of the labeling of u, allows us to reduce this to (u, v), producing edge $u_k v$ in $dom^c(D)$ and path $u', ..., u_k, v, ...v'_l$. So, (u, v) must be an oriented edge in D. There are only two remaining interior vertices that are not on the path created by edge $u_k v$ in $dom^c(D)$. They are the vertices u'_k and v_l. They must form the second edge, creating the second path of i vertices. This can be done if D contains oriented edges (u', v_l) or (v, u'_k) or both. Figure 6(c) gives the example where (u, v) and both of the other two oriented edges are in D.

($\Longleftarrow$) By Lemma 5.3, parts 1(a), (b) and (c), when used separately, create edges in $dom^c(D)$ between $V(P_i)$ and $V(P_j)$ that result in $dom^c(D)$ consisting of two disjoint paths with $i = j$ even vertices each. Thus, $UG^c(D) \cong dom^c(D)$ and $UG(D) \cong dom(D)$. ■

Following is the proof for **Theorem 5.10.**

Proof. The arbitrary selection of u and v generates all nonisomorphic digraphs.

($\Longrightarrow$) If $UG(D) \cong dom(D)$, then P_i in $dom^c(D)$ must be the generated subpath $V_1 = v, ..., v'$ on $V(P_j)$. Thus, subpaths $V_2 = v_l, ..., v'_l$, U_1 and U_2 must form P_j in $dom^c(D)$. Either U_1 and U_2 are connected by an edge or they are not.

1. Say that U_1 and U_2 are connected by an edge. This implies that oriented edges must be used as stated in Lemma 2.2. When i is odd, path $U_o = u, ..., u', u_k, ..., u'_k$ is formed. When i is even, path $U_e = u, ..., u'_k, u_k, ..., u'$ is formed. In both U_o and U_e, the arbitrary vertex u is an end vertex, so the edge uv_l can be created in both cases. Oriented edge (v, u) in D is the only one that creates edge uv_l in $dom^c(D)$. For U_e, there is no other nonisomorphic edge that connects it to V_2. However, in U_o, $u'_k v_l$ is an option. It is nonisomorphic since u'_k is an interior vertex, and u is not. Oriented edges (u', v_l) or (v, u'_k) create edge $u'_k v_l$ in $dom^c(D)$, and can be used simultaneously. This gives us part (1) of the theorem.

2. If U_1 and U_2 are not connected, this implies that V_2 must have one end vertex adjacent to U_1 and the other to U_2. Whether i is odd or even,

the edge uv_l can be used to connect U_1 to V_2, so (v, u) can be an oriented edge in D. When it is, U_2 must be connected to v_l'. When i is either odd or even, the edge $u_k v_l'$ does this, so (u, v_l') or (v', u_k) or both may be oriented edges in D. Additionally, if i is odd, using $u_k' v_l'$ instead of $u_k v_l'$ is a nonisomorphic construction connecting U_2 and V_2, where (u', v_l') or (v', u_k') or both are oriented edges in D. Likewise, if i is even, using $u' v_l'$ instead of $u_k v_l'$ connects U_2 and V_2, where (v', u') is an oriented edge of D. This gives us part (2)(a) of the theorem.

All other choices when i is odd are isomorphic within the labeling. However, uv_l does not have to be an edge if i is even. Both interior vertices, u_k and u_k', may be used to connect U_1 and U_2 to V_2 respectively. Thus, edges $u_k v_l$ and $u_k' v_l'$ in $dom^c(D)$ will create path P_j. To do this, D must have oriented edges (u, v_l) or (v, u_k) or both, and oriented edges (u', v_l') or (v', u_k') or both. This gives us part (2)(b) of the theorem.

3. From Corollary 2.4, we are guaranteed that the arcs in part 3 will not alter the relationship $UG(D) \cong dom(D)$ as long as vertices u, u', v, and v' are each the origin of at most one oriented edge in D.

($\Longleftarrow$) In all constructions for parts (1) and (2), $P_i = V_1$ in $dom^c(D)$. P_j is formed as follows, creating $UG(D) \cong dom(D)$ with an edge between $V(P_i)$ and $V(P_j)$. In part (1)(a), $P_j = v_l', \ldots, v_l, U_1 U_2$. In part (1)(b), $P_j = U_1 U_2 V_2$. In part (2)(a)(i), $P_j = u', \ldots, u, V_2 U_2$. In part (2)(a)(ii), $P_j = u', \ldots, u, V_2, u_k', \ldots, u_k$. In part (2)(a)(iii), $P_j = U_2, v_l', \ldots, v_l, U_1$. In part (2)(b), $P_j = U_1, v_l', \ldots, v_l, U_2$. In all cases, the oriented edges in Corollary 2.3 may be used and only generate edges already in the generated subpaths. Since u, u', v, and v' are each the origin of at most one oriented edge, $UG(D) \cong dom^c(D)$. ■

Following is the proof for **Theorem 5.11**.

Proof. The arbitrary selection of u and v generates all nonisomorphic digraphs.

($\Longrightarrow$) $UG(D) \cong dom(D)$ so P_i in $dom^c(D)$ must be the generated subpath $V_1 = v, \ldots, v_l'$ or $V_2 = v_l, \ldots, v'$. Without loss of generality, say $P_i = V_2$. Thus, subpaths V_1, U_1 and U_2 must form P_j in $dom^c(D)$. Either U_1 and U_2 are connected by an edge or they are not.

1. Say that U_1 and U_2 are connected. This implies that oriented edges must be used as stated in Lemma 2.2. When i is odd, path $U_o = u, \ldots, u', u_k, \ldots, u_k'$ is formed. When i is even, path $U_e = u, \ldots, u_k', u_k, \ldots, u'$ is formed. To connect V_1 to U_o or U_e, edge uv_l' can be formed when arbitrarily choosing u and v. In both instances, oriented edge (v', u) in D will create uv_l' in $dom^c(D)$. There is no other way to connect V_1 to U_e since the only other vertex choices are end vertices that cannot form edges in $dom^c(D)$. However, V_1 can be connected to U_o using edge $u_k' v$. Oriented edge (u', v) in D is the only way to create that edge, and gives us part (1) of the theorem.

2. If U_1 and U_2 are not connected, this implies that V_1 must have one end vertex adjacent to U_1 and the other to U_2. When i is odd, this can be done nonisomorphically in two ways. In each, we create edge uv_l' by using oriented edge (v', u) in D. In addition, either u_k or u_k' can be connected to v. These are nonisomorphic choices since u_k is adjacent to our chosen u in $UG^c(D)$, and u_k' is not. We obtain edge $u_k v$ or edge $u_k' v$ only by creating oriented edges (u, v) or $(u'v)$ respectively in D. This gives us part (2)(a) of the theorem. When i is even, at least one of u_k or u_k' must be connected to v since neither u nor u' can be. Without loss of generality, say that $u_k v$ is the edge, which implies the (u, v) is an oriented edge in D. The remaining subpath may connect to v_l' using either vertex u or u_k'. Edge uv_l' is formed in one way, and that is by creating oriented edge (v', u) in D. Edge $u_k' v_l'$ can be formed in two ways, using oriented edge (u', v_l') or (v', u_k') or both. This gives us part (2)(b) of the theorem.

3. From Corollary 2.4, we are guaranteed that the arcs in part 3 will not alter the relationship $UG(D) \cong dom(D)$ as long as vertices u, u', v, and v' are each the origin of at most one oriented edge in D.

($\Longleftarrow$) In all constructions for parts (1) and (2), $P_i = V_2$ is an arbitrary choice. P_j is formed as follows, creating $UG(D) \cong dom(D)$ with an edge between $V(P_i)$ and $V(P_j)$. In part (1)(a), $P_j = V_1 U_1 U_2$. In part (1)(b), $P_j = U_1 U_2 V_1$. In part (2)(a)(i), $P_j = u_k', \ldots, u_k, V_1 U_2$. In part (2)(a)(ii), $P_j = U_2 V_1 U_1$. In part (2)(b)(i), $P_j = u', \ldots, u_k, V_1 U_1$. In part (2)(b)(ii), $P_j = U_1, v_l', \ldots, v, U_2$. In all cases, the oriented edges in Corollary 2.3 may be used and only generate edges already in the generated subpaths. Since u, u', v, and v' are each the origin of at most one oriented edge, $UG(D) \cong dom^c(D)$. ■

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