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Characterizing the Arc

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1. The Framework

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1. The Framework. Arcs, as well as other well known topological spaces, have been characterized in many ways; e.g.

Theorem 1.1: [R. L. Moore, 1920] *The arc is unique among metrizable continua with exactly two noncut points.*

In this famous characterization, the arc is compared to other members of a *class of peers* (the metrizable continua) and is characterized in *terms* of cut points. In this talk we present a first order logic framework in which we may specify the "terms of characterization." One advantage of this extra precision is that we may also make sense of when an object is *not* characterizable in certain terms, relative to a given peer class.

The main ingredients in our framework are the following:

(1) A **class** $\mathcal{P}$ of topological spaces, the *peers* to which a given member of $\mathcal{P}$ is to be compared for the purposes of characterization.

(2) An **alphabet** L of finitary relation and function symbols, with an assignment

$$Y \mapsto Y_{\mathsf{L}}$$

from spaces in $\mathcal{P}$ to L structures, such that homeomorphic spaces are assigned isomorphic structures.

(3) A **characterization language** Ψ of first order sentences over L.

For any $X \in \mathcal{P}$, a set Φ_X of first order sentences over L is a Ψ *characterization* of X, relative to $\mathcal{P}$, if: (i) $\Phi_X \subseteq \Psi$; (ii) $X_{\mathsf{L}} \models \Phi_X$; and (iii) whenever $Y \in \mathcal{P}$ is such that $Y_{\mathsf{L}} \models \Phi_X$, it follows that Y is homeomorphic to X.

We then say that $X \in \mathcal{P}$ is Ψ *characterizable* if such a Φ_X exists.

In this talk we are concerned mainly with characterizing the arc relative to various peer classes, using various characterization languages. Not surprisingly, restriction of the peer class of a space makes characterization of the space easier; restriction of the characterization language, on the other hand, makes life harder.

2. Results Involving Lattices. In this section L is the alphabet of bounded lattices, and X_{L} is the closed set lattice of a space X. The first result known to us, couchable in our framework, is the following, where the characterization language consists of all first order sentences.

Theorem 2.1 [C. W. Henson, et al, 1979]: *The arc is first order lattice characterizable, relative to the class of metrizable spaces.*

In the remainder of this section, we restrict both the peer class and the characterization language, and give both positive and negative results on characterizing the arc.

A *lattice base* for a space X is a closed set base that is also a sublattice of X_{L}. An L sentence φ is *base free* if whenever X is a compactum and $\mathcal{A}$ is a lattice base for X, then $\mathcal{A} \models \varphi$ iff $X_{\mathsf{L}} \models \varphi$.

Theorem 2.2 [R. Gurevič, 1988]: *The arc is not base free lattice characterizable, relative to the class of metrizable continua.*

Define two compacta to be *lattice base related* if there is a lattice base of one and a lattice base of the other, both satisfying the same first order sentences. What Gurevič did in Theorem 2.2 was to show that the arc, as well as every nondegenerate metrizable continuum, is lattice base related to a metrizable continuum that is not locally connected. This led to the following positive result.

Theorem 2.3 [P. B., 1988]: *The arc is base free lattice characterizable, relative to the class of locally connected metrizable compacta.*

The positive result in Theorem 2.3 raised the question of whether other Peano continua could be so characterized, and several small extensions of the technique culminated in

Theorem 2.4 [P. B., 2011]: *Any topological graph is base free lattice characterizable, relative to the class of locally connected metrizable compacta.*

The Gurevič result 2.2 also suggested that certain nonlocally connected continua might still be base free characterizable; a natural candidate was the pseudoarc, famously characterized by R. H. Bing as being unique among the nondegenerate hereditarily indecomposable chainable metrizable continua.

J. Krasinkiewicz and P. Minc (1977) gave a characterization of hereditary indecomposability which, years later, K. P. Hart noticed to be expressible as a base free condition. From this, the following is a direct consequence.

Theorem 2.4: *The pseudoarc is base free lattice characterizable, relative to the class of chainable metrizable continua.*

Meanwhile, in [T. Banach, P. B., B. Raines, W. Ruitenburg, 2006] it was possible to obtain an analogue of the Gurevič result to show that every nondegenerate metrizable continuum is lattice base related to a metrizable continuum that is not chainable. As a direct consequence of this, we have:

Theorem 2.5 *The pseudoarc is not base free characterizable, relative to the class of metrizable continua.*

The final *coup de grâce* to the search for base free characterizable continua was delivered in a recent (2010) paper by K. P. Hart, who proved that every nondegenerate metrizable continuum is lattice base related to a topologically distinct metrizable continuum. His proof made essential use of the famous result (1934) of Z. Waraszkiewicz, to the effect that no metrizable continuum has every metrizable continuum as a continuous image. So immediately, we have

Theorem 2.6 [K. P. Hart, 2010]: *No nondegenerate metrizable continuum is base free lattice characterizable, relative to the class of metrizable continua.*

The results so far, together with the fact that the Cantor space is a metric compactum whose classic characterization as a zero-dimensional metrizable compactum with no isolated points easily translates into base free terms, suggests the following

Conjecture: *An infinite metrizable compactum is base free chacterizable relative to the class of metrizable compacta if and only of that compactum is the free union of a Cantor set with a finite set.*

3. Results Involving Betweenness. In this section L is the alphabet of betweenness, and X_{L} is the betweenness structure whose elements are the points of X; and we say that the triple $\langle a, b, c \rangle \in X^3$ satisfies the condition that a *lies between* b *and* c just in case every connected closed subset of X containing both b and c contains a as well. We now consider characterizations in terms of first order sentences involving equality and the ternary betweenness predicate as the only nonlogical symbols.

Characterizations involving betweenness, and the study of betweenness in a topological setting in general, are in fairly preliminary stages, but we can announce the following analogue of Theorem 2.1 above.

Theorem 3.1: *The arc is first order betweenness characterizable, relative to the class of metrizable spaces.*