

8-1-2010

# Noise Assumptions in Complex-Valued SENSE MR Image Reconstruction

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*Noise Assumptions in Complex Valued SENSE MR Image Reconstruction*. Presented at the [2010 Joint Statistical Meeting](#), sponsored by the American Statistical Association. Vancouver, Canada, August 4, 2010.

# Noise Assumptions in Complex-Valued SENSE MR Image Reconstruction

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August 4, 2010

## **OUTLINE**

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**3. Methods**

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**5. Discussion**

## Motivation

In MRI k-space for images is not measured instantaneously.

In parallel imaging, sub-sampled k-space points are measured in parallel and combined to form a single image.

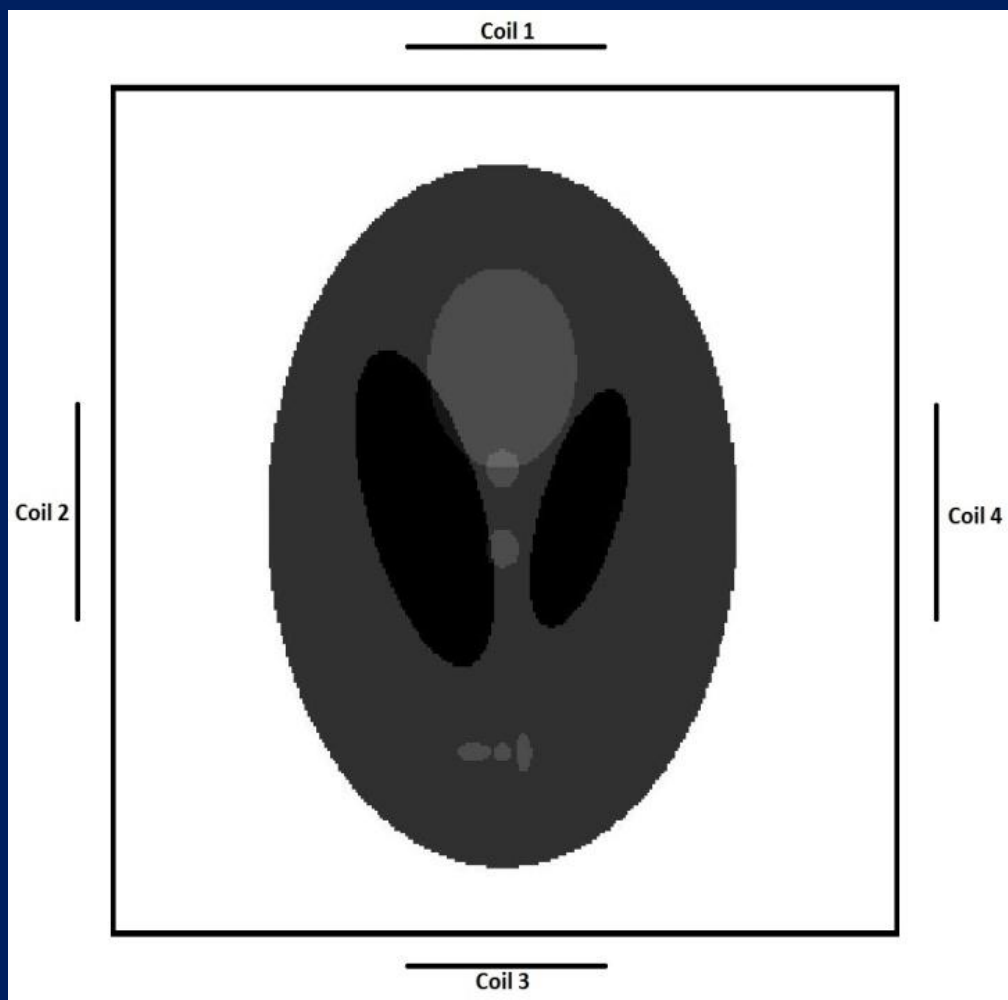
Image and volume measurement time is decreased at the expense of increased image reconstruction difficulty and time.

The SENSE parallel imaging reconstruction technique utilizes a complex-valued least squares estimation process.

However, in SENSE the covariance is not properly modeled.

## Background

In parallel imaging there is more than one receive coil.

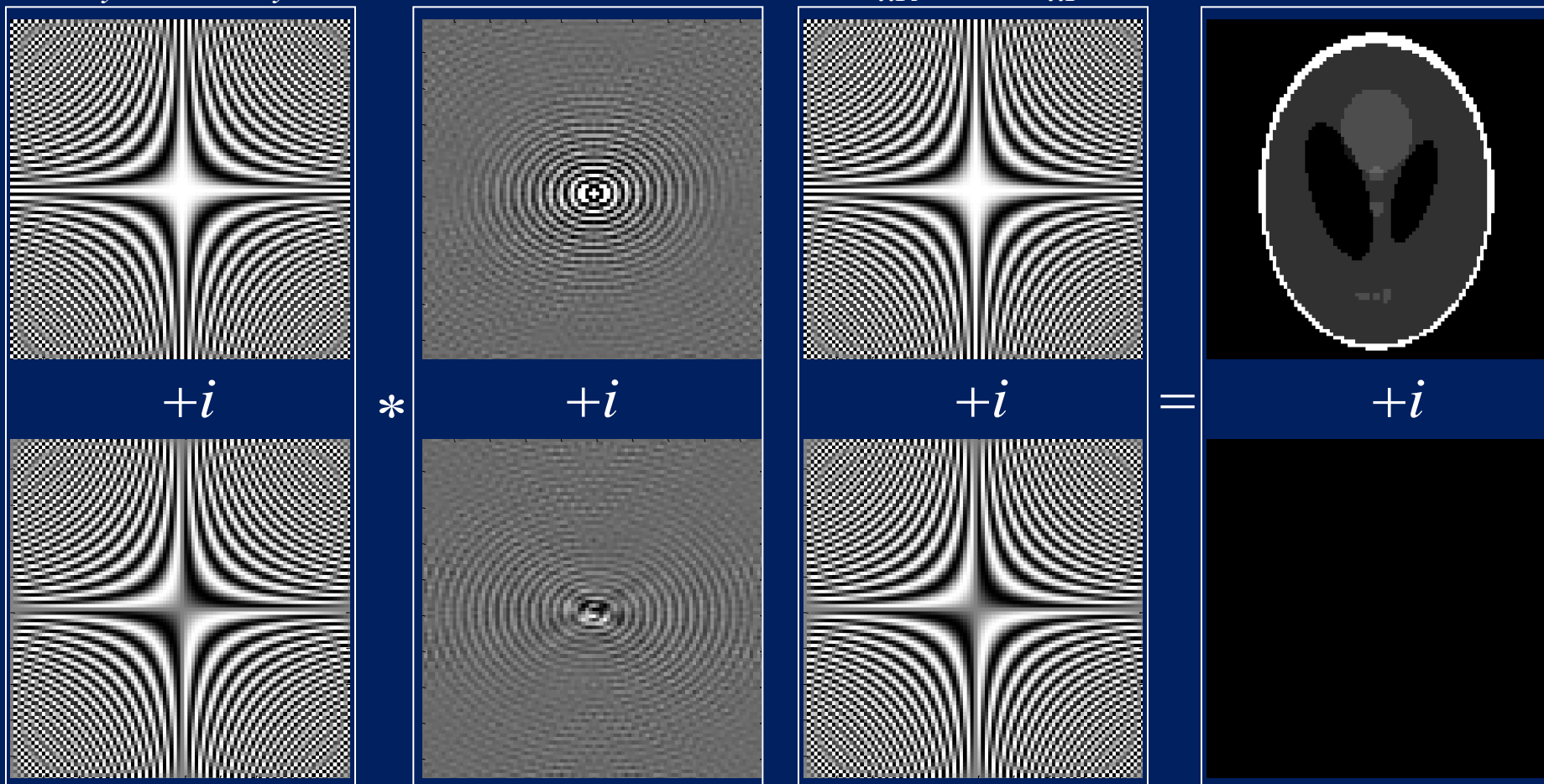


Each coil measures a  $k$ -space array that is reconstructed into an aliased image then combined to form a single unaliased image.

## Background

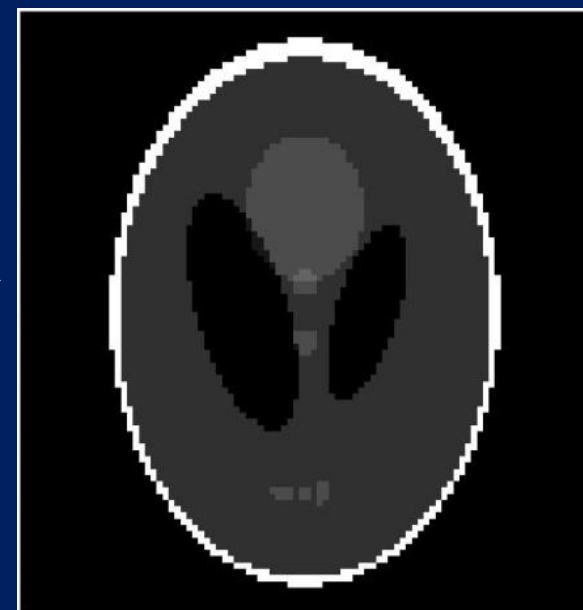
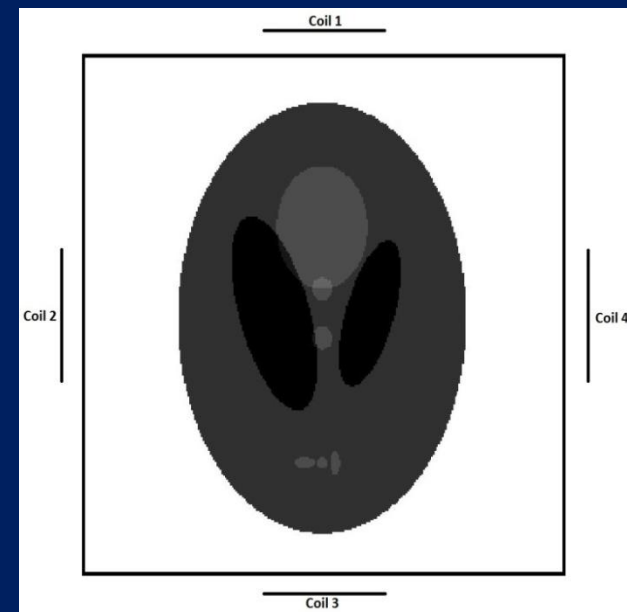
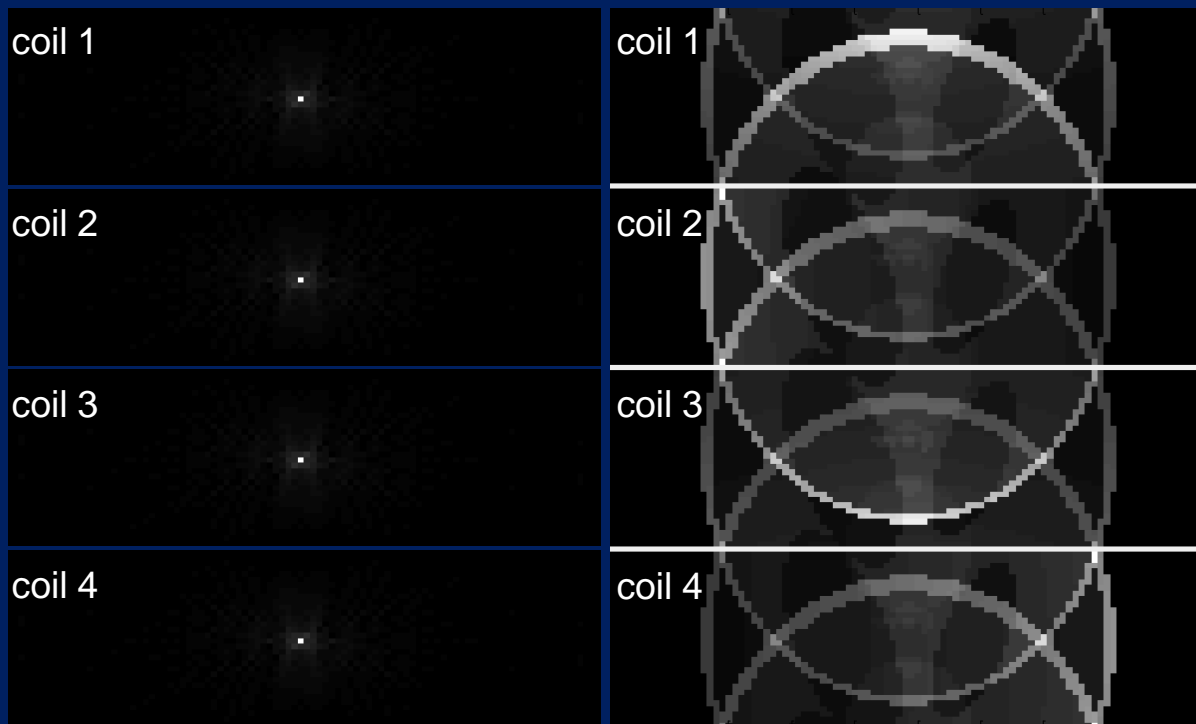
Image inverse Fourier Reconstruction for single coil.

$$(\Omega_{yR} + i\Omega_{yI}) * (F_R + iF_I) * (\Omega_{xR} + i\Omega_{xI})^T = (V_R + iV_I)$$



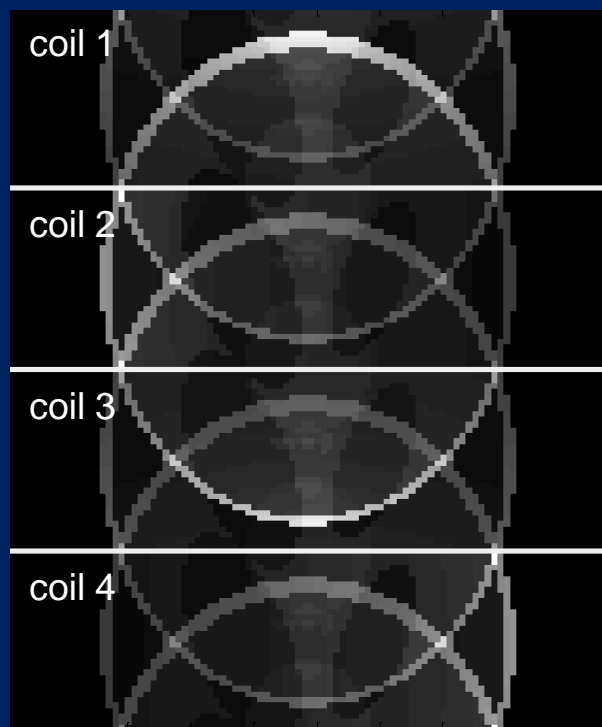
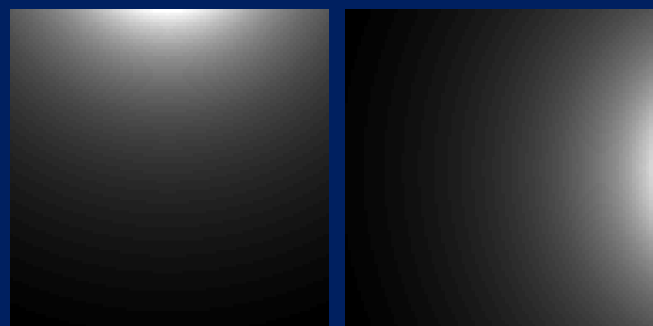
## Background

Each coil measures a  $k$ -space array that is reconstructed into an aliased image then combined to form a single image.



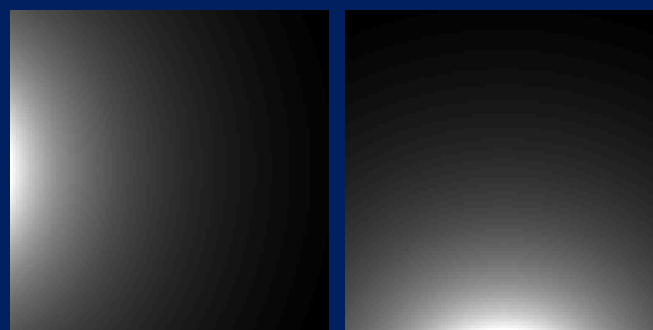
## Background

Each coil measures a  $k$ -space array that is reconstructed into an aliased image then combined to form a single image.


 $a_C$ 


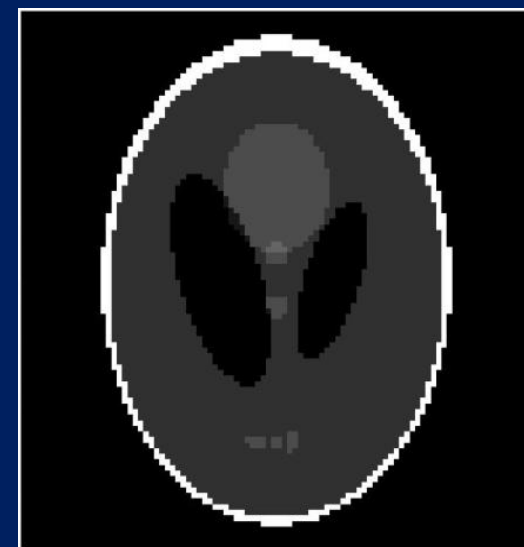
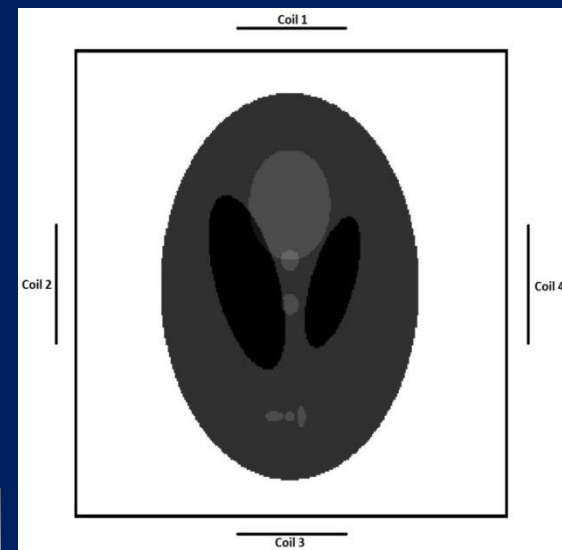
coil 1

coil 4



coil 2

coil 3

 $S_C$ 

 $v_C$



## Methods

The SENSE model for aliased voxel values from  $n$  coils is

$$\underset{n \times 1}{a_C} = \underset{n \times A}{S_C} \underset{A \times 1}{v_C} + \underset{n \times 1}{\varepsilon_C}, \quad \varepsilon_C \sim CN(0, \Psi_C)$$

where for each voxel

$$\Psi_C = \Psi_R + i\Psi_I$$

$a_C$  is a vector of the  $n$  complex-valued aliased voxel values

$$a_C = a_R + ia_I$$

$v_C$  is a vector of the  $A$  unaliased voxel values

$$v_C = v_R + iv_I$$

$S_C$  is an  $n \times A$  matrix of complex-valued coil sensitivities

$$S_C = S_R + iS_I$$

$\varepsilon_C$  is a vector of the  $n$  complex-valued error values

$$\varepsilon_C = \varepsilon_R + i\varepsilon_I$$

## Methods

The SENSE process

$$a_C = S_C v_C + \varepsilon_C, \quad \varepsilon_C \sim \text{CN}(0, \Psi_C)$$

$$\Psi_C = \Psi_R + i\Psi_I$$

$n \times 1$        $n \times A$        $A \times 1$        $n \times 1$

uses the complex normal distribution

$$f(\varepsilon_C) = (2\pi)^{-n} |\Psi_C|^{-1} e^{-1/2 \varepsilon_C^H \Psi_C^{-1} \varepsilon_C}, \quad \begin{array}{l} H \text{ is the conjugate} \\ \text{transpose (Hermetian)} \end{array}$$

and for  $N_C$  coil measurements

$$f(a_C) = (2\pi)^{-n} |\Psi_C|^{-1} e^{-1/2 (a_C - S_C v_C)^H \Psi_C^{-1} (a_C - S_C v_C)}$$

Pruessmann et al.: SENSE: Sensitivity Encoding for Fast MRI. MRM 42:952–962, 1999.

Wooding The multivariate distribution of complex normal variables. Biometrika 43:212–215, 1956.

Bruce and Rowe: In progress.

## Methods

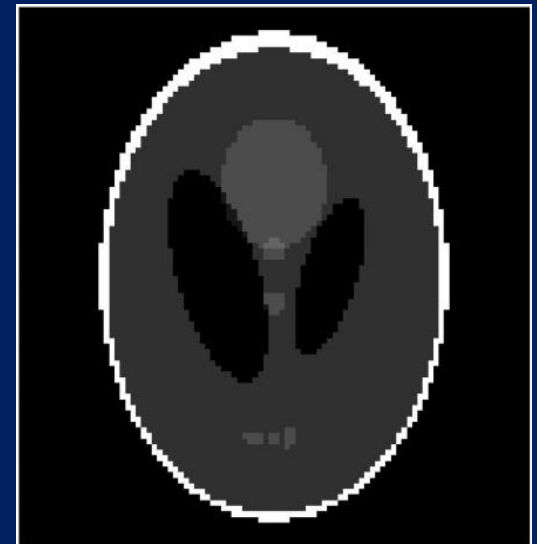
From the distribution for the  $n$  coil measurements

$$f(a_C) = (2\pi)^{-n} |\Psi_C|^{-1} e^{-1/2(a_C - S_C v_C)^H \Psi_C^{-1} (a_C - S_C v_C)}$$

the voxel values can be estimated as

$$v_C = (S_C^H \Psi_C^{-1} S_C)^{-1} S_C^H \Psi_C^{-1} a_C$$

with knowledge of  $S_C$  and  $\Psi_C$ .



Pruessmann et al.: SENSE: Sensitivity Encoding for Fast MRI. MRM 42:952–962, 1999.

Wooding The multivariate distribution of complex normal variables. Biometrika 43:212–215, 1956.

Bruce and Rowe: In progress.

## Methods

Instead of writing the model with complex numbers as

$$\mathbf{a}_C = \mathbf{S}_C \mathbf{v}_C + \boldsymbol{\varepsilon}_C, \quad \begin{matrix} n \times 1 \\ n \times A \\ A \times 1 \\ n \times 1 \end{matrix}$$

$$\mathbf{a}_C = \mathbf{a}_R + i\mathbf{a}_I \quad \mathbf{S}_C = \mathbf{S}_R + i\mathbf{S}_I \quad \mathbf{v}_C = \mathbf{v}_R + i\mathbf{v}_I \quad \boldsymbol{\varepsilon}_C = \boldsymbol{\varepsilon}_R + i\boldsymbol{\varepsilon}_I$$

we can write the model using an isomorphism as

$$\mathbf{a} = \mathbf{S} \mathbf{v} + \boldsymbol{\varepsilon} \quad \begin{matrix} 2n \times 1 \\ 2n \times 2A \\ 2A \times 1 \\ 2n \times 1 \end{matrix}$$

$$\mathbf{a} = \begin{pmatrix} \mathbf{a}_R \\ \mathbf{a}_I \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} \mathbf{S}_R & -\mathbf{S}_I \\ \mathbf{S}_I & \mathbf{S}_R \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} \mathbf{v}_R \\ \mathbf{v}_I \end{pmatrix} \quad \boldsymbol{\varepsilon} = \begin{pmatrix} \boldsymbol{\varepsilon}_R \\ \boldsymbol{\varepsilon}_I \end{pmatrix}$$

Pruessmann et al.: SENSE: Sensitivity Encoding for Fast MRI. MRM 42:952–962, 1999.

Wooding: The multivariate distribution of complex normal variables. Biometrika 43:212–215, 1956.

Bruce and Rowe: In progress.

## Methods

Then the distribution for  $n$  coil measurements is

$$f(\mathbf{a}) = (2\pi)^{-n} |\Psi_{SE}|^{-1/2} e^{-1/2(\mathbf{a}-S\mathbf{v})' \Psi_{SE}^{-1} (\mathbf{a}-S\mathbf{v})}$$

with

$$\mathbf{a} = \begin{pmatrix} a_R \\ a_I \end{pmatrix} \quad S = \begin{pmatrix} S_R & -S_I \\ S_I & S_R \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} v_R \\ v_I \end{pmatrix} \quad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_R \\ \varepsilon_I \end{pmatrix}$$

and the complex normal distribution imposes skew-symmetric

$$\Psi_{SE} = \begin{pmatrix} \Psi_R & -\Psi_I \\ \Psi_I & \Psi_R \end{pmatrix}$$

## Methods

The skew-symmetric covariance structure

$$\Psi_{SE} = \begin{pmatrix} \Psi_R & -\Psi_I \\ \Psi_I & \Psi_R \end{pmatrix} \text{ is incorrect.}$$

What this says is that  $\text{cov}(I, I) = \text{cov}(R, R)$

and that  $\text{cov}(I, R) = -\text{cov}(R, I)$ .

The proper covariance structure should be

$$\Psi_{SI} = \begin{pmatrix} \Psi_R & \Psi_{RI} \\ \Psi'_{RI} & \Psi_I \end{pmatrix}$$

(SE for SENSE and SI for new covariance model SENSE-ITIVE)

## Methods

Examine the difference between the two covariance structures

$$\Psi_{SE} = \begin{pmatrix} \Psi_R & -\Psi_I \\ \Psi_I & \Psi_R \end{pmatrix} \quad \Psi_{SI} = \begin{pmatrix} \Psi_R & \Psi_{RI} \\ \Psi'_{RI} & \Psi_I \end{pmatrix}$$

in the distribution

$$f(\mathbf{a}) = (2\pi)^{-n} |\Psi_{SE/SI}|^{-1/2} e^{-1/2(\mathbf{a}-S\mathbf{v})' \Psi_{SE/SI}^{-1} (\mathbf{a}-S\mathbf{v})}$$

through estimates

$$\mathbf{v}_{SE} = (S' \Psi_{SE}^{-1} S)^{-1} S' \Psi_{SE}^{-1} \mathbf{a}$$

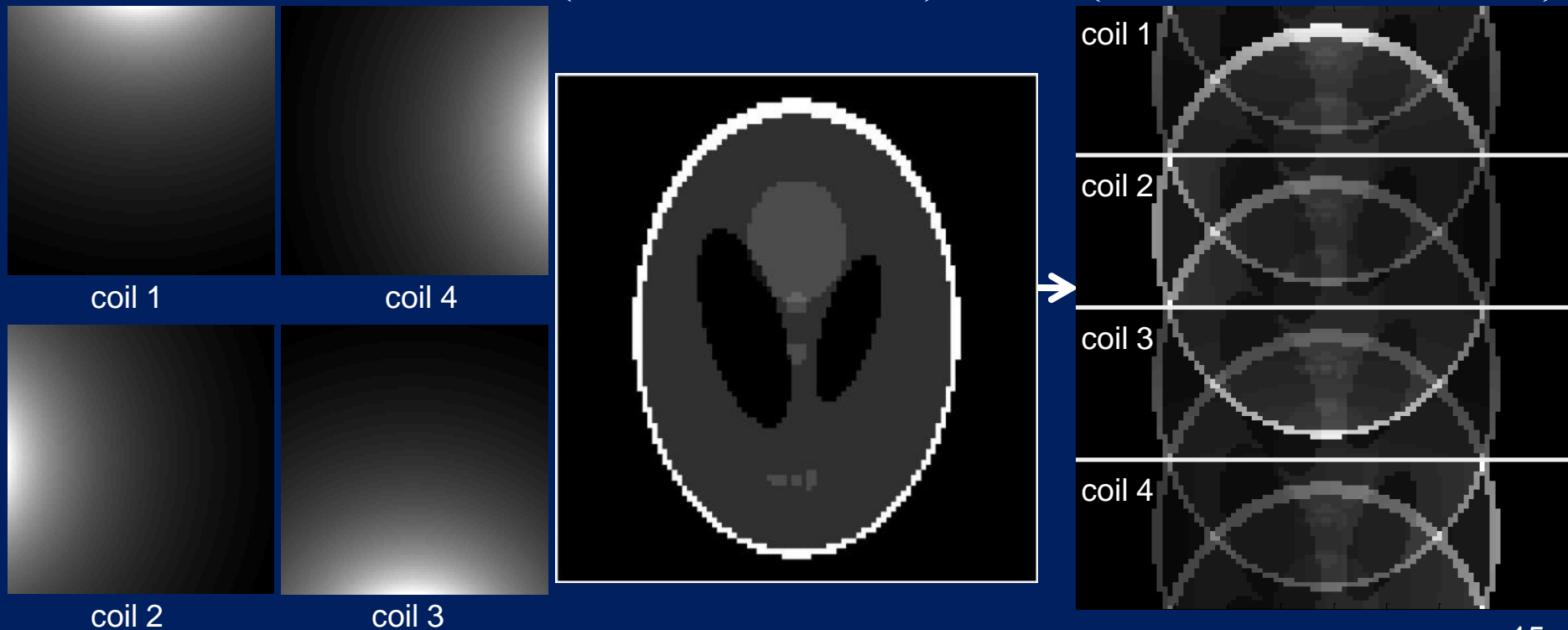
$$\mathbf{v}_{SI} = (S' \Psi_{SI}^{-1} S)^{-1} S' \Psi_{SI}^{-1} \mathbf{a}$$

## Results

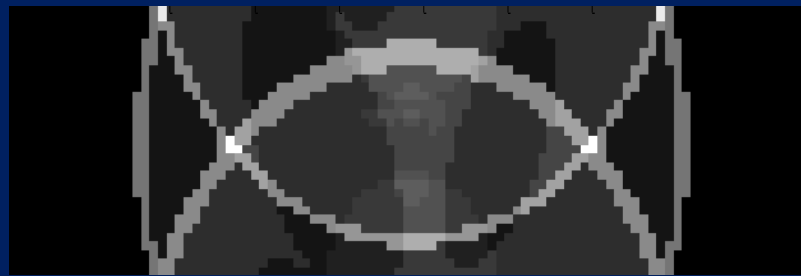
Noiseless multi-coil spatial frequency arrays are with

$$\Psi_{SI} = \begin{pmatrix} \Psi_R & \Psi_{RI} \\ \Psi'_{RI} & \Psi_I \end{pmatrix} \quad \Psi_R = \begin{pmatrix} 1 & .33 & .11 & .33 \\ .33 & 1 & .33 & .11 \\ .11 & .33 & 1 & .33 \\ .33 & .11 & .33 & 1 \end{pmatrix} \quad \Psi_{RI} = \begin{pmatrix} 0 & -.11 & -.07 & -.11 \\ .33 & 0 & -.11 & -.07 \\ .42 & .26 & 0 & -.11 \\ .26 & .42 & .26 & 0 \end{pmatrix}$$

$$\Psi_I = \Psi_R$$







**Results**  
**Magnitude**



**SENSE**

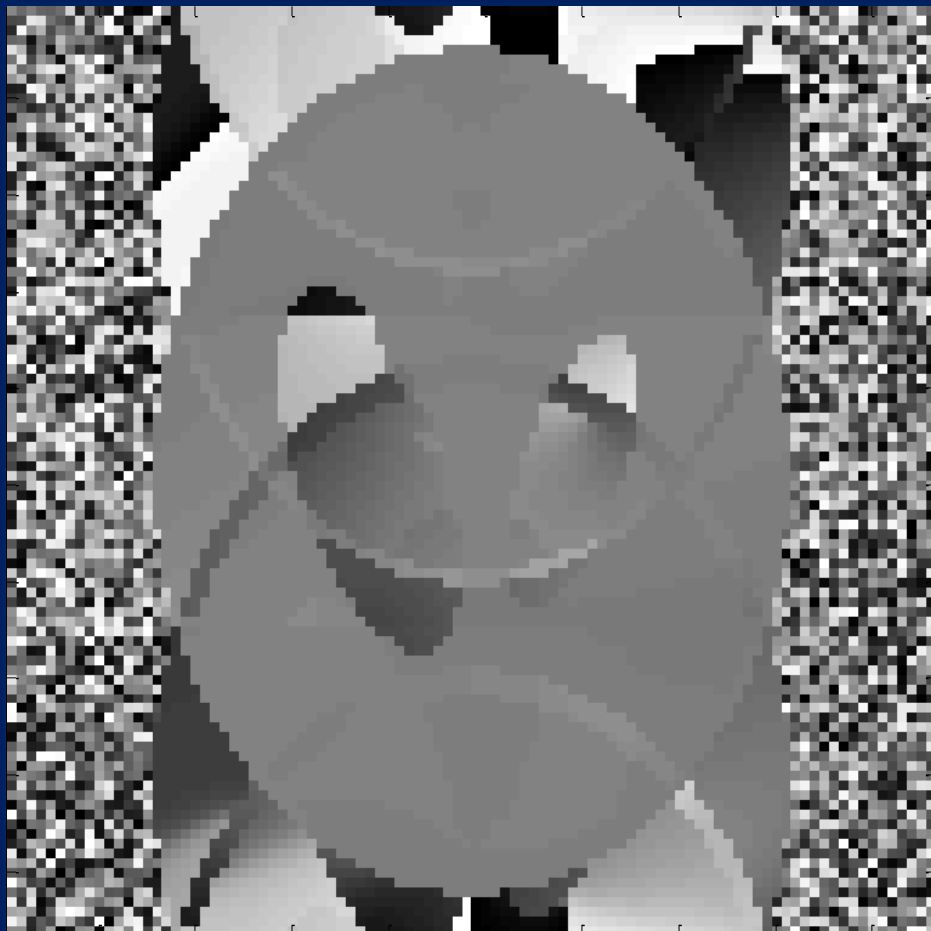


**SENSE-ITIVE**

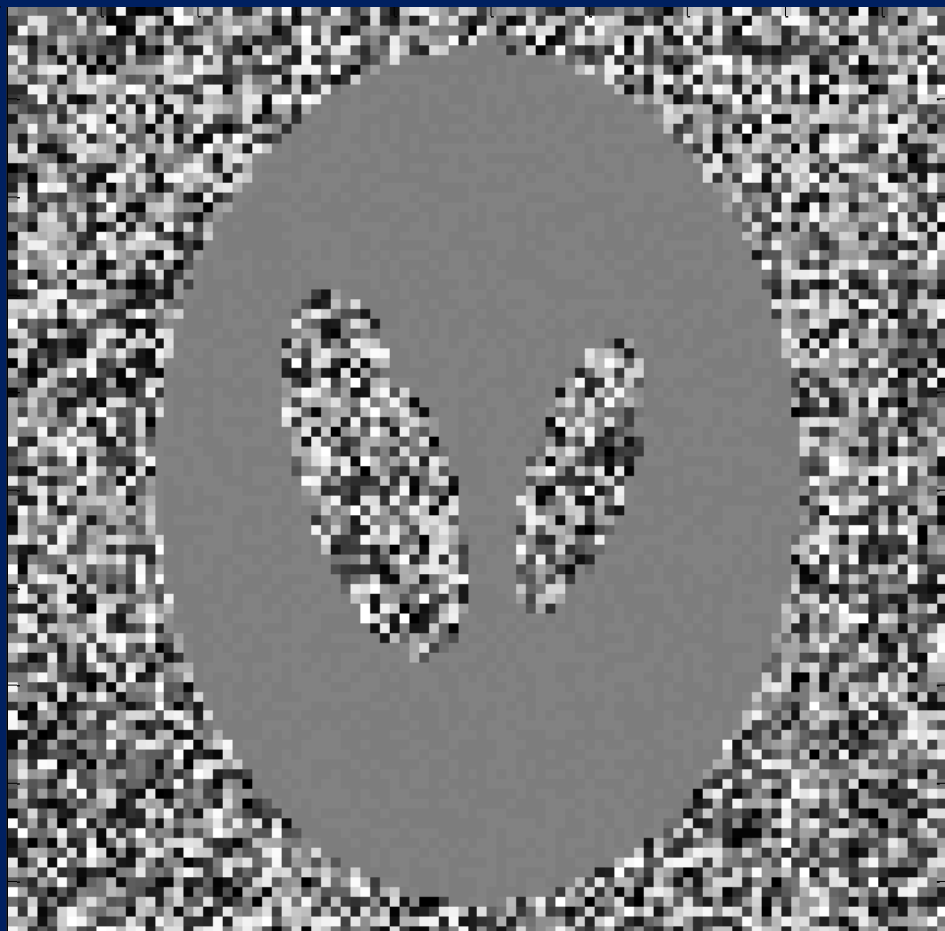


# Results

Phase



SENSE



SENSE-ITIVE

## Discussion

The SENSE image reconstruction method was described.

The SENSE reconstruction written with an isomorphism.

The covariance structure of complex SENSE described.

New SENSE-ITIVE method described with proper covariance.

Results of SENSE & SENSE-ITIVE reconstruction presented.

Ghosting present in SENSE magnitude and phase images.

Better reconstruction in SENSE-ITIVE reconstruction especially phase used for complex-valued time series activation.

# Thank You

## Acknowledgements:

Iain Bruce, Marquette University

Muge Karaman, Marquette University