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A Proposal for a Problem-Driven Mathematics Curriculum Framework

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A Proposal for a Problem-Driven Mathematics Curriculum Framework

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Abstract: A framework for a problem-driven mathematics curriculum is proposed, grounded in the assumption that students learn mathematics while engaged in complex problem-solving activity. The framework is envisioned as a dynamic technologically-driven multi-dimensional representation that can highlight the nature of the curriculum (e.g., revealing the relationship among modeling, conceptual, and procedural knowledge), can be used for programmatic, classroom and individual assessment, and can be easily revised to reflect ongoing changes in disciplinary knowledge development and important applications of mathematics. The discussion prompts ideas and questions for future development of the envisioned software needed to enact such a framework.

Keywords: Problem-based Mathematics, Curriculum frameworks, Mathematical Modeling, Model-Eliciting Activities.

Introduction

Curriculum frameworks are commonly organized around categories of mathematical topics (e.g., number, geometry), such as in the new Common Core School Mathematics Standards (NGA & CCSSO, 2011) and the National Council of Teachers of Mathematics (NCTM) standards documents (1989, 2000) for the United States (U.S.). Oftentimes, to convey the nature of mathematics teaching and learning, the content topics are cross-referenced with other types of mathematical behaviors, such as the "process standards" (e.g., problem solving, reasoning and proof) of the NCTM documents, and the "practices" (e.g., model with mathematics, attend to precision) of the CCSSM document.

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Another approach is to formulate mathematics curriculum frameworks based on assumptions about learning mathematics, such as the Dutch curriculum framework described by van den Heuvel-Panhuizen (2003) (e.g., informal to formal, situated to generalized, individual to social). The developers of mathematics curriculum frameworks choose their organization and structure in order to communicate a mathematics curriculum to broad audiences (e.g., teachers, administrators, parents, students). The choices for content and the representation of curricula made by the framework developers, in turn, convey a distinctive perspective on mathematics curriculum, accompanied by inevitable (some intended, some unintended) consequences when users of the framework transform the represented curriculum into prescriptions for classroom experiences and assessment. A proposal for framing and representing a problem-driven mathematics curriculum is described in this article. The proposal envisions a framework that grows out of Lesh and colleague's work on models-and-modeling, which has focused on using modeling problems as sites for revealing and assessing students' thinking (e.g., Lesh, Cramer, Doerr, Post & Zawojewski, 2003), and more recently by Richard Lesh to teach data modeling (personal conversation, Dec. 21, 2012). The proposal also envisions a representational system that builds on a one originally posed by Lesh, Lamon, Gong and Post (1992), and is particularly poignant today because technology is now available that could carry out the proposal.

Why an Alternative Framework?

Assumptions about Curriculum Frameworks

Curriculum frameworks convey a view of mathematics learning to stake holders in education, influencing the full range of mathematics education activity—from implementation to assessment. For example, the two foundational NCTM curriculum documents (1989, 2000) contributed to a huge shift in views of mathematics curriculum in the U. S. Prior to the publication of these documents, schools, districts and state curriculum guides predominantly listed expected mathematical competencies by grade level, commonly referred to as *scope and sequence* documents. The NCTM standards documents introduced a process dimension (problem solving, reasoning, connections, communication) in addition to the common practice of describing mathematics competencies and performance expectations. Further, discussions about the mathematical processes and expected mathematical performances were embedded in the context of illustrative problems, teaching and learning scenarios, and ways of thinking about mathematics. These standards documents impacted not only state curriculum standards, but also resulted in the development of the now-famous NSF curricula (described in Hirsch, 2007a). Research on the standard-based curricula suggests that students using these curricula demonstrate enhanced learning of mathematical reasoning and problem solving (Hirsch, 2007b).

The new Common Core State Standards in Mathematics (NGA & CCSSO, 2011), adopted by 45 of the United States and 3 territories, lists mathematical learning objectives, or standards, organized by grade level, and is accompanied by a completely separate discussion of eight mathematical practices. There is no discussion in the document to help the practitioner envision what the implementation of the intended curriculum will look like—leaving the accomplished curriculum more dependent on professional development and local school culture to fill in the picture. One advantage to the separation of mathematics competencies from the mathematical practices may be to avoid representing the mathematics curriculum as an array, which can inadvertently convey a view of mathematics curriculum as disaggregated into bits and pieces represented by each cell.

Consider, for example, the Surveys of Enacted Curriculum (SEC) (Porter, 2002), which are intended to drive assessment of student performance. The SEC is organized in a two-dimensional framework of cognitive demand (memorize, perform procedures, demonstrate understanding, conjecture/generalize/prove, and solve non-routine problems) vs. disciplinary topics (e.g., functions, data analysis, rational expressions). It divides the (K-12) mathematics topics dimension into 19 general categories, each of which is then divided into 4 to 19 smaller mathematical topics. "Thus, for mathematics, there are 1,085 distinct types of content contained in categories represented by the cells" (Porter, McMaken, Hwant, & Yang, 2011, p. 104). Porter's fine-grained representation of curriculum is intended to ensure coverage of mathematical topics and types of cognitive demand while minimizing gaps and overlaps. However, such a representation may lead to an enacted curriculum prescribed by the "pieces" (i.e., the cells), and if educators are prompted to "teach to the test" an unintended emphasis on disconnected mathematics education may result. Further, once a framework like this is codified by formal external assessments, the mathematics content becomes more difficult to revise in response to the needs of evolving fields of science, engineering and technology.

An alternative may be found in the Dutch mathematics curriculum, rooted in Realistic Mathematics Education (RME) learning theory, initially developed by the wellrespected Dutch mathematics educator, Freudenthal (1991), and continued at the Freudenthal institute today. The work in RME portrays a vision of mathematics as a human activity that combines learning and problem solving as a simultaneous activity. Smith & Smith (2007) describe the three dimensions around which the RME-based mathematics curriculum framework is organized: informal to formal; situated to generalized; and individual to social. In practice, RME emphasizes curriculum designed to encourage students' development via progressive mathematization. van den Heuvel-Panhuizen (2003) describes progressive mathematization as the growth of an individual's mathematical knowledge from informal and connected to the local context, to an increasing understanding of solutions designed to reach some level of schematization (making shortcuts, discovering connections between concepts and strategies, making use of these new findings in a new way), and finally to an increasing understanding of formal mathematical systems.

The work on such progressive mathematization is growing (e.g., hypothetical learning trajectories as described by Clements & Sarama, 2004a; 2004b). But, questions have been raised by Lesh and Doerr (in press): Do all students optimally learn along a particular normalized path (learning line, learning trajectory)? Do all students learn the "end product" in the same way? Likely not. Rather than describing a particular learning objective or standard as a goal for learning, they use Vygotsky's (1978) "zone of proximal development" to describe particular goals for students' learning as regions around those goals that are individualistic and dependent on a variety of interacting factors. Such might include the scaffolding provided by the teacher, the language that the student has and the teacher uses, and the technology or manipulatives that may or may not be available during the learning episode. Further, Lesh and Doerr, using Piaget's (1928, 1950) notion of decalage, describe how apparent learning of an objective may mask the partial development of an idea when "operational thinking" for one concept may occur years earlier or later than comparable levels of "operational thinking" for another closely related concept (Lesh & Sriraman, 2005). Lesh and Doerr emphasize that individuals learn in different ways and develop their understandings along different paths. They argue that intended "final products" (i.e., identified as standards or learning objectives) are likely to be in intermediate stages of development in most students, and open to revision and modification as they encounter new situations for which they need to form a mathematical interpretation.

Assumptions about Mathematics Learning

Lesh and Zawojewski (2007) refer to the work of various theorists and researchers (e.g., Lester & Charles, 2003; Lester & Kehle, 2003; Schoen & Charles, 2003; Silver, 1985; Stein, Boaler, & Silver, 2003) to establish a close relationship between the development of mathematical understandings and mathematical problem-solving. Their perspective on learning "treats problem solving as important to developing an understanding of any given mathematical concept or process . . . . [and]. . . the study of problem solving needs to happen in the context of learning mathematics . . ." (p. 765). In particular, Lesh and Harel (2003), and Lesh and Zawojewski's (1992) description of "local concept development" highlight the simultaneous increase in an understanding of a specific problem situation and the development of one's mathematization of the problem. "[S]tudents begin these type of learning/problem-solving experiences by developing [local] conceptual systems (i.e., models) for making sense of real-life situations where it is necessary to create, revise or adapt a mathematical way of thinking" (Lesh & Zawojewski, 2007, p 783).

What is meant by local concept development and learning? Consider the Grant Elementary School Reading Certificate activity described in Figure 1, in which students are asked to create a set of "rules for awarding certificates" (i.e., a decision model). As described in Figure 1, the students generate a variety of models as an answer to this problem, and their answers provide windows to their mathematical thinking and learning—their local concept development.

Grant Elementary School Reading Certificates Problem[1]

In this activity, third grade students are asked to create and apply a set of decision rules for awarding certificates to readers who read a lot and who read challenging books. The students are given sample sets of individual reader's accomplishments, each presented in a table including the title of each book read, the number of pages for each book, and the difficulty level of the book (labeled as easy, medium, hard). The tension between the two criteria for earning a certificate (reading a lot of books and reading challenging material) was intentional, in order to enhance the potential for various reasonable models to be developed.

Summary of Group #1 Response:

- Students should read either 10 books, or more than 1000 pages.
- At least 2 of the books read should be *hard* books.

This group clearly communicates the decision rules (i.e., model) and takes into account both required conditions: reading many books, and reading challenging books. Readers can readily apply the rules to the given data sets. For example, in one data set, the reader had read 5 books (two of which were hard), and a total of 722 pages. Given the clarity of the decision rules, a reader can figure what he or she needs to do to earn a certificate. In this illustrative case, one way for the reader to earn a certificate is to read 5 more books (even if they are all easy). Another way is to pick one long book that has at least 279 pages.

Summary of Group #2 Response:

- A student gets 1 point per page for easy-to-read books.
- A student gets 2 points per page for hard-to-read books.
- A student has to earn 1000 points to get a reading certificate.

This set of decision rules is clearly communicated, and a reader could easily apply the decision rules and self-assess. However, a reader could earn a reading certificate award by reading *only easy books*, not meeting the criteria that readers must read both hard and easy books. Therefore, the set of rules does not meet the requirements for a "good" set of rules.

Figure 1. Two Illustrations of Local Concept Development

1 This activity, in full, can be found in Yeatts, C. L., Battista, M. T., Mayberry, S., Thompson, D., & Zawojewski, J. S. (2004). Navigating through problem solving and reasoning in grade 3. Reston, VA: National Council of Teachers of Mathematics.

The mathematical goals of the activity are three-fold. First, each group of problem solvers is expected to generate a mathematical model, meaning they must develop a procedure or algorithm that meets the criteria given—that those earning a certificate must read a lot of books *and* read challenging books. In the generation of a model, many students engage in other types of mathematical knowledge development, such as quantifying qualitative information and differentially weighing and/or rank ordering factors. Each of the two responses described in Figure 1 represents different locally developed concepts, which are represented in the groups' model (i.e., a set of rules). Note that the first response meets the criteria, whereas the second does not. Note, also, how in each case, the model developed is situated in the context of the problem, and is also dependent on the knowledge that individuals bring to the group—about mathematics, about reading programs, about meaning of "challenging books" and meaning of "reading a lot." A second goal is for students to practice basic skills, such as recognizing the need for and carrying out calculations, and comparing and ordering numbers. These take place as the students test their proposed models, and in the full activity, students are given further sets of data to conduct additional tests of the model they have generated. A third area of learning is generalization, which is driven by the design of the problem. In particular, a good response to this problem is one in which the model produced is re-usable (reliably produces the same results for a given set of data), share-able (the decision rules are clearly and precisely communicated to all of the students, the teachers, and the parents, resulting in reliable application of the model across users), and modifiable (rationales and assumptions on which the model is built are articulated so others can make intelligent adjustments for new situations). Without assumptions or rationales explained, intelligent modification of models can be quite difficult, if not impossible. Notice that neither of the two sample responses in Figure 1 meets the modifiability criteria for generalization, but they have addressed the re-usability and the share-ability criteria for generalization.

Over the years, Lesh and colleagues have reported on the local concepts developed by small groups of students as they engage in various problems, such as the one described in Figure 1. They indicate that individual students often pose initially primitive solutions, and as a result of social interactions, challenges, testing and revision, their initial solutions typically move toward a consensus model that is more stable. The learning of mathematics is described as an iterative process of expressing, testing and revising one's conceptual model. In particular, by using mathematical modeling as a way to think about mathematics learning, Lesh and Doerr (2003) describe a move away from behaviorist views on mathematics learning based on industrial age hardware metaphors in which the whole is viewed simply as a sum of the parts and involving simple causal relationships. Their perspective on mathematics learning also moves beyond software-based information processing metaphors, which involve layers of recursive interactions leading at times to emergent phenomena at higher levels that are not directly derived from the characteristics of lower levels. Instead, they align their models-and-modeling perspective on mathematics learning with a biology-based "wetware" metaphor, in which "neurochemical interactions . . . involv[e] logics that are 'fuzzy,' partially redundant, and partly inconsistent and unstable" (Zawojewski, Hjalmarson, Bowman, & Lesh, 2008, p. 4). Assumed is that students arrive to school with dynamic mathematical conceptual systems already in place, that these conceptual systems are active and evolving before, during and after problem solving and learning episodes, and that students must be motivated to engage in experiences by intellectual need (Harel, 2007) in order to learn. Thus, even when two students in a group may appear to have the same end product knowledge on one task, changing the task slightly, but keeping it mathematically isomorphic with the original, often reveals that the two students are thinking about the intended mathematical ideas in significantly different ways (Lesh, Behr, & Post, 1987; Lesh, Landau, & Hamilton, 1983).

What is the role of the small group in learning? Social aspects of acquiring knowledge from communities have been characterized in society over the decades (e.g., Mead, 1962, 1977, Thayer, 1982), and more recent work describes learning in communities of practice in various trades and occupations (Greeno, 2003; Boaler, 2000; Wenger & Snyder, 2000; Lave & Wenger, 1991; Wenger, 1998). These situations of social learning are characterized by the presence of a teacher, tutor, or mentor who models, teaches and collaborates with novices while engaged in the specific context of practice, rather than in a classroom. Other social aspects of learning have also been documented in situations where there is no teacher/tutor/mentor available. For example, researchers have documented successful collaborations among groups of diverse experts, where any needed leadership emerges flexibly from within the group in response to emerging challenges and opportunities (Cook & Yanow, 1993; Wenger, 2000; Wenger & Snyder, 2000; Yanow, 2000). Both perspectives on social aspects of learning are based on the assumption that all members of a group bring *some* understanding to the table, that the knowledge each brings is idiosyncratic, that the knowledge elicited by the problem is specific to the context, and that local concept development takes place among the group members while simultaneously each individual in the group is adapting and modifying one's own understanding.

Social aspects of problem solving and learning are also related to the development of representational fluency, because interactions among collaborators require representations be used to communicate. When presenting initial solution ideas to peers, a problem solver typically describes one's own model using spoken words, written narratives, diagrams, graphs, dynamic action (e.g., gestures or using geometric software), tables, and other representations. The interpreting peer, who works to make sense of these representations, may request clarification, an additional explanation, or may point out inconsistencies, misrepresentations or other flaws. The peers, thus, iteratively negotiate a consensus meaning. Lesh and Zawojewski (2007) describe various social mechanisms that can elicit the use of representations, leading to the development of representational fluency, including: problem solvers making explanations to each other; groups or individuals keeping track of ideas they have tried; problem solvers making quick reference notes for new ideas to try as they continue in a current line of thinking; and problem solvers documenting their current line of reasoning when they must temporarily disrupt the work. These types of mechanisms, based largely on communication with others and oneself, provide the need to generate and use representations, and develop representational fluency.

Toward an Alternative Framework

Given the assumptions about learning grounded in problem solving, a number of challenges face the development of a framework for a problem-driven mathematics curriculum. How can a curriculum framework feature problem-solving activity as the center of learning, while national and state standards documents highlight specific mathematical content as the central feature? How can a curriculum framework accommodate both the multi-topic nature of realistic mathematics problems and the pure mathematical nature of other mathematics investigations? How can a framework be represented to convey the complexities implied by the previous questions, yet be practical in meeting practical classroom needs? How can a framework be represented to inform the static nature of various standards documents, while also being responsive to changing societal needs and demands?

Envisioning a Curriculum Framework and It's Representation

What is Meant by a Problem-Driven Framework?

The development of problem-driven mathematics text series gained momentum in the U.S. in response to the 1989 NCTM *Curriculum and Evaluation Standards for School Mathematics*. In general, the NSF-funded texts (described in Hirsch, 2007a) are comprised of units of study organized around applied problems or mathematical themes. In many cases, these curricula use mathematical problems to launch and motivate learning sequences that progress toward development of understanding and proficiency for specified mathematical goals. For example, two of the design principles for developing the *Mathematics in Context* text series, which is based on the Dutch RME, are that the starting point of any instructional sequences "should involve situations that are experientially real to students" and "should . . . be justifiable by the potential end point of a learning sequence" (Web & Meyer, 2007, p. 82). The commitment to an experiential basis reflects the commitment to problem-solving as a means to learning, while the well-defined mathematical end points correspond to a commitment to a curriculum framework organized around specific mathematical standards or learning objectives. In contrast, the problem-driven curriculum, *Mathematics: Modeling Our World*, described by Garfunkel (2007), is characterized by using mathematical models as end points. The dilemma for the *Mathematics: Modeling Our World* development team was coordinating the mathematics content naturally emerging from their model-based problem-driven curriculum with a standard mathematics topic driven curriculum framework. Garfunkel describes how the team grappled with the need to "cover" the scope and sequence of the required curriculum:

> "[W]e believed (and still believe) that if we could not find, for a particular mathematics topic, a real problem to be modeled, that that topic would not be included in our curriculum.. . . Instead of 'strands' as they are usually defined we chose to organize curriculum around modeling themes such as Risk, Fairness, Optimization. We made an explicit decision . . . not to create a grid with boxes for mathematical and application topics. Instead, within the themes we chose areas and problems that we believed would carry a good deal of the secondary school curriculum. . . . For example, it was decided that one of the major mathematical themes of Course 1 was to be Linearity, so that each of the units in the course had to carry material leading to a deepening understanding not only of linear functions and equations, but also of the underlying concept of linearity." (pp.161-162).

Garfunkel's dilemma illuminates a fundamental mismatch between a curriculum framework that identifies a list of specific mathematical learning objectives or standards as outcomes, and the development of a curriculum framework driven by problem solving, and in particular, modeling. The "coverage" issue seems to force the enacted problem-driven curriculum to be a mix of problem-driven units accompanied by a collection of gap fillers to address missed content objectives. Thus, while *Mathematics: Modeling Our World* began the journey toward a problem-driven curriculum, it was challenged by the coverage constraint, speaking to the question about what content *should* be included in a mathematics curriculum framework.

As a result, questions are raised about envisioning a problem-driven curriculum framework. Should a problem-driven curriculum framework have as final goals students' deep understanding of mathematical ideas that support certain types of problems, models or themes, or to demonstrate abilities about certain big mathematical ideas that were initiated in problem-solving settings? If the goal of a problem-driven curriculum is to cover certain mathematical models or themes, should the designers of a curriculum cover only those areas that naturally emerge in modeling or problem-solving work? If, on the other hand, the goal of a problem-driven curriculum framework is to accomplish certain big mathematical ideas, is the power of learning those ideas through problem solving to some extent defeated?

What is the Nature of Mathematics Content in a Problem-Driven Framework?

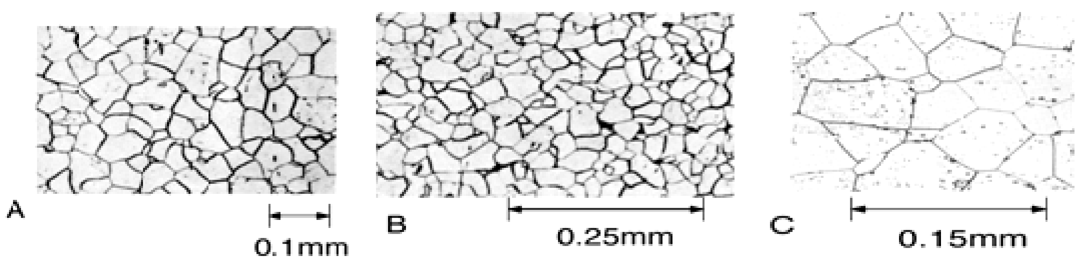
This larger question raises at least three issues about what mathematics to include in a problem-driven framework: What type of problems will the curriculum framework accommodate? What are the boundaries on the mathematics content to be covered? And how does the curricular framework adapt to evolving societal, scientific and technological needs concerning what mathematics is important?

A problem-driven curriculum framework would need to incorporate pure mathematical investigations, real-world applications, and modeling problems, among others. Whereas some problems nicely map onto a single mathematical big idea, others, especially applied and modeling problems emphasize multiple mathematical big ideas—

adding to the complexity of developing such a framework. Consider, for example, the Aluminum Crystal Size[2] MEA, included in Figure 2, as an illustration.

Aluminum Crystal Size Problem Description

The activity is situated in the context of the manufacture of softball bats that would resist denting, but also won't break. In materials science, one learns that the larger the typical size of crystals in a metal, the more prone to bending, and the smaller the typical size the more brittle the metal. A problem was posed that had two purposes. The first was to motivate the problem solver to quantify crystal size. The second was to establish a context where a client needs a procedure to quantifying crystal size as part of their quality control. The client in the problem "hires" the problem solver to create a way to measure, or quantify, aluminum crystal size using two-dimensional images, such as the ones here:



The images are given in different scales, making visual comparison of crystals in the three samples difficult. Therefore, the mathematical procedure would need to take scale into account.

A number of different approaches typically emerge, including:

- Draw a rectangular region to designate a sample within each image. Calculate the area of the rectangle in which the crystals are enclosed. Count the number of crystals in a rectangle drawn. Compute the average area per crystal. Compare samples.
- Select a sample of crystals within each image and estimate the area of each crystal (e.g., by measuring the distance across the widest part of a crystal, and the length of the distance perpendicular to that widest part, and then finding the product of those lengths). Compute the average area per crystal. Compare samples.

Figure 2. Aluminum Crystal Size Problem

In the Aluminum Crystal Size Problem, multiple big ideas in mathematics are relevant to producing a good solution. Spatial reasoning is important as the problem solver needs to figure out ways to quantify regions that are not consistently shaped nor consistently sized, yet must be considered collectively as a "class" tending toward a

[2] For the full activity, see: Diefes-Dux, H. A., Bowman, K. J., Zawojewski, J. & Hjalmarson, M. (2006). Quantifying aluminum crystal size part 1: The model-eliciting activity. *Journal of STEM Education and Research (7)* 1&2, 51-63.
Hjalmarson, M., Diefes-Dux, H. A., Bowman, K. J. & Zawojewski, J. S. (2006). Quantifying aluminum crystal size part 2: The Model-Development Sequence. *Journal of STEM Education and Research h (7)* 1&2, 64-73.

certain size. The problem solver must also consider what parts of the regions to use in the quantification. Measurement is another big idea addressed, since a definition of crystal *size* needs to be generated and mathematized. Proportional reasoning is needed because the micrographs are all shown to different scales, which needs to be accounted for in the development of the mathematical model for crystal size. Sampling is important when deciding what regions of the micrograph to use to determine the size of the crystals in the full image; a good solution will incorporate a method for selecting samples to include in the mathematical model. Measures of centrality are likely to emerge because quantifiable characteristics of the various crystals need to be summarized in some way to come up with a single measurement of crystal size. Finally, mathematical modeling is the centerpiece of the activity. If the Aluminum Crystal Size activity is used as the centerpiece of a unit of study, the problem context drives what mathematical topics are encountered. A framework, then, is needed to help make decisions about which topics to investigate more deeply, whether to stay within the problem context in those investigations, and whether or when to incorporate other more conventional lessons or purely mathematical investigations on the conventional topics.

The second consideration concerns the boundaries of mathematics curricular topics. For example, an economics problem may require designing a mathematical model that optimizes costs while producing the highest quality possible. An engineering problem may have ethical ramifications, where the “best” possible mathematical solution to attain cost-effectiveness may not meet equity considerations. A problem may lend itself to an elegant mathematical solution that uses cutting-edge technology, but the solution may not work with the *commonly* available technology. In the real world, when clients want quantitative-based solutions that are cost-effective yet most powerful, thoroughly but quickly produced, and usable by a wide audience yet secure from abusers, the mathematical and non-mathematical considerations are inseparable. A collaboration of engineering educators have grappled with such an issue in the context of engineering education, where the goal has been to teach foundational engineering principles through mathematical modeling problems that carry competing constraints when considering ethical components (e.g., Yildirim, Shuman, Besterfield-Sacre, 2010).

The content of mathematics curriculum needs to be an entity that can evolve, and can be flexible and nimble as problems faced in the workplace and society evolve—the third consideration. To illustrate, two hundred years ago the computational algorithms needed for bookkeeper's math were appropriately the main focus of school mathematics content. Now-a-days, research on current professional use of mathematics in fields such as engineering (e.g., Ginsburg, 2003, 2006), health sciences (Hoyles, Noss, & Pozi, 2001; Noss, Holyes & Pozi, 2002) and finance (Noss & Hoyles, 1996) reports an increasing need for students to develop or adapt mathematical models to solve novel problems and to flexibly interpret and generate representations. Zawojewski, Hjalmarson, Bowman, & Lesh (2008) indicate that "the real world uses of mathematics are described [in the studies referenced above] as often requiring that mathematical knowledge be created or reconstituted for the local [problem] situation and that content knowledge be integrated across various mathematics topics and across disciplines" (p. 3).

A Proposal for an Alternative Problem-Driven Curriculum Framework

Major dilemmas of constructing on over-arching curriculum framework were illuminated using the two problem driven curriculum frameworks described above. But, even when considered together, the RME and *Mathematics: Modeling Our World* do not necessarily accommodate all aspects of important mathematics to be learned. In particular, the RME framework is driven by problem-solving launches followed by a sequence of activities and instruction that lead to an increased understanding of formal mathematical systems. Garfunkel's *Mathematics: Modeling Our World* is organized around themes such as risk, fairness, optimization and linearity, each representing important areas of mathematics associated with formal mathematical modeling. Both generally aim toward formal mathematical goals, but do not have as end goals mathematics deeply embedded within broad contextual situations and areas such as ethics or equity. The alternative proposed here is based on a notion of model-development sequences that broadens the one described by Lesh, Cramer, Doerr, Post, & Zawojewski (2003). Like RME and Garfunkels' curricula's development, the underlying assumption is that powerful learning of mathematics emerges from students' mathematization of problematic situations. Going beyond RME and Garfunkel, a problem-driven mathematics curriculum framework built around model-development sequences has the potential to incorporate both formal mathematical big ideas/models and real world messy models that are intertwined with non-mathematical constraints.

Lesh, Cramer et al. (2003) describe model-development sequences as beginning with model-eliciting activities (MEAs), which are instantiated in the two problems presented so far (Figures 1 & 2). The main characteristic of a MEA is that the problem requires students to create a mathematical model in response to the task posed, which could be extended to the production of smaller parts of formal mathematical systems. MEAs have traditionally been designed using six specific design principles (Lesh, Hoover, Hole, Kelly, & Post, 2000) to devise "authentic" contexts, involving a client with a specified need for a mathematical model that facilitates making a decision, making a prediction, or explaining a reoccurring type of event in a system. Following the initial MEA are planned model-exploration activities (MXAs), which vary from comparing and contrasting trial models posed by peers in a class, to more conventional meaning-based instruction on various mathematical aspects of the model. For example, the Aluminum Crystal Size problem may be followed up with a lesson on the role and power of random sampling for making inferences, or an opportunity for students to compare and contrast their procedures for determining typical crystal size in micrograph samples. Similarly, one of the authors interviewed a teacher who enacted an MXA activity with her third grade students who had completed the Grant Avenue Reading Certificate Problem (Figure 1). After the teacher asked the third grade students to present their rules to each other, she asked students to identify similarities and differences among the sets of rules, and probed students perceptions of the pros and cons. By asking questions about what aspects of the situation each set of rules attended to and ignored, and how the choice of variables influenced the impacts the outcomes, she was teaching foundational ideas of modeling. For example, Group 2's response (Figure 1) ignores the number of books in the data—depending only on the number of pages to represent "reading a lot." Group 1 (Figure1), on the other hand, used all three types of data (number of pages, number of books and the rating of easy/medium/hard). Even though Group 2's response did not fully meet the criteria articulated in the problem, the use of page numbers only, and not the number of books, is defendable as an indicator of amount of reading. Helping the students articulate rationales for their decisions supports the development of an initial understanding that models are systems that represent larger systems, and inevitably capture some features of the original system, while ignoring other aspects.

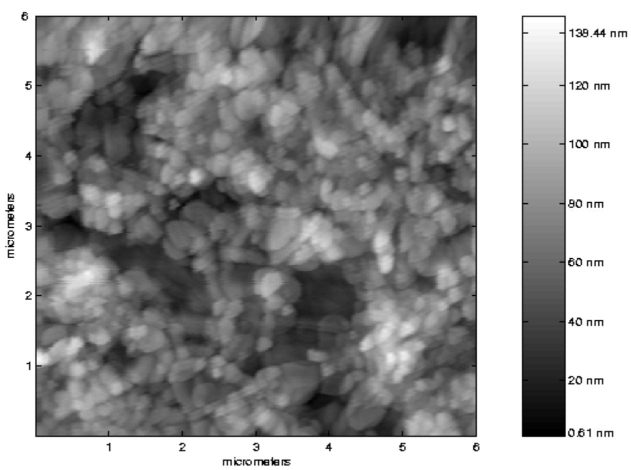
A model-development sequence closes with a model-adaptation or model-application activity (MAA). To illustrate the power of a MAA, consider the full sequence of activities that has been used in the first-year engineering course (with students fresh out of high school) at Purdue University[3]. The opening Nano Roughness[4] MEA, (see Figure 3) is "set in the context of manufacturing hip-joint replacements where the roughness of the surface determines how well the joint replacement moves and wears within the hip socket" (Hjalmarson, et al., p. 41). Given digital images of the molecular surface of different samples of metal, students were asked to create a procedure for quantifying the roughness of each sample, which resulted in a variety of models. The subsequent MXA introduced students to a conventional engineering model for quantifying roughness, the *average maximum profile (AMP)* method, and then asked them to compare their model for quantifying roughness to the conventional engineering model. The goal for this MXA was to enable students to identify and understand trade-

[3] Purdue's first-year engineering program has been using MEAs and model-development sequences for the past 9 years with approximately1500 student per year in West Lafayette, Indiana, USA (Hjalmarson, Diefes-Dux & Moore, 2008).
[4] The full activity can be found in J. S. Zawojewski, H. A. Diefes-Dux, & K. J. Bowman (Eds.), (2008). *Models and modeling in engineering education: Designing experiences for all students* (pp. 317-322). Sense: Rotterdam, The Netherlands. The lead author of the activity is Tamara J. Moore.

offs between models, and to identify and understand rationales and assumptions underlying different models.

Nano Roughness MEA

This goal of this activity is to produce a procedure to quantify roughness of metal surfaces at a microscopic level. Students are given atomic force microscope (AFM) images, similar to the one below, of three different samples of metal surfaces. At the atomic level, the lighter parts of the image represent higher surface, and the darker parts of the image represent lower surface. The gray scale indicator, to the side, provides information about the height of the surface. To motivate the problem situation, the students learn that the company, who is their client, specializes in biomedical applications of nanotechnology. They are planning to produce synthetic diamond coatings for use in orthopedic and biomedical implants, and need to have a way to quantify roughness of the coating surfaces. Given three top-view images of gold samples (illustrated in the one sample below), the modelers are asked to develop a procedure for quantifying the roughness of the material so the procedure could be applied to measure roughness in other types of metal samples.



Sample of an AFM image of gold surface (AFM data courtesy of Purdue University Nanoscale Physics Lab)

Figure 3. Nano Roughness MEA Description

The model-development sequence closes with a Model-Adaptation Activity (MAA) that requires students to adapt either *their* model for measuring roughness, or the conventional model, to a new situation. To do the work in the Purdue example, students were given a raw data set that had been used to produce a sample digital image. These raw data had been gathered by using an atomic force microscope (AFM), which uses a nano-scale probe dragged along the surface of the metal sample in lines at regular intervals, measuring the relative heights along the bumps of the molecules. The students were asked to generate, using MATLAB, a cross sectional view of any line segment drawn on an image of the gold surface. In particular, they produced graph-like products that portrayed the relative heights of the bumps and valleys for any line segment drawn on an image. The mathematical learning goals for this MAA were to conduct 2-dimensional array manipulations of the data and to incorporate statistical reliability considerations into the process. Broader learning goals for the Nano Roughness problem include programming and fundamental engineering principles—illustrating how mathematics learned may be embedded and intertwined to what traditionally has been considered non-mathematical topics.

While MEAs, and their accompanying model-development sequences have traditionally been tied to authentic realistic modeling contexts, the basic concept of eliciting a mathematical model can be broadened to incorporate the more traditional modeling work, such as described by Garfunkel (2007). The model-development sequence framework can also be envisioned to include the elicitation of aspects of formal mathematical systems, such as what is the aim in RME. In other words, model-development sequences have a great deal of potential to serve as an umbrella framework that provides a way to unify problem-driven curricula frameworks, especially when considering the flexibility of Learning Progress Maps (LPMs), which is proposed as a possible way to represent problem-driven curriculum frameworks.

Envisioning a Representational System for Problem-Driven Curriculum

Using a metaphor of topographical maps, Learning Progress Maps (LPMs) can be thought of as a dynamic representation of mathematics curriculum *and* students' learning (Lesh, Lamon, et al., 1992; Lesh, unpublished manuscript). Lesh's goal in developing this concept has been to help teachers readily answer practical classroom questions such as: What concepts do my students still need to address in this unit I am teaching? Which topics would be strategic to address next? What are concepts or topic areas where my students appear to require more experience? Which students are having difficulty with specific concepts, and which have demonstrated learning in those areas? Single score results from large-scale measures do not provide useful information for these questions, whereas item-by-item information for every student might be overwhelming to use as an everyday tool to make decisions about classroom instruction. Portfolio assessment is difficult to define and standardize, let alone use for day-to-day classroom decisions about instruction. On the other hand, good teachers *do* develop their own personal methods to keep track of individual students' progress in a variety of ways, although their systems are idiosyncratic to the teacher, often very detailed, and usually perceived by others as too time consuming to maintain.

How Might a Learning Progress Map (LPM) Represent a Problem-driven Curriculum?

Consider a hypothetical topographical map representing a curriculum organized around mathematical big ideas, important mathematical models, or formal mathematical systems, presented in Figure 4. Lesh, Lamon, et al. (1992) describe the mountains of the landscape as corresponding to the "big mathematical ideas" of a given course (6 to 10 big ideas in this case), and the surrounding terrain of foothills and valleys as depicting facts and skills related to the big ideas. Using the topographical maps metaphor, one can think about the height of the mountain as representing the relative importance of big mathematical ideas in the course, while relationships among the big ideas can be expressed by the proximity of the mountains to each other. The tops of the mountains would represent deep understanding of the big idea, abilities supporting the big ideas can be represented on the sides of the mountains, and associated tool skills (e.g., manipulations, skills, facts) can be represented by the regions of the surrounding foothills and valleys.

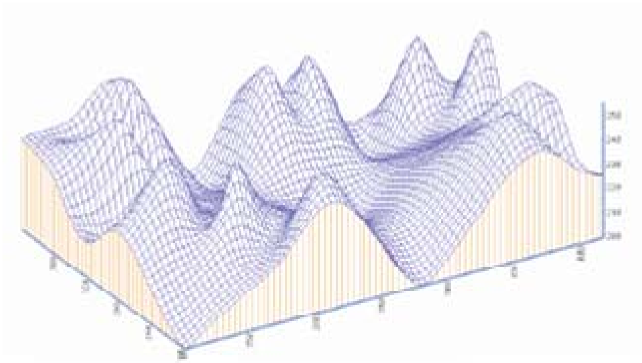


Figure 4. Representation of Big Ideas, Supporting Abilities, and Tools in a Course

A top-down view of the topographical curriculum map (illustrated in Figure 5), might delineate the interplay of the big mathematical ideas, supporting abilities and tool skills to be "covered" in the given course, in a way analogous to a traditional scope-and-sequence document.

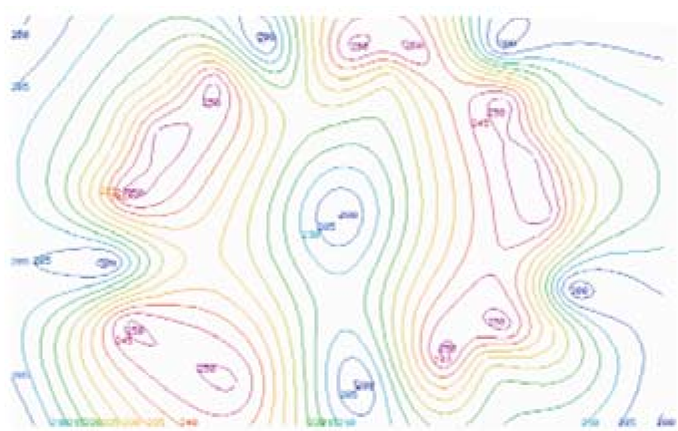


Figure 5. Top Down View of Curriculum Scope and Sequence

On a LPM, problem-solving, modeling, deep insights into a designated mathematical big idea, and higher-order mathematical thinking about the idea would be designated in regions on the tops of the mountains. Thus, problems that involve multiple big mathematical ideas, such as the Nano Roughness MEA, could be represented by multiple mountains (e.g., 3-d geometry, proportions, sampling, measurement, mathematical models). The height of the mountains, and the arrangement of the regions around them, would represent the relative importance of the major mathematical areas with respect to the MEA. Supporting concepts and procedures would correspond to the sides of the mountain, and needed skills and facts would correspond to the adjoining valleys around each mountain. For example, in the Nano Roughness MEA, the fluent interpretation and manipulation of the scales would be an important component of proportional reasoning, and thus represented on the sides of a proportional reasoning mountain. The valleys nearby each mountain would represent the automatic skills and concepts that might be thought of as the tools of the trade for that big idea, such as masterful and precise computation or algebraic manipulation. Another illustration might be the linearity theme of *Mathematics: Modeling out World*, as described by Garfunkel (2007). Linearity might be the name of the mountain, and the idea of representing linear expressions in various forms (as narratives of situations, as tables, as graphs) may each correspond to regions along the side of that mountain, and fluent manipulation of linear equations might be represented in the valley nearby the mountain.

The potential flexibility of the proposed representational system is greatly enabled by the power of technology. For example, given that the concept of linearity is a major theme in Course 1 of *Mathematics: Modeling out World*, linearity may be important to a number of units, and emerge in a variety of contexts. In the mountain representational system for each unit, the theme of linearity may be represented as an overlay of a particular colors or textures (e.g., striping, dotting) on all terrains. Further, theoretical perspectives on learning may also be represented using different intensities of colors to illustrate the three dimensions in RME, or an activity's classification in modeling sequences i.e., MEAs, MXAs and MAAs. The envisioned representation of a curriculum framework could provide teachers with the opportunity to manipulate the map, providing varied views of the curriculum. For example, a teacher may want to see how linearity emerges across chapters within a course by viewing any and all mountains that represent linearity across chapters. While one can imagine many useful scenarios of manipulation, the greatest potential for LPMs, however, is probably representing students' progress through the curriculum.

How Might a Learning Progress Map (LPM) Serve Assessment?

In a problem-driven curriculum framework, assessment of big ideas and models would be supported by LPMs which are envisioned as providing manipulable representations of students' attained curriculum. Specific assessment data can be used to "fill in" appropriate regions of a LPM for a particular student in a course. Since in a problem-driven curriculum, the students' mathematical experiences begin in problem-solving environments, and supporting skills may be learned or mastered later and at various levels, record-keeping is potentially very challenging. Planning assessment points to correspond with particular regions of the map would be a strategy for input points that would in turn help keep track of accomplishments by individuals, while also potentially providing a visual picture that organizes the assessment data for the individual students.

Assessment data points can be drawn from students' responses to problem-solving or modeling activities and used to guide subsequent instructional activity. To illustrate, consider the work of Diefes-Dux and colleagues, who have been very active in documenting students' modeling performance on iterations of revised solutions to MEAs. They have developed systematic ways to evaluate the development of mathematical models that students generate (e.g., Carnes, Diefes-Dux, & Cardella, 2011; Diefes-Dux, Zawojewski, & Hjalmarson, 2010). Their assessment rubric (Diefes-Dux, Zawojewski, Hjalmarson, & Cardella, 2012) that addresses four general characteristics of the models is made into task-specific versions for each MEA. Their recent work has focused on the challenge of identifying and implementing feedback to students with a goal of prompting students to rethink and revise their solution model to be more powerful and efficient (personal conversation with Diefes-Dux, January 17, 2012). One can imagine that this line of research would be enhanced with the proposed framework and representational system. For example, Diefes-Dux and colleagues' evaluate the generalizability of students' models based on three criteria. Assessment of a model's "re-usability" documents the stability of the model over its independent applications; that is, whenever the model is re-applied to a given data set the model will produce the same results each time. Assessment of model's "share-ability" documents whether the model is communicated well enough so that other users can apply the model independently and reliably. Finally, the assessment of the model's "adaptability" focuses on the articulation of critical rationales and assumptions on which the model is constructed, so that an external user would be able to intelligently modify the model for new, somewhat different, circumstances. These three dimensions could be easily represented and manipulated in the envisioned framework to look for patterns and trends in students' series of revised models.

In a problem-driven curriculum framework, assessment of students' performance on concepts, skills, and procedures that support big ideas and models can be facilitated by LPMs and guided by available mathematics education research. For example, in the Ongoing Assessment Project (OGAP), Petit and colleagues (e.g., Petit, Laird, & Marsden, 2010) examined all available mathematics education research in selected domains, identified important benchmarks and "trouble spots" of understanding, and targeted those specific concepts and skills for the development of assessment items and activities. They have completed the work on fractions, multiplication and proportions. Such assessment items can be used as data points in the side regions and valleys of mountains corresponding to the big mathematical ideas. Further, in conjunction with the growing body of research on learning progressions (e.g., Clements, 2004), assessment points that have been embedded in the learning trajectories can become benchmarks that are carefully placed to track general progress as students eventually abstract from their variety of situations to generalized mathematical ideas.

How Might a Learning Progress Map (LPM) be Used to Inform Practice and Programs?

The envisioned dynamic LPMs would provide a means for teachers to quickly and easily identify information relevant to day-to-day questions for teaching and students' learning. For example: *What concepts do my students still need to address? Which topics would be strategic to address next? Which students are (or are not) having difficulty with specific concepts?* Using a keystroke, summarized students' assessment data could displayed on the LPM, providing opportunities for nimble decision-making about classroom practice. By illuminating the whole class's attainment on the LPM curriculum, teachers would be able to see what yet needs to be addressed in the course, and what may need some reteaching. Profiles of individual students' attainment could help teachers plan to group students for differentiated instructional experiences. LPMs could, for example, help teachers to form problem-solving groups by identifying students with a variety of expertise relevant to the problem. Individual profiles, when displayed side-by-side, could also inform teachers' decisions about students access to limited resources (e.g., volunteer tutors, particular technological assistance, advanced placement coursework).

Self-assessment could become a major component of classroom experience. Students could use their own individual profiles to self-assess their own progress, and perhaps even select problems through which they can address their own areas of need. In an advanced version of LPM, where the curriculum topics are linked to appropriate problems, perhaps students could select a context they like to think about (e.g., sports, health care), and be assigned an appropriate problem from the targeted area of need. By integrating an assessment system with the curricular map, LPMs could be used as a tool to guide students' selection of problems that have the potential to move them forward mathematically.

Professional development and program evaluation can also be enhanced through LPMs. Lesh, Lamon, et al., (1992) describe a variety of program level assessments that could be accomplished by dynamic LPMs. For example, a summary class attainment map that looks like the one in Figure 5, suggests instruction that is highly skill-based, and thus provides an opportunity to for a teacher to confront one's own (perhaps unconscious) assumption that problem solving and deep conceptual understanding can only be addressed after all of the "basics skills" have been accomplished. On the other hand, a summary class attainment map that looks like the one illustrated in Figure 6 might suggest that a teacher is effectively implementing a problem-driven curriculum, given that the attained map illustrates splashes and spreads from multiple points near the tops of the mountains, and oozing downward to the sides of mountains and surrounding valleys.

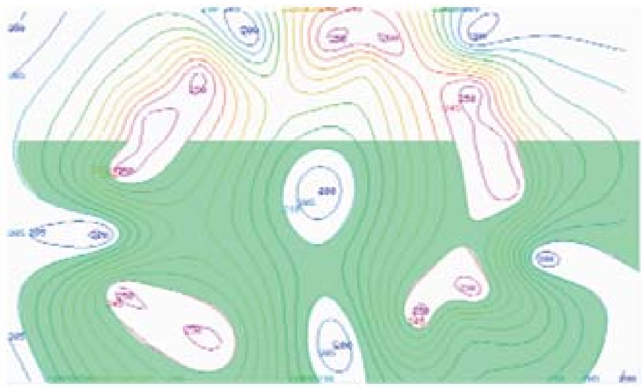


Fig 5: LPM (green) in Skill-based Attainment by Students

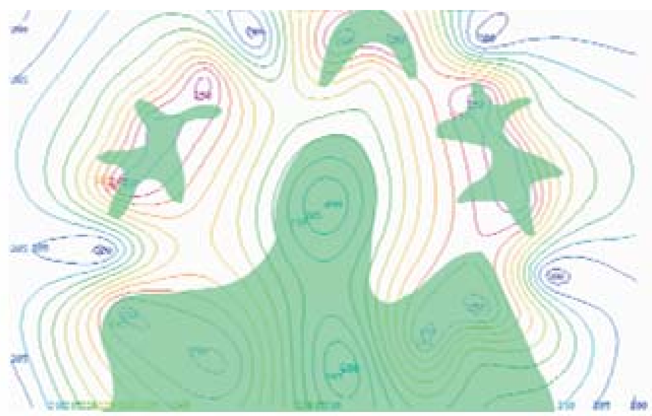


Fig 6: A LPM (green) in Multi-level Attainment by Students

Reflections

The envisioned problem-driven mathematics curriculum framework supported by a dynamic representational system, LPM, seems feasible. Given the potential of today's technologies, design research (Kelly, Lesh, Baek, 2008) methodologies could be used to simultaneously build, study and revise theoretical, pedagogical, and practical considerations of a problem-driven curriculum framework and its representational system. The LPM could be manipulated and revised quickly and easily in response to various changing conditions, such as changes in what constitutes important mathematics, changes in important problem context, changes in new content-driven state standards, and changes in interdisciplinary and social considerations. While the representational system has yet to be actualized, many aspects of problem-driven curricular frameworks are already under research and development. Imagining future work that links technology-driven LPMs and problem-driven curriculum frameworks brings a variety research questions and potential issues for investigation.

Given that problem-driven mathematics curriculum frameworks are grounded in the assumption that students learn mathematics while engaged in complex problem-solving activity, a question arises about how LPMs could be used to represent such curricula. What would a LPM look like for a course, or a unit of study? What will be identified as the "big ideas" or mountains around which the mathematical terrain is developed? What variables need to be represented in the LPM, beyond content topics? What needs to be fixed and what needs to be flexible in the software? These are only a few of the questions that need to be answered in interdisciplinary teams of mathematics educators, curriculum developers, assessment experts, and software developers in a design process.

How can LPMs be used to identify when, and the extent to which, problem-based instruction supports the given problem-based curriculum? Collaborative research and development would be needed to design software to display an image, such as the one in

Figure 5 that represents successful implementation of problem-based instruction. The design of the software would require the identification of variables and development of models to show the splashes and spreads from multiple points near the tops of the mountains, oozing downward, and eventually filling in the valleys. The needed data include the curriculum specifications, student assessment data, and teacher input about experiences implemented. The goal would be to provide real-time information to teachers and their support personnel concerning what students are learning, and to use that information to adjust instructional strategies to align with those appropriate for problem-based learning.

How might LPMs assist classroom teachers in their enactment of a problem-driven curriculum, yet help to keep an eye on "content coverage" as potentially required by other stake holders? To support implementation of problem-driven curricula in environments that are driven by standards and emphasize content coverage, teachers' need to have tools that help them traverse the challenges of real world implementation. The envisioned LPMs must have embedded in them the ability to manipulate the representations so that teachers can easily check on "content coverage" while teaching a problem-based curriculum. Further, they need to be able to easily check on individual student progress in order to plan for reasonable differentiation. Challenges in implementing a problem-based curriculum must be addressed by well-designed LPMs that are easily used by teachers to inform *their* questions and issues.

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