1-1-2012

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Published version. *Houston Journal of Mathematics*, Vol. 38, No. 1 (2012): 295-310. Publisher Link.

Houston Journal of Mathematics
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Volume 38, No. 1, 2012

CATEGORICITY AND TOPOLOGICAL GRAPHS

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Communicated by Charles Hagopian

Abstract. Let X be a topological graph; i.e., a union of finitely many points and arcs, with arcs joined only at end points. If Y is any locally connected metrizable compactum that is co-elementarily equivalent to X, then Y is homeomorphic to X. In particular, X and Y are homeomorphic if some lattice base for one is elementarily equivalent to some lattice base for the other.

1. introduction

This paper is about the model-theoretic topology of compact Hausdorff spaces—also referred to as *compacta*—and our aim is to show that any topological graph is categorical, relative to the class of compacta that are locally connected and metrizable.

As the term *categorical* has a range of interpretations, we begin with a general description of how it is used here. Suppose $\mathfrak{K}$ is a class of objects, together with two reflexive symmetric relations, one finer than the other. To keep things straight, call the finer relation *indistinguishability* (always an equivalence relation) and the coarser one *similarity* (usually an equivalence relation). An object $X \in \mathfrak{K}$ is defined to be *categorical*, relative to $\mathfrak{K}$, if any member of $\mathfrak{K}$ that is similar to X is actually indistinguishable from X.

2000 *Mathematics Subject Classification.* 03C20,03C35,54F25,54F50.

Key words and phrases. Categoricity, compacta, continua, locally connected, topological graphs, ultracoproducts, co-elementary equivalence, lattice bases.

The author would like to thank the anonymous referee for pointing out some important errors in the initial submitted draft of this paper.

In practice indistinguishability is isomorphism in some appropriate category; and, in the case of model theory, similarity is *elementary equivalence*, the sharing of the same first-order properties [6]. Here are some classical examples.

Examples 1.1. *We consider $\mathfrak{K}$ to be a class of linear orderings.*

(i) *If $\mathfrak{K}$ is the class of all linear orderings, then only the finite linear orderings are categorical. Any infinite linear ordering is elementarily equivalent to a linear ordering of a different cardinality, by the Löwenheim-Skolem theorem.*

(ii) *If $\mathfrak{K}$ is the class of countable linear orderings, then the rational ordering $\mathbb{Q}$ is categorical. By Hausdorff's "back-and-forth" method for constructing order isomorphisms, any countable dense linear ordering without end points is isomorphic to $\mathbb{Q}$.*

(iii) *If $\mathfrak{K}$ is the class of well orderings, then the ordered set $\mathbb{N}$ of natural numbers is categorical. Any well ordering in which every element has an immediate successor and only the first element has no immediate predecessor is isomorphic to $\mathbb{N}$. Note that $\mathbb{N}$ is* not *categorical relative to the class of countable linear orderings because there are countable linear orderings that are elementarily equivalent to $\mathbb{N}$, but which are not well ordered.*

So categoricity is not an intrinsic property of an object; rather it is an expression of how "distinguished" that object is within a class of its peers. Categoricity becomes increasingly rare as the peer class is broadened or as the similarity relation is coarsened. In the setting of the present paper, the ambient class consists of locally connected metrizable compacta, indistinguishability is homeomorphism, and similarity is co-elementarily equivalence, a topological dualization of elementary equivalence (see below, and, in more detail, in [4]). In this context, topological graphs will be shown to be categorical.

2. PRELIMINARIES

We first explain co-elementary equivalence, first introduced in [1]. Briefly, two compacta X and Y are *co-elementarily equivalent* if they have homeomorphic ultracopowers, a relationship expressed in the following diagram.

$$\begin{array}{ccc} X_{\mathcal{D}} & \xrightarrow{h} & Y_{\mathcal{E}} \\ p_{X,\mathcal{D}} \downarrow & & \downarrow p_{Y,\mathcal{E}} \\ X & & Y \end{array}$$

The horizontal mapping is a homeomorphism, the vertical mappings are canonical ultracopower projections. $\mathcal{D}$ and $\mathcal{E}$ are ultrafilters on sets I and J, respectively, and the *ultracopower* $X_{\mathcal{D}}$ is the subspace of the Stone-Čech compactification $\beta(X \times I)$ (I is a discrete topological space) consisting of points that are sent to $\mathcal{D} \in \beta(I)$ under the Stone-Čech lift q^β of the standard coordinate projection $q : X \times I \to I$. With $p : X \times I \to X$ denoting the other coordinate projection, we have the following diagram.

$$\begin{array}{ccccc} X_{\mathcal{D}} & \xrightarrow{\subseteq} & \beta(X \times I) & \xrightarrow{q^\beta} & \beta(I) \\ & \searrow p_{X,\mathcal{D}} & \downarrow p^\beta & & \\ & & X & & \end{array}$$

The mapping $p_{X,\mathcal{D}}$, the restriction of p^β to $X_{\mathcal{D}} \subseteq \beta(X \times I)$, is easily shown to be surjective, and is the prototypical *co-elementary map* [1]. It is not immediately obvious that co-elementary equivalence is indeed a transitive relation, but this fact is established in [1].

As mentioned above, the topological ultracopower is a dualization of the model-theoretic notion of ultrapower; and to do this claim justice, we need to discuss closed-set lattices of compacta.

For a topological space X, we denote by $F(X)$ the collection of closed subsets of X, viewed as a bounded lattice under the usual Boolean operations. More precisely, $F(X)$ is the L-structure $\langle F(X); \cup, \cap, \emptyset, X\rangle$, where $\mathsf{L} := \{\sqcup, \sqcap, \bot, \top, =\}$ is the standard first-order alphabet, with equality, for bounded lattices. By a *lattice base* for X, we mean a bounded sublattice of $F(X)$ that is also a closed set base. Stemming from the work of H. Wallman [11] (see also [4, 7]), there is a particularly useful model-theoretic result regarding compacta and their lattice bases.

Theorem 2.1 (Representation)**.** *There is a sentence ρ in the first-order language over alphabet L such that an L-structure satisfies ρ if and only if that structure is isomorphic to a lattice base for a unique compactum.*

While the specific formulation of the sentence ρ above is not of primary importance here, it simply says that the structure is a bounded distributive lattice for which two more properties hold: it is *normal*, in the obvious sense of topological normality phrased purely in terms of closed sets; and it is *disjunctive*, in the sense that for any two elements, one of them dominates a non-bottom element disjoint from the other. We call an L-structure satisfying ρ a *normal disjunctive lattice.*

If A is a normal disjunctive lattice, let $w(A)$—the *Wallman space* of A—be the compactum promised in Theorem 2.1. The elements of $w(A)$ are maximal filters of A, and a lattice base for $w(A)$ consists of sets $a^\sharp := \{x \in w(A) : a \in x\}$. The assignment $a \mapsto a^\sharp$ is a lattice isomorphism, and this gives us the representation.

The assignment $A \mapsto w(A)$ is Stone duality when restricted to the *Boolean* lattices, those normal disjunctive lattices that are complemented. In general, however, it is easy to find examples where the assignment is not "one-one;" indeed, $F(X)$ is always atomic, but may contain atomless lattice bases of strictly smaller cardinality. Thus A and B having homeomorphic Wallman spaces does not ensure that A and B are either elementarily equivalent or equinumerous.

By Theorem 2.1 and standard model theory, an ultrapower $\mathcal{A}^{\mathcal{D}}$ of a lattice base for compactum X is again a lattice base for some compactum. Indeed, it is a fundamental result of [1] that $w(\mathcal{A}^{\mathcal{D}})$ is canonically homeomorphic to $X_{\mathcal{D}}$. The ultrapower of the lattice base $\mathcal{A}$ gives rise to a lattice base for $w(\mathcal{A}^{\mathcal{D}})$, consisting of *internal ultracoproducts* $\sum_{\mathcal{D}} A_i := \overline{\bigcup_{i\in I}(A_i \times \{i\})} \cap X_{\mathcal{D}}$ of I-indexed collections from $\mathcal{A}$, where overline indicates closure in $\beta(X \times I)$. Since elements of $\mathcal{A}^{\mathcal{D}}$ may be viewed as ultraproducts $\prod_{\mathcal{D}} A_i$ of such collections, the internal ultracoproduct $\sum_{\mathcal{D}} A_i$ is just $(\prod_{\mathcal{D}} A_i)^\sharp$.

Thus, if $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent lattice bases for compacta X and Y, respectively, then we may find, by the Keisler-Shelah ultrapower theorem [6], isomorphic ultrapowers $\mathcal{A}^{\mathcal{D}}$ and $\mathcal{B}^{\mathcal{E}}$. Any such isomorphism directly gives rise to a homeomorphism between $X_{\mathcal{D}}$ and $Y_{\mathcal{E}}$; hence X and Y are co-elementarily equivalent. In the zero-dimensional case, Stone duality then tells us that two Boolean spaces are co-elementarily equivalent if and only if their lattices of clopen sets are elementarily equivalent. This adds further credibility to the assertion that co-elementary equivalence is the "right" analogue of elementary equivalence in the compact Hausdorff context.

The least infinite cardinal κ such that a space X has a lattice base of cardinality $\leq \kappa$ is known as the *weight* of X, and when we are dealing with compacta, having countable weight is tantamount to being metrizable. Our main objects of study in the sequel are the metrizable compacta that are locally connected, and one of the most important of these is the *arc*. This space is defined to be any homeomorphic copy of the usual closed unit interval in the real line, but is topologically characterized (Theorem 6.17 in [10]) as being the unique metrizable *continuum*—i.e., connected compactum—that has precisely two points with connected complement. These points are the ones that are not *cut* points, and every nondegenerate metrizable continuum has at least two non-cut points (Theorem 6.6 in [10]).

The simplest spaces using arcs and isolated points as building blocks are the *topological graphs*, those compacta that may be decomposed into a finite union of points and arcs, no two arcs of which intersect in a cut point of either. Topological graphs are clearly locally connected and metrizable, so the main result of this paper is the following.

Theorem 2.2 (Graph categoricity). *Every topological graph is categorical, relative to the class of locally connected metrizable compacta.*

Remarks 2.3. (i) The first result along the lines of categoricity in the topological setting is in [9], where the arc was shown to be categorical, relative to the class of *all* metrizable spaces, when the similarity relation is taken to be elementary equivalence of full closed-set lattices. The question was then posed in [1] whether categoricity of the arc still holds when the similarity relation is coarsened to co-elementary equivalence, and—in the interests of having the question make sense—the ambient class is narrowed to the metrizable compacta. R. Gurevič provided a negative answer [7] by showing the arc is co-elementarily equivalent to a metrizable continuum that is not locally connected. This prompted the result in [2] that arcs and simple closed curves are categorical in the locally connected compact metrizable environment. (And in [5] this result was extended to topological graphs that are n-ods.)

(ii) The Cantor space, the unique zero-dimensional compact metrizable space without isolated points, is categorical relative to the class of metrizable compacta; but K. P. Hart [8] has shown that no nondegenerate metrizable continuum is so categorical. It is still an open question whether there are any metrizable compacta of positive dimension that are categorical in this wide sense.

(iii) The class of locally connected compacta is quite restrictive, and it is natural to ask whether metrizability must be considered when using co-elementary equivalence to compare such spaces. The answer is *yes* because (Proposition 2.4 in [3]) any two generalized arcs (i.e., linearly ordered continua) are co-elementarily equivalent. Since such spaces are locally connected, this tells us that the arc is co-elementarily equivalent to locally connected compacta of any given weight.

3. PROOF OF GRAPH CATEGORICITY

The proof of Theorem 2.2 is divided into three independent steps. The first is an immediate corollary of Theorem 2.11 in [5], and provides a major simplification of the task at hand.

Lemma 3.1. *If two metrizable locally connected compacta are co-elementarily equivalent and one of them is a topological graph, then so is the other.*

So, in view of Lemma 3.1, all we need show is that two co-elementarily equivalent topological graphs are homeomorphic.

In the second step we introduce an equivalence relation between compacta, called *G-equivalence*, and show that this relation is a consequence of co-elementary equivalence. Then, in the third step, we show that two G-equivalent topological graphs are homeomorphic.

To start the second step, define a finite cover $\mathcal{K}$ of a compactum X to be a *G-cover* if the following three conditions hold.

(1) Each member of $\mathcal{K}$ has nonempty interior, and is a *subcontinuum*, i.e., a closed connected subset, of X.
(2) $\mathcal{K}$ is a *minimal* cover; i.e., for each $K \in \mathcal{K}$, $\mathcal{K} \setminus \{K\}$ is not a cover of X.
(3) No point of X lies in more than two members of $\mathcal{K}$.

Given a G-cover $\mathcal{K}$, we denote by $N(\mathcal{K})$ the *nerve* of $\mathcal{K}$; i.e., the abstract (finite simple) graph whose vertices are the sets $K \in \mathcal{K}$, and whose adjacency relation consists of all pairs of distinct vertices with nonempty intersection.

Given two G-covers $\mathcal{K}$ and $\mathcal{L}$ of X, we say that $\mathcal{L}$ is a *perfect refinement* of $\mathcal{K}$, and that the pair $\langle \mathcal{K}, \mathcal{L} \rangle$ is a *perfect pair* for X, if we have the following two conditions.

(4) Each member of $\mathcal{L}$ is contained in a unique member of $\mathcal{K}$.
(5) Each member of $\mathcal{K}$ is a union of members of $\mathcal{L}$.

When $\langle \mathcal{K}, \mathcal{L} \rangle$ is a perfect pair and $K \in \mathcal{K}$, we denote by $\mathcal{L}_K$ the set of members of $\mathcal{L}$ contained in K. Then $\mathcal{L}_K$ is a minimal cover of K. Furthermore $\{\mathcal{L}_K : K \in \mathcal{K}\}$ forms a partition of $\mathcal{L}$.

Given two compacta X and Y, we say Y *G-dominates* X to mean the following.

(6) For any perfect pair $\langle \mathcal{K}, \mathcal{L} \rangle$ for X, there exist:
 - a perfect pair $\langle \mathcal{K}', \mathcal{L}' \rangle$ for Y; and
 - abstract graph isomorphisms $f : N(\mathcal{K}) \to N(\mathcal{K}')$ and $g : N(\mathcal{L}) \to N(\mathcal{L}')$ such that for each $K \in \mathcal{K}$, $f(K) = \bigcup\{g(L) : L \in \mathcal{L}_K\}$. (Equivalently, $\mathcal{L}'_{f(K)} = \{g(L) : L \in \mathcal{L}_K\}$.)

If f and g are as in (6) above, we call $\langle f, g\rangle$ an *isomorphism of perfect pairs*, and write $\langle f, g\rangle : \langle \mathcal{K}, \mathcal{L}\rangle \to \langle \mathcal{K}', \mathcal{L}'\rangle$. Two compacta are called *G-equivalent* if each G-dominates the other.

The remainder of the second step of the proof of graph categoricity is to show co-elementary equivalence implies G-equivalence. As preparation for this, we first recall that a *component* of a topological space is a maximally-connected subset of the space; we then define a lattice base $\mathcal{A}$ for compactum X to satisfy the *component property* if whenever $A \in \mathcal{A}$ and $U \subseteq A$ is a nonempty open subset of X, there is a component C of A such that $C \in \mathcal{A}$ and $C \cap U \neq \emptyset$. (In particular, $F(X)$ always satisfies the component property.)

Lemma 3.2. *There is a sentence γ in the first-order language over alphabet L such that an L-structure satisfies γ if and only if that structure is isomorphic to a lattice base with the component property, for some (unique) compactum.*

Proof. We use the well-known result that, in a compactum, the component containing a given point is the intersection of all the clopen neighborhoods of the point. We also use the easy fact that if A is a member of a lattice base $\mathcal{A}$ for a compactum X, and if A is disconnected, then A has a disconnection consisting of members of $\mathcal{A}$. Thus, in addition to saying that an L-structure is a normal disjunctive lattice, γ says the following of a lattice base $\mathcal{A}$ for compactum X: Given A and B in $\mathcal{A}$ such that $A \cup B = X$ and $B \neq X$, there is a $C \in \mathcal{A}$ such that:

- $C \subseteq A$ and $C \not\subseteq B$;
- C is connected; and
- for each $D \in \mathcal{A}$ such that $D \subseteq A$ and $D \cap C = \emptyset$, there exist $U, V \in \mathcal{A}$ such that $C \subseteq U$, $D \subseteq V$, $U \cup V = A$, and $U \cap V = \emptyset$.

It is easy to check that this gives an L-sentence that captures the component property for lattice bases of compacta. □

Before we can use Lemma 3.2, we need to connect the component property with G-covers.

Lemma 3.3. *Let $\mathcal{A}$ be a lattice base for compactum X, and suppose $\mathcal{A}$ satisfies the component property. If $\langle \mathcal{K}, \mathcal{L}\rangle$ is a perfect pair for X, then there exists a perfect pair $\langle \mathcal{K}', \mathcal{L}'\rangle$ for X, satisfying:*

(i) *each set $K \in \mathcal{K}$ (resp., $L \in \mathcal{L}$) is contained in a unique $K' \in \mathcal{K}'$ (resp., $L' \in \mathcal{L}'$);*
(ii) *the sets in $\mathcal{K}'$ and $\mathcal{L}'$ are members of $\mathcal{A}$; and*

(iii) *the maps* $K \mapsto K'$ *and* $L \mapsto L'$ *define an isomorphism* $\langle f, g\rangle : \langle \mathcal{K}, \mathcal{L}\rangle \to \langle \mathcal{K}', \mathcal{L}'\rangle$ *of perfect pairs.*

Proof. We use the fact that $\mathcal{A}$ is a lattice base for a compactum, plus induction on the size of finite covers, to define $\mathcal{L}^* := \{L^* : L \in \mathcal{L}\}$ as follows. For each $L \in \mathcal{L}$, L^* is chosen from $\mathcal{A}$ so that:

(a) for any $L_1, L_2 \in \mathcal{L}$, $L_1 \subseteq L_2^*$ if and only if $L_1 = L_2$, and $L_1^* \cap L_2^* = \emptyset$ if and only if $L_1 \cap L_2 = \emptyset$;
(b) $L_1^* \cap L_2^* \cap L_3^* = \emptyset$ for each three distinct $L_1, L_2, L_3 \in \mathcal{L}$; and
(c) $\mathcal{L}^*$ is a minimal cover of X.

Next, by the definition of *perfect pair*, each $K \in \mathcal{K}$ is minimally covered by $\mathcal{L}_K$. Thus there is no ambiguity when we define $K^* \in \mathcal{A}$ to be $\bigcup\{L^* : L \in \mathcal{L}_K\}$.

We now use the component property to find, for each $L \in \mathcal{L}$, a component L' of L^* such that $L' \in \mathcal{A}$ and L' intersects the interior of L. Then, because L is a connected subset of L^*, L' is a maximally connected subset of L^*, and $L \cap L' \neq \emptyset$, we have $L \subseteq L' \subseteq L^*$ for each $L \in \mathcal{L}$.

Since no subset of $\mathcal{L}$ besides $\mathcal{L}_K$ is a cover of K for any $K \in \mathcal{K}$, we may define $K' \in \mathcal{A}$ unambiguously to be $\bigcup\{L' : L \in \mathcal{L}_K\}$.

This gives us our pair $\langle \mathcal{K}'\mathcal{L}'\rangle$, and we need to check that this choice does as claimed.

First note that $\mathcal{L}'$ covers X because $\mathcal{L}$ does; every member of $\mathcal{L}'$ is, by construction, a subcontinuum of X with nonempty interior; and for each three distinct sets $L_1, L_2, L_3 \in \mathcal{L}$, $L_1' \cap L_2' \cap L_3' \subseteq L_1^* \cap L_2^* \cap L_3^* = \emptyset$. So conditions (1) and (3) hold for $\mathcal{L}'$. Finally, for any $L \in \mathcal{L}$, we know from (c) above that $\mathcal{L}^* \setminus \{L^*\}$ is not a cover of X; hence neither is $\mathcal{L}' \setminus \{L'\}$. We therefore know that (2) also holds, and thus that $\mathcal{L}'$ is a G-cover of X.

Next we verify that $\mathcal{K}'$, clearly a cover of X, is also a G-cover. Note that, for each $K \in \mathcal{K}$ and each $L \in \mathcal{L}_K$, L' is connected and intersects the connected set K. Hence $K' = \bigcup\{L' : L \in \mathcal{L}_K\}$ is a connected superset of K, and so condition (1) holds for $\mathcal{K}'$. Suppose $x \in K_1^* \cap K_2^* \cap K_3^*$, where $K_1, K_2, K_3 \in \mathcal{K}$ are distinct. Then, by (5) and (a) above, there are $L_1, L_2, L_3 \in \mathcal{L}$ such that $L_i \subseteq K_i$, $i = 1, 2, 3$, and $x \in L_1^* \cap L_2^* \cap L_3^*$. Since $\mathcal{L}$ is a perfect refinement of $\mathcal{K}$, it follows, by (3) and (4), that the sets L_1, L_2, L_3 are distinct and must have empty intersection. But then, by (b), the sets L_1^*, L_2^*, L_3^* also have empty intersection, giving us a contradiction. Thus $K_1' \cap K_2' \cap K_3' \subseteq K_1^* \cap K_2^* \cap K_3^* = \emptyset$ whenever $K_1, K_2, K_3 \in \mathcal{K}$ are distinct, and so (3) holds for $\mathcal{K}'$. Finally, for each $K \in \mathcal{K}$, $\mathcal{K}' \setminus \{K'\}$ is not a cover of X because its union is contained in the union of $\mathcal{L}' \setminus \{L'\}$ for any $L \in \mathcal{L}$ contained in K. Since (2) holds for $\mathcal{L}'$, it therefore holds for $\mathcal{K}'$.

To check that $\mathcal{L}'$ is a perfect refinement of $\mathcal{K}'$, we first note that condition (5) holds by how we constructed the sets K' for $K \in \mathcal{K}$. As for condition (4), pick $L \in \mathcal{L}$. If $L \subseteq K$, where $K \in \mathcal{K}$, then $L' \subseteq K'$ by definition of K'. If $L \not\subseteq K$, then, by (c) above, $L \not\subseteq \bigcup\{L^* : L \in \mathcal{L}_K\}$. Hence $L \not\subseteq \bigcup\{L' : L \in \mathcal{L}_K\}$, and hence $L' \not\subseteq K'$. So if L' is contained in both K_1' and K_2', then L is contained in both K_1 and K_2. Since (4) holds for the pair $\langle \mathcal{K}, \mathcal{L}\rangle$, we have $K_1' = K_2'$.

We need to verify conditions (i)-(iii) above; (ii) is already taken care of. If $L \in \mathcal{L}$, then, by (a), L cannot lie in L_1' for any $L_1 \neq L$. Thus (i) holds for $\mathcal{L}$. If $K_1 \neq K_2$ and $L_1 \subseteq K_1$, then, as in the last paragraph, $L_1 \not\subseteq K_2^*$; and so $K_1 \not\subseteq K_2'$. The verification of condition (iii) is now quite easy: the maps $K \mapsto K'$ and $L \mapsto L'$ are one-to-one because of (ii); the adjacency/nonadjacency relations are easily checked, using condition (a) above. $\square$

The next lemma completes the second step.

Lemma 3.4. *If two compacta are co-elementarily equivalent, then they are G-equivalent.*

Proof. From the assumption that compacta X and Y are co-elementarily equivalent, we fix ultrafilters $\mathcal{D}$ on I and $\mathcal{E}$ on J, and a homeomorphism $h : X_{\mathcal{D}} \to Y_{\mathcal{E}}$. By symmetry it is enough to show that Y G-dominates X; so let the perfect pair $\langle \mathcal{K}, \mathcal{L}\rangle$ be given for X. For each $A \in F(X)$, we denote the internal ultracoproduct $\sum_{\mathcal{D}} A_i$, where each A_i equals A, by $A_{\mathcal{D}} = \overline{A \times I} \cap X_{\mathcal{D}}$ (closure with respect to $\beta(X \times I)$). Then the assignment $A \mapsto A_{\mathcal{D}}$ is a lattice embedding of $F(X)$ into $F(X_{\mathcal{D}})$ [1]; hence $\langle\{K_{\mathcal{D}} : K \in \mathcal{K}\}, \{L_{\mathcal{D}} : L \in \mathcal{L}\}\rangle$ is a perfect pair for $X_{\mathcal{D}}$, witnessing the fact that $X_{\mathcal{D}}$ G-dominates X. And, since h is a homeomorphism, we have the perfect pair $\langle\{h[K_{\mathcal{D}}] : K \in \mathcal{K}\}, \{h[L_{\mathcal{D}}] : L \in \mathcal{L}\}\rangle$ to witness that $Y_{\mathcal{E}}$ G-dominates X as well.

Now $F(Y)$ is a lattice base that trivially satisfies the component property. Therefore, by Lemma 3.2 and the Łoś ultraproduct theorem [6], so does the ultrapower lattice $F(Y)^{\mathcal{E}}$. This lattice is naturally isomorphic to the lattice base for $Y_{\mathcal{E}}$ consisting of internal ultracoproducts of J-indexed collections from $F(Y)$, so we may now apply Lemma 3.3 to obtain a perfect pair $\langle \mathcal{K}', \mathcal{L}'\rangle$ for $Y_{\mathcal{E}}$ such that:

- (i)' for each $K \in \mathcal{K}$ (resp., $L \in \mathcal{L}$) $h[K_{\mathcal{D}}]$ (resp., $h[L_{\mathcal{D}}]$) is contained in a unique $K' \in \mathcal{K}'$ (resp., $L' \in \mathcal{L}'$);
- (ii)' the sets in $\mathcal{K}'$ and $\mathcal{L}'$ are internal ultracoproducts; and
- (iii)' $\langle \mathcal{K}, \mathcal{L}$ and $\langle \mathcal{K}', \mathcal{L}'\rangle$ are isomorphic perfect pairs.

For each $K \in \mathcal{K}$ (resp., $L \in \mathcal{L}$), we may write $K' = \sum_{\mathcal{E}} A_{K,j}$ (resp., $L' = \sum_{\mathcal{E}} B_{L,j}$). Since $\mathcal{K}$ and $\mathcal{L}$ are finite, and since the property of being a perfect pair that is isomorphic to $\langle \mathcal{K}, \mathcal{L} \rangle$ is expressible as a first-order sentence over the lattice alphabet, we use the Łoś theorem once again to infer that the set of $j \in J$ such that $\langle \{A_{K,j} : K \in \mathcal{K}\}, \{B_{L,j} : L \in \mathcal{L}\} \rangle$ is a perfect pair for Y that is isomorphic to $\langle \mathcal{K}, \mathcal{L} \rangle$ is a set in $\mathcal{E}$, and hence nonempty. Thus Y G-dominates X. □

This brings us to the third and final step in the proof of graph categoricity; it remains to show that two topological graphs are homeomorphic if they are G-equivalent.

If G is an abstract graph (i.e., finitely many vertices, no loops or multiple edges), we define the *topological realization of* G to be the topological graph $T(G)$ that has one isolated point for each isolated vertex of G; and otherwise is the union of arcs $A_{u,v}$, one for each doubleton set $\{u, v\}$ of adjacent vertices, where each arc $A_{u,v}$ has end points u and v, and no two such arcs intersect in any points other than end points. Clearly isomorphic graphs have homeomorphic topological realizations.

We say a G-cover $\mathcal{K}$ of a topological graph X is *sufficiently fine* if X is homeomorphic to $T(N(\mathcal{K}))$. It thus suffices to show that if X and Y are topological graphs that are G-equivalent, then X and Y respectively have sufficiently fine G-covers whose nerves are isomorphic.

Given a Hausdorff space X and a point $a \in X$, we define the *order of a in X*, to be the least cardinal number α such that a has a neighborhood base of open sets with boundaries of cardinality at most α. All points in totally disconnected compacta have order 0; in locally connected compacta the points of order 0 are the isolated points. A point of order 1 (resp., of finite order ≥ 3) in X is called an *end point* (resp., a *branch point*) of X.

Clearly, if X is a topological graph, then all its points have finite order, and only finitely many of them have order different from 2. And if v is a vertex in an abstract graph G, then the *degree* of v in G—i.e., the number of vertices of G adjacent to v—is the same as the order of v when considered as a point in $T(G)$.

For n any positive integer, an *n-od* is the union of n arcs, all intersecting at one common end point, called the *center* of the n-od. An n-od may also be described as the cone over a discrete set of cardinality n; a 3-od is commonly called a *triod*. A topological graph is called a *star* if it is an n-od for some $n \geq 3$. Note that the center of a star is topologically unique, as the only point of order ≥ 3. The following definition is inspired by the old Tinker Toy sets of childhood. A G-cover $\mathcal{K}$ of a topological graph X is called *proper* if the following six conditions hold.

(7) $\mathcal{K}$ consists of singletons, arcs, and stars.
(8) The branch points of X are the centers of the stars in $\mathcal{K}$.
(9) Any two stars are disjoint, as are any two arcs intersecting the same star.
(10) Every end point of X is contained in a unique arc in $\mathcal{K}$, called an *end*; no end in $\mathcal{K}$ contains more than one end point of X.
(11) The arcs in $\mathcal{K}$ that contain no end points of X are called *connectors*; every connector in $\mathcal{K}$ intersects exactly two other members of $\mathcal{K}$.
(12) Every end in $\mathcal{K}$ intersects exactly one other member of $\mathcal{K}$, but not another end.

Lemma 3.5. *Every topological graph has a proper G-cover; every proper G-cover for a topological graph is sufficiently fine.*

Proof. If the topological graph X has isolated points, place a singleton in $\mathcal{K}$ for each of these. If A is a constituent arc of X (*à la* the definition of *topological graph*), we decompose A into $K_A \cup M_A \cup L_A$, a union of three arcs, joined end-to-end, where one end point of A is contained in K_A, the other in L_A. The "middle" arcs M_A are connectors in $\mathcal{K}$; an "outer" arc K_A becomes an end in $\mathcal{K}$ just in case the end point k_A of A, contained in K_A, is an end point of X. If k_A is a point of order $n \geq 2$ in X, $B_1, \ldots, B_{n-1}$ are the other constituent arcs of X sharing k_A as an end point, and $k_A = k_{B_i}$, $1 \leq i \leq n-1$, then we take the n-od $K_A \cup \bigcup_{i=1}^{n-1} K_{B_i}$ to be a set in $\mathcal{K}$. (If $n = 2$ we get a new connector; if $n \geq 3$ we get a star.) Clearly $\mathcal{K}$ is a proper G-cover of X.

For the second part of the proof, we use induction based on the number $b(X)$ of branch points of X. If $b(X) = 0$, then X is a finite disjoint union of singletons, arcs, and simple closed curves. Let $\mathcal{K}$ be a proper G-cover for X; and, for each component C of X, let $\mathcal{K}_C$ be $\{K \in \mathcal{K} : K \subseteq C\}$. Then $\mathcal{K}_C = \{K \in \mathcal{K} : K \cap C \neq \emptyset\}$, and hence $\mathcal{K}_C$ is a proper G-cover of C. We therefore lose no generality in assuming that X is connected and nondegenerate, hence either an arc or a simple closed curve. By (7), $\mathcal{K}$ consists only of arcs. If X itself is an arc, (10) tells us that $\mathcal{K}$ contains exactly two ends; so by (12), $\mathcal{K}$ consists of two disjoint ends and at least one connector. Thus $N(\mathcal{K})$ is a connected graph with at least three vertices, each vertex has degree either 1 or 2, and precisely two vertices have degree 1. $T(N(\mathcal{K}))$ must therefore be an arc. If X is a simple closed curve, then X has no end points, and (11) tells us that $\mathcal{K}$ consists of at least three connectors. So $N(\mathcal{K})$ is a connected graph where each vertex has degree 2. $T(N(\mathcal{K}))$ is therefore a simple closed curve..

Note that in the two cases above, it is easy to arrange the homeomorphism $h : T(N(\mathcal{K})) \to X$ in such a way that the image under h of the vertex $K \in N(\mathcal{K})$

is a member of the set K (i.e., $h(K) \in K$) for each $K \in \mathcal{K}$. So, for our induction step, assume $k \geq 0$ is fixed, and that the following induction hypothesis holds:

> If Y is a topological graph with $b(Y) \leq k$, and $\mathcal{M}$ is a proper G-cover of Y, then there is a homeomorphism $h : T(N(\mathcal{M})) \to Y$ such that $h(M) \in M$ for each $M \in \mathcal{M}$.

Now let X be a topological graph with $b(X) = k+1$, and suppose $\mathcal{K}$ is a proper G-cover of X. Fix $H \in \mathcal{K}$, where H is an n-od for some $n \geq 3$. Then—by (8,9,10)—$\{K \in \mathcal{K} : H \cap K \neq \emptyset\}$ consists of precisely n pairwise disjoint arcs $A_1, \ldots, A_n$. For $1 \leq i \leq n$, let $A_i = B_i \cup C_i$, where B_i and C_i are subarcs intersecting in a single point and C_i is disjoint from H. Let $\mathcal{K}' = (\mathcal{K} \setminus \{A_1, \ldots, A_n\}) \cup \{B_1, \ldots, B_n, C_1, \ldots, C_n\}$. Then $N(\mathcal{K}')$ is a graph-theoretic subdivision of $N(\mathcal{K})$, and hence $T(N(\mathcal{K}'))$ and $T(N(\mathcal{K}))$ are homeomorphic. Let $\mathcal{M} = \mathcal{K}' \setminus \{H\}$, with $Y = \bigcup \mathcal{M}$. Then Y is a topological graph, $b(Y) = k$, and $\mathcal{M}$ is a proper G-cover of Y. [This claim would be false if $\mathcal{M}$ were the result of taking H away from $\mathcal{K}$, and some A_i happened to be an end of $\mathcal{K}$.] By our induction hypothesis, we may pick a homeomorphism $h : T(N(\mathcal{M})) \to Y$ in such a way that $h(M) \in M$ for each $M \in \mathcal{M}$. In particular, because the vertex $B_i \in N(\mathcal{M})$ is an end point of the space $T(N(\mathcal{M}))$, $h(B_i)$ is the unique end point e_i of B_i that is a member of H, as well as an end point of Y, $1 \leq i \leq n$. Let $c \in H$ be the center of H. We may extend $h : T(N(\mathcal{M})) \to Y$ to $\overline{h} : T(N(\mathcal{K}')) \to X$ by setting $\overline{h}(H) = c$, and by mapping the open arc in $T(N(\mathcal{K}))$ joining (the points) H to B_i homeomorphically onto the open arc in X that joins c to e_i. This completes the induction, and the proof of the lemma. □

The next lemma is all that is left to establish graph categoricity.

Lemma 3.6. *Suppose X and Y are G-equivalent topological graphs. Then X and Y respectively have proper G-covers whose nerves are isomorphic.*

Proof. We begin by defining an *n-wheel* in a space Z to be a collection $\{H, S_1, \ldots, S_n\}$ of subcontinua of Z, all with nonempty interior, such that each *spoke* S_i intersects—but is not contained in—the *hub* H, and no two spokes intersect each other.

Let $\mathcal{K}$ be a proper G-cover of X, constructed as in the proof of Lemma 3.5, and let $\mathcal{L}$ be a perfect refinement of $\mathcal{K}$, obtained as follows:

- If $K \in \mathcal{K}$ is a singleton, then $\mathcal{L}_K = \{K\}$.
- If K is an arc, then $\mathcal{L}_K$ is a 2-wheel consisting of three arcs, joined end-to-end, and whose hub is disjoint from $\bigcup(\mathcal{K} \setminus \{K\})$.

- If K is an n-od, $n \geq 3$, then $\mathcal{L}_K$ is an n-wheel whose hub is an n-od that is disjoint from $\bigcup(\mathcal{K} \setminus \{K\})$, and whose spokes each intersect the hub in a single point.

Since Y G-dominates X, there is a perfect pair $\langle \mathcal{K}', \mathcal{L}' \rangle$ for Y and a perfect pair isomorphism $\langle f, g \rangle : \langle \mathcal{K}, \mathcal{L} \rangle \to \langle \mathcal{K}', \mathcal{L}' \rangle$. We are done once we show that $\mathcal{K}'$ is a proper G-cover of Y.

Let us first address the issue of singleton sets. Note that x is an isolated point of X if and only if $\{x\}$ is a member of any G-cover of X. So if $K \in \mathcal{K}$ is not a singleton, then, because $|\mathcal{L}_K| > 1$, $f(K)$ is not a singleton either. Thus, if Y has exactly m isolated points (and $\mathcal{K}'$ has exactly m singletons), then there must be at least m singletons in $\mathcal{K}$. But it is also the case that X G-dominates Y, enabling a reversal of the argument. Thus there must be exactly m isolated points in X, and m singletons in $\mathcal{K}$. From this it follows that $f(K)$ is a singleton in $\mathcal{K}'$ if and only if K is a singleton in $\mathcal{K}$.

We next deal with branch points. We know the branch points of X are the centers of the stars in $\mathcal{K}$, so let $H \in \mathcal{K}$ be a star, say an n-od for $n \geq 3$. Let $\mathcal{L}_H$ be the n-wheel $\{M, S_1, \ldots, S_n\}$, as specified above. We first show $g(M)$ contains a branch point of Y. Note that $\{g(M), g(S_1), \ldots, g(S_n)\}$ is an n-wheel whose union is $f(H)$. $g(M)$ may contain a branch point b of its own, in which case b is a branch point of Y. If $g(M)$ has no branch point itself, then—because subcontinua of topological graphs are also topological graphs [10]—it is either an arc or a simple closed curve. Since $n \geq 3$, there must be some $1 \leq i \leq n$ such that $g(S_i) \cap g(M)$ consists only of points of order 2 in $g(M)$. Let A be an arc in $g(M)$ which contains $g(S_i) \cap g(M)$ in its (relative) interior. Since $g(S_i)$ is not contained in $g(M)$, there must be an arc $B \subseteq g(S_i) \cup g(M)$ with one end point in A and the other not in A. Thus the connected topological graph $A \cup B$ has at least three end points, and must therefore have a branch point of its own somewhere in A. Thus $g(M)$ still contains a branch point of Y.

So if H is any star in $\mathcal{K}$ and M_H is the hub of $\mathcal{L}_H$, then there is a point $b_H \in g(M_H)$ that is a branch point of Y. Moreover, if H_1 and H_2 are any two stars in $\mathcal{K}$, then they are disjoint, so $b_{H_1} \neq b_{H_2}$. Thus there are at least as many branch points in Y as there are in X; and, since X also G-dominates Y, we conclude that the numbers of branch points in X and in Y are equal. In particular, b_H is the only branch point of Y that is contained in $f(H)$, and all the branch points of Y are of the form b_H for some star $H \in \mathcal{K}$.

Now suppose $A \in \mathcal{K}$ is an arc. A contains no branch point of X, by (8); so, as above, neither does $f(A)$, which is therefore either an arc or a simple closed curve. But $f(A)$ is also decomposable as the 2-wheel $\{g(M), g(S_1), g(S_2)\}$, where $\mathcal{L}_A = \{M, S_1, S_2\}$, and this forces $f(A)$ to be an arc. If $A \in \mathcal{K}$ contains an end point e of X, suppose $e \in S_1$. Then M is the only set in $\mathcal{L}$ intersecting S_1; hence $g(M)$ is the only set in $\mathcal{L}'$ intersecting $g(S_1)$. Since $f(A)$ is an arc, so are its nondegenerate subcontinua; hence $g(S_1)$ contains an end point of Y. Then, because X also G-dominates Y, we infer that $K \in \mathcal{K}$ is an end in $\mathcal{K}$ if and only if $f(K) \in \mathcal{K}'$ is an arc in $\mathcal{K}'$ that contains a unique end point of Y. This argument also tells us that if A is a connector in $\mathcal{K}$, then $f(A)$ is an arc in $\mathcal{K}'$ that has degree 2 in $N(\mathcal{K}')$ and contains no end (or branch) point of Y.

What is left to prove is that $f(H)$ is a star in $\mathcal{K}'$ whenever H is a star in $\mathcal{K}$. What we already know is that $f(H)$ is a topological graph with a unique branch point, so suppose H is an n-od, $n \geq 3$. Let $A_1, \ldots, A_n$ be the n pairwise disjoint arcs in $\mathcal{K}$ that intersect H. Let $\mathcal{L}_H = \{M, S_1, \ldots, S_n\}$; and for $1 \leq i \leq n$, Let $\mathcal{L}_{A_i} = \{M_i, S_{i1}, S_{i2}\}$. Assume that S_{i1} is the subarc of A_i that shares an end point with S_i $1 \leq i \leq n$. Since each $f(A_i)$ is an arc, so too are the members of the 2-wheels $\{g(M_i), g(S_{i1}), g(S_{i2})\}$. As discussed above, $g(S_i)$ must contain an end point of $g(S_{i1})$, otherwise we introduce a second branch point in $f(H)$. Similarly, $g(M_i)$, disjoint from $g(S_i)$, must contain the other end point of $g(S_{i1})$. In particular, exactly one end point of the arc $g(S_{i1})$ is contained in $f(H)$, and $f(H) \cup \bigcup_{i=1}^{n} g(S_{i1})$ is a topological graph with at least n end points and exactly one branch point b_H. We claim that the order of b_H is $\geq n$, and we argue this combinatorially.

Sublemma. *Suppose G is a connected abstract graph with n end vertices (i.e., of degree 1) and exactly one vertex of degree $m \geq 3$. Then $m \geq n$. And if $m = n$, then the graph is a tree.*

Proof of sublemma. Let v be the unique vertex in G with degree $d(v) = m \geq 3$, and let $\{v_1, \ldots, v_m\}$ be the set of vertices adjacent to v. Our proof is by induction on the number k of additional edges in G. If $k = 0$, then clearly $m = n$; so suppose G has $k+1$ additional edges. Let E be any one of these, represented by its doubleton set $\{u, w\}$ of terminal vertices. If E is part of a cycle in G, let G' result by removing E from G. (Vertices remain; we just declare u and w to be nonadjacent.) Then G' is connected and has k additional edges and $n+2$ end vertices. By our induction hypothesis, we have $m \geq n+2 > n$. If E is not part of a cycle, then removal of E decomposes G into two connected parts G' and G'', say with G' containing the vertex v.

(G'' may be empty if E is an end arc.) Since G'' is connected with no branch vertices, the number of end vertices in G' is n. Since we have lowered the number of extra edges by at least 1, our induction hypothesis still tells us that.

So if G is a connected abstract graph with exactly one vertex of degree $n \geq 3$, with the number of end vertices being n as well, and if there is a cycle in G, then the removal of one edge—as described above—increases the number of end vertices without destroying connectedness. This contradiction ensures that G must be a tree. □

We have established that if $H \in \mathcal{K}$ is an n-od, then $f(H)$ is a topological graph that contains exactly one branch point b_H, and that point has order $m \geq n$. By the second clause of the sublemma; if we can show $m = n$, then we may infer that $f(H)$ is an n-od

First note that if y is any point of order $k \geq 3$ in Y, then, because X G-dominates Y, there must be a point y^* of order $\geq k$ in X; moreover, this assignment $y \mapsto y^*$ is a one-to-one function between finite sets, and is therefore a bijection. There is nothing to prove if there are no branch points in X; so suppose n_1 is the maximal order of a point in X, say $H \in \mathcal{K}$ is an n_1-od, $n_1 \geq 3$. If the order of b_H is $m > n_1$, then the point b_H^* has order $> n_1$ in X, a contradiction. Thus $m = n_1$ in this case. The converse also holds, so we know that b_H has order n_1 in Y if and only if H is an n_1-od in $\mathcal{K}$, $n_1 \geq 3$.

If there are no branch points other than those of maximal order n_1, we are done. Otherwise, suppose $3 \leq n_2 < n_1$, where there are branch points in X of order n_2, but none of any order $n_2 < k < n_1$. Let $H \in \mathcal{K}$ now be an n_2-od. Then the order of b_H in Y is some m, $n_2 \leq m < n_1$. But then the order of b_H^* in X is some k, $n_2 \leq m \leq k < n_1$; hence $k = m = n_2$. We thus come to the same conclusion as we did in the last paragraph: b_H has order n_2 in Y if and only if H is an n_2-od in $\mathcal{K}$, $n_2 \geq 3$.

This procedure may be continued for as long as necessary, treating ever smaller orders of branch points until they run out. We therefore conclude that the order of b_H in Y equals the order of the center of H for every star $H \in \mathcal{K}$. This shows that $f(H)$ is an n-od, and completes the proof of the graph categoricity theorem. □

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Received March 8, 2009

Revised version received September 28, 2009

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