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Characterizations of Distributions of Ratio of Certain Independent Random Variables

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CHARACTERIZATIONS OF DISTRIBUTIONS OF RATIOS OF CERTAIN INDEPENDENT RANDOM VARIABLES

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Abstract

Various characterizations of the distributions of the ratio of two independent gamma and exponential random variables as well as that of two independent Weibull random variables are presented. These characterizations are based, on a simple relationship between two truncated moments ; on hazard function ; and on functions of order statistics.

Mathematics Subject Classification 2000: 46F10

1. INTRODUCTION

As pointed out by Nadarajah and Kotz (2006), the distribution of U / V for independent random variables U and V have been studied by several authors when these random variables come from the same family of distributions (see their pertinent references). Recently, Shakil and Ahsanullah (2011) also pointed out that the distribution of the ratio of independent random variables arises in many fields of studies such as biology, economics, engineering, genetics and order statistics, to name a few. For the detailed explanation of the importance of the distribution of the ratio of independent random variables, we refer the interested reader to Shakil and Ahsanullah (2011) where they consider the distributional properties of record values of the ratio of independent Rayleigh random variables. Nadarajah (2010) studied the distributional properties as well as estimation of the ratio of independent Weibull random variables. For the detailed discussion, domain of applicability and practical examples we refer the interested reader to Nadarajah (2010). A more interesting case, however, is when the random variables have different distributions. Nadarajah and Kotz (2006) considered the case when the independent random variables U and V have different but similar distributions (gamma and Weibull) and obtained the exact distribution of the ratio U / V. The goal of the present work is to establish various characterizations of certain sub-family of distributions of the ratios given in Nadarajah and Kotz (2006) and Nadarajah (2010).

The problem of characterizing a distribution is an important problem which has recently attracted the attention of many researchers. Thus, various characterizations have been established in many different directions. In Section 2, we present characterizations of the distributions of the ratios of two independent gamma and exponential random variables as well as two Weibull random variables in different

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directions. Our results in subsections 2.1-2.3, will be based: on two truncated moments ; on hazard function ; and on truncated moments of certain functions of order statistics, respectively.

Let U and V have a gamma and an exponential distributions respectively. The *pdf 's* (probability density functions) f_U and f_V are given respectively by

$$f_U(u) = \frac{\mu^\alpha}{\Gamma(\alpha)} u^{\alpha - 1} e^{-\mu u}, \quad u > 0$$

and

$$f_V(v) = \lambda e^{-\lambda v}, \quad v > 0$$

where α , μ and λ are positive parameters.

The *pdf* f and *cdf* (cumulative distribution function) F of the distribution of the ratio, $X = \frac{U}{V}$, of two independent gamma and exponential random variables U and V (given above) are respectively (see Nadarajah and Kotz (2006))

$$f(x) = f(x;\alpha,\gamma) = \alpha\,\gamma\, x^{\alpha - 1} (x+\gamma)^{-(\alpha+1)}, \quad x > 0, \tag{1.1}$$

and

$$F(x) = x^\alpha (x+\gamma)^{-\alpha}, \quad x \geq 0, \tag{1.2}$$

where $\gamma = \frac{\lambda}{\mu}$ reducing the number of parameters which is desirable as far as characterization results are concerned. We denote the random variable X with *cdf* (1.2) by *RGE*.

Let U and V have Weibull distributions with *pdf 's* given respectively by

$$f_U(u) = \beta\,\lambda\, u^{\beta - 1} e^{-\lambda u^\beta}, \quad u > 0$$

and

$$f_V(v) = \beta\,\mu\, v^{\beta - 1} e^{-\mu v^\beta}, \quad y > 0$$

where β , λ and μ are positive parameters.

The *pdf* f , *cdf* F and hazard function ξ_F of the ratio, $X = \frac{U}{V}$, of two independent Weibull random variables U and V (given above) are respectively (see Nadarajah and Kotz (2006))

$$f(x) = f(x;\beta,\lambda,\mu) = \beta\lambda\mu\, x^{\beta\,-\,1}\left(\lambda x^{\beta} + \mu\right)^{-2}, \quad x > 0\,, \tag{1.3}$$

$$F(x) = \lambda x^{\beta}\left(\lambda x^{\beta} + \mu\right)^{-\,1}, \quad x \geq 0\,, \tag{1.4}$$

and

$$\xi_F\ (x) = \beta\lambda\, x^{\beta\,-\,1}\left(\lambda x^{\beta} + \mu\right)^{-1}, \quad x > 0\,. \tag{1.5}$$

We denote the random variable X with *cdf* (1.4) by RWW.

2. Characterization Results

As pointed out in the introduction, the distribution of the ratio of independent random variables has applications in many fields of study. So, an investigator will be vitally interested to know if their model fits the requirements of RGE or RWW distributions. To this end, the investigator relies on characterizations of these distributions, which provide conditions under which the underlying distribution is indeed that of RGE or of RWW. In this section we will present several characterizations of these distributions.

Throughout this section we assume, where necessary, that the distribution function F is twice differentiable on its support.

2.1. Characterization based on two truncated moments

In this subsection we present characterizations of RGE and RWW distributions in terms of truncated moments. We like to mention here the works of Galambos and Kotz (1978), Kotz and Shanbahag (1980), Glänzel (1987, 1988, 1990), Glänzel et al. (1984, 1994), Glänzel and Hamedani (2001) and Hamedani (1993, 2002, 2006) in this direction. Our characterization results presented here will employ an interesting result due to Glänzel (1987) (Theorem G below).

Theorem G. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $a < b$ $(a = -\infty\ ,\ b = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let g and h be two real functions defined on H such that

$$\mathbf{E}\left[g(X)\ \mid X \geq x\right] = \mathbf{E}\left[h(X)\ \mid X \geq x\right]\eta(x), \qquad x \in H\ ,$$

is defined with some real function η . Assume that g , $h \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H . Finally, assume that the equation $h\eta = g$ has no real solution in the interior of H . Then F is uniquely determined by the functions g , h and η , particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)h(u) - g(u)} \right| \exp(-s(u))\ du ,$$

where the function s is a solution of the differential equation $s' = \frac{\eta'\ h}{\eta\ h\ -\ g}$ and C is a constant, chosen to make $\int_H dF = 1$.

Remarks 2.1.1. (a) In Theorem G, the interval H need not be closed. (b) The goal is to have the function η as simple as possible. For a more detailed discussion on the choice of η , we refer the reader to Glänzel and Hamedani (2001) and Hamedani (1993, 2002, 2006).

Proposition 2.1.2. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h(x) = x^{-(\alpha-1)}$ and $g(x) = x^{-2\alpha}(x+\gamma)^{\alpha+1}$ for $x \in (0,\infty)$. The *pdf* of X is (1.1) if and only if the function η defined in Theorem G has the form

$$\eta(x) = \left(1 + \gamma x^{-1}\right)^{\alpha} , \quad x > 0 .$$

Proof. Let X have *pdf* (1.1) , then

$$(1 - F(x))\ \mathbf{E}[h(X) \mid X \geq x] = \gamma (x+\gamma)^{-\ \alpha} , \qquad x > 0 ,$$

and

$$(1 - F(x))\ \mathbf{E}[g(X) \mid X \geq x] = \gamma x^{-\ \alpha}, \qquad x > 0 ,$$

and finally

$$\eta(x) h(x) - g(x) = -\gamma x^{-2\ \alpha} (x+\gamma)^{\alpha} < 0 , \quad for \ \ x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\ \ h(x)}{\eta(x)\ \ h(x) - g(x)} = \ \alpha (x+\gamma)^{-1} , \qquad x > 0 ,$$

and hence

$$s(x) = \ln((x+\gamma)^{\alpha}).$$

Now, in view of Theorem G (with $C = \gamma$) , X has *cdf* (1.2) and *pdf* (1.1) .

Corollary 2.1.3. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h(x) = x^{-(\alpha-1)}$ for $x \in (0, \infty)$. The *pdf* of X is (1.1) if and only if there exist functions g and η defined in Theorem G satisfying the differential equation

$$\frac{\eta'(x)\ x^{-(\alpha-1)}}{\eta(x)\ x^{-(\alpha-1)} - g(x)} = \alpha (x+\gamma)^{-1}, \qquad x > 0.$$

Remark 2.1.4. The general solution of the differential equation given in Corollary 2.1.3 is

$$\eta(x) = \ (x+\gamma)^{\alpha} \left[- \int g(x)\ \alpha\ x^{\alpha\ -\ 1} (x+\gamma)^{-(\alpha+1)}\, dx + D \right],$$

for $x > 0$, where D is a constant. One set of appropriate functions is given in Proposition 2.1.2 with $D = 0$.

Proposition 2.1.5. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h(x) = x^{-\ (\beta+1)} \left(\lambda x^{\beta} + \mu\right)^2$ and $g(x) = x^{-2\beta} \left(\lambda x^{\beta} + \mu\right)^2$ for $x \in (0, \infty)$. The *pdf* of X is (1.3) with $\beta \neq 1$ if and only if the function η defined in Theorem G has the form

$$\eta(x) = \frac{1}{\beta} x^{1-\beta}\ , \quad x > 0\ .$$

Proof. Let X have *pdf* (1.3) , then

$$(1 - F(x))\ \mathbf{E}[h(X)\ |\ X \geq x] = \beta \lambda \mu x^{-1}\ , \qquad x > 0\ ,$$

and

$$(1 - F(x))\ \mathbf{E}[g(X)\ |\ X \geq x] = \lambda \mu x^{-\ \beta}, \qquad x > 0\ ,$$

and finally for $\beta \neq 1$

$$\eta(x) h(x) - g(x) = \frac{(1-\beta)}{\beta} x^{-2\ \beta} \left(\lambda x^{\beta} + \mu\right)^2 \neq 0\ , \quad for\ \ x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\ h(x)}{\eta(x)\ h(x) - g(x)} = x^{-1}, \qquad x > 0\ .$$

Integrating both sides of the above equation from $x_0 > 0$ to x we have

$$s(x) = \ln(\frac{x}{x_0}),$$

for which, without loss of generality, we take $s(x_0) = 0$. Now, in view of Theorem G (with $C = x_0 = \beta\lambda\mu$), X has *cdf* (1.4) and *pdf* (1.3) .

Corollary 2.1.6. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h(x) = x^{-(\beta+1)}\left(\lambda x^{\beta} + \mu\right)^2$ for $x \in (0, \infty)$. The *pdf* of X is (1.3) if and only if there exist functions g and η defined in Theorem G satisfying the differential equation

$$\frac{\eta'(x)\ x^{-(\beta+1)}\left(\lambda x^{\beta} + \mu\right)^2}{\eta(x)\ x^{-(\beta+1)}\left(\lambda x^{\beta} + \mu\right)^2 - g(x)} = \frac{1}{x}, \qquad x > 0.$$

Remark 2.1.7. The general solution of the differential equation given in Corollary 2.1.6 is

$$\eta(x) = x\left[-\int g(x)\ x^{\beta-1}\ \left(\lambda x^{\beta} + \mu\right)^{-2} dx + D\right],$$

for $x > 0$, where D is a constant. One set of appropriate functions is given in Proposition 2.1.5 with $D = 0$.

Remark 2.1.8. Clearly there are other triplet functions (h, g, η) satisfying conditions of Proposition 2.1.2 or those of Proposition 2.1.5.

2.2. Characterization based on hazard function

For the sake of completeness, we state the following simple fact.

Let F be an absolutely continuous distribution with the corresponding *pdf* f . The hazard function corresponding to F is

$$\xi_F(x) = \frac{f(x)}{1 - F(x)}\ , \qquad x \in Supp\ F\ , \tag{2.2.1}$$

where $Supp\ F$ is the support of F .

It is obvious that the hazard function of a twice differentiable distribution function satisfies the first order differential equation

$$\frac{\xi_F'(x)}{\xi_F(x)} - \xi_F(x) = k(x) \ , \tag{2.2.2}$$

where $k(x)$ is an appropriate integrable function. Although this differential equation has an obvious form since

$$\frac{f'(x)}{f(x)} = \frac{\xi_F'(x)}{\xi_F(x)} - \xi_F(x)\,\alpha \ ,$$

for many univariate continuous distributions (2.2.2) seems to be the only differential equation in terms of the hazard function. The goal here is to establish a differential equation which has as simple form as possible and is not of the trivial form (2.2.2). For some general families of distributions this may not be possible. Here is our characterization result for RWW distribution.

Proposition 2.2.1. Let $X : \Omega \to (0,\infty)$ be a continuous random variable. The *pdf* of X is (1.3) if and only if its hazard function ξ_F satisfies the differential equation

$$\xi_F'(x) - (\beta - 1)\ x^{-1}\xi_F(x) + \xi_F^2(x) = 0, \quad x > 0. \tag{2.2.3}$$

Proof. If X has *pdf* (1.3) , then obviously (2.2.3) holds. If ξ_F satisfies (2.2.3) , then

$$\frac{d}{dx}\left(x^{\beta-1}\ (\ \xi_F(x))^{-1}\right) = x^{\beta-1} \ ,$$

or after integrating both sides with respect to x from $x_0 > 0$ to x and some grouping of terms, we arrive at

$$\xi_F(x) = \frac{f(x)}{1-F(x)} = \frac{\beta x^{\beta-1}}{x^\beta + C} \ ,$$

where C is a constant. Upon another integration with respect to x from 0 to x , we obtain (1.4) with $C = \frac{\mu}{\lambda}$.

Remark 2.2.2. For characterizations of other well-known continuous distributions based on hazard function, we refer the reader to Hamedani (2004) and Hamedani and Ahsanullah (2005) .

2.3. Characterization based on truncated moments of certain functions of order statistics

Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be n order statistics from a continuous *cdf* F. We present here characterization results base on some functions of these order statistics. We refer the reader to Ahsanullah and Hamedani (2007) , Hamedani et al. (2008) and Hamedani (2010) , among others , for characterizations of other well-known continuous distributions in this direction. The proof of the following proposition is similar to that of Theorem 2.2 of Hamedani (2010) in which $k(x) = x^{\gamma}$ for some $\gamma > 0$. We give a brief proof, however, for the sake of completeness.

Proposition 2.3.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable with *cdf* F and $k(x)$ be a differentiable function such that $\lim_{x\to\infty} k(x)(1 - F(x))^n = 0$. Let $g(x, n)$ be a real-valued function which is differentiable with respect to x on $[0, \infty)$ and $\int_0^\infty \frac{k'(x)}{g(x,n)} dx = \infty$. Then

$$\mathbf{E}\left[k\left(X_{1:n}\right) \mid X_{1:n} > t\right] = k(t) + g(t, n) \ , \quad t > 0 \ , \tag{2.3.1}$$

implies that

$$F(x) = 1 - \left(\frac{g(0, n)}{g(x, n)}\right)^{\frac{1}{n}} e^{-\int_0^x \frac{k'(t)}{n\ g(t,n)} dt} \ , \quad x \geq 0.$$

Proof. Condition (2.3.1) and the assumption $\lim_{x\to\infty} k(x)(1 - F(x))^n = 0$ imply that

$$\int_t^\infty k'(x)(1 - F(x))^n dx = g(t, n)(1 - F(t))^n . \tag{2.3.2}$$

Differentiating (2.3.2) with respect to t, we have

$$\begin{aligned} & -k'(t)(1 - F(t))^n \\ = & \left[\frac{\partial}{\partial t} g(t, n)(1 - F(t)) - n\ g(t, n)\ f(t)\right](1 - F(t))^{n-1} \ , \end{aligned}$$

from which

$$\frac{f(t)}{1 - F(t)} = \frac{\frac{\partial}{\partial t} g(t, n)}{n\ g(t, n)} + \frac{k'(t)}{n\ g(t, n)}. \tag{2.3.3}$$

Integrating (2.3.3) with respect to t , from 0 to x , results in

$$F(x) = 1 - \left(\frac{g(0,n)}{g(x,n)}\right)^{\frac{1}{n}} e^{-\int_0^x \frac{k'(t)}{n\ g(t,n)}dt}, \qquad x \geq 0.$$

Remarks 2.3.2. (i) For $k(x) = x^{\beta}$ and $g(x,n) = \frac{1}{(n-1)\lambda}\left(\lambda x^{\beta} + \mu\right)$, Proposition 2.3.1 gives a characterization of (1.4). (ii) For $\alpha = 1$, $h(x) = (x+\gamma)^{n-1}$ and $g(x,n) = (n-1)(x+\gamma)^{n-1}$, Proposition 2.3.1 gives a characterization of (1.2).

Let X_j, $j = 1,2,...,n$ be n *i.i.d.* random variables with *cdf* F and corresponding *pdf* f and let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be their corresponding order statistics. Let $X^*_{1:n-i+1}$ be the $1st$ order statistic from a sample of size $n-i+1$ of random variables with *cdf* $F_t(x) = \frac{F(x)-F(t)}{1-F(t)}$, $x \geq t$ (t is fixed) and corresponding *pdf* $f_t(x) = \frac{f(x)}{1-F(t)}$, $x \geq t$. Then

$$(X_{i:n} \mid X_{i-1:n} = t) \stackrel{d}{=} X^*_{1:n-i+1} \quad (\stackrel{d}{=} \textit{ means equal in distribution}),$$

that is

$$f_{X_{i:n}\ \mid\ X_{i-1:n}}(x|t) = f_{X^*_{1:n-i+1}}(x) = (n-i+1)\left(1-F_t(x)\right)^{n-i}\ \frac{f(x)}{1-F(t)},\ x \geq t\ .$$

Now we can state the following characterizations of REE (RGE with $\alpha = 1$) and RWW distributions in yet somewhat different direction. The proofs are similar to that of Propositions 2.3.1 and hence will be omitted.

Corollary 2.3.3. Let $X : \Omega \to (0,\infty)$ be a continuous random variable with *cdf* F. Then

(i)

$$\mathbf{E}\left[(X_{i:n}+\gamma)^{n-i} \mid X_{i-1:n} = t\right] = (t+\gamma)^{n-i} + (n-i)\ (t+\gamma)^{n-i}, \quad t > 0,$$

for $n > i$ and some $\gamma > 0$ if and only if X has *cdf* (1.2);

(ii)

$$\mathbf{E}\left[X^{\beta}_{i:n} \mid X_{i-1:n} = t\right] = t^{\beta} + \frac{1}{(n-i)\lambda}\ \left(\lambda t^{\beta} + \mu\right), \quad t > 0,$$

for $n > i$ and some $\beta > 0$, $\lambda > 0$, $\mu > 0$ if and only if X has *cdf* (1.4).

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