1-1-2009

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Published version. *Houston Journal of Mathematics*, Vol. 35, No. 4 (2009): 1091-1102. Publisher Link.

Houston Journal of Mathematics
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Volume 35, No. 4, 2009

DENDRITES, TOPOLOGICAL GRAPHS, AND 2-DOMINANCE

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Communicated by Charles Hagopian

Abstract. For each positive ordinal α, the reflexive and transitive binary relation of α-dominance between compacta was first defined in [6] using the ultracopower construction. Here we consider the important special case $\alpha = 2$, and show that any Peano compactum 2-dominated by a dendrite is itself a dendrite (with the same being true for topological graphs and trees). We also characterize the topological graphs that 2-dominate arcs (resp., simple closed curves) as those that have cut points of order 2 (resp., those that are not trees).

1. introduction

In [6] we initiated the study of α-dominance between compacta (i.e., compact Hausdorff spaces), where α is a positive ordinal. Here we continue that study, focusing on the case $\alpha = 2$. The basic definition revolves around the topological ultracopower construction (see, e.g., [6]), which we briefly describe as follows: Given a compactum X and an ultrafilter $\mathcal{D}$ on a (discrete) set I, the *ultracopower of X via $\mathcal{D}$* is denoted $XI\backslash\mathcal{D}$, and is the inverse image of the point $\mathcal{D} \in \beta(I)$ under the Stone-Čech lift $q^\beta : \beta(X \times I) \to \beta(I)$ of the standard projection $q : X \times I \to I$. If $p : X \times I \to X$ is the other standard projection map, we also have the *ultracopower (codiagonal) projection* $p_{X,\mathcal{D}} : XI\backslash\mathcal{D} \to X$, defined by restricting $p^\beta : \beta(X \times I) \to X$ to the ultracopower. In the language of commutative mapping diagrams, this may be expressed as follows.

2000 *Mathematics Subject Classification.* Primary 54C05, 54F25, 54F50; Secondary 03C20, 54F55.

Key words and phrases. Peano compacta, Peano continua, dendrites, topological graphs, topological trees, ultracoproducts, co-existential maps, 2-dominance, 2-equivalence.

$$\begin{array}{ccccc} XI\backslash\mathcal{D} & \stackrel{\subseteq}{\longrightarrow} & \beta(X \times I) & \stackrel{q^\beta}{\longrightarrow} & \beta(I) \\ & \searrow p_{X,\mathcal{D}} & \downarrow p^\beta & & \\ & & X & & \end{array}$$

$p_{X,\mathcal{D}}$ is always a continuous surjection, and we define a mapping $f : X \to Y$ between compacta to be *co-existential* if there is an ultracopower $YI\backslash\mathcal{D}$ and a continuous surjection $g : YI\backslash\mathcal{D} \to X$ such that $f \circ g = p_{Y,\mathcal{D}}$. Finally, we define a compactum Y to *2-dominate* a compactum X (in symbols, $X \leq_2 Y$) if X is a co-existential image of some ultracopower of Y. So to witness the 2-dominance of X by Y, there must be a commutative diagram such as the following.

$$\begin{array}{ccc} XJ\backslash\mathcal{E} & \stackrel{g}{\longrightarrow} & YI\backslash D \\ p_{X,\mathcal{E}} \downarrow & f \swarrow & \downarrow p_{Y,\mathcal{D}} \\ X & & Y \end{array}$$

A sufficient—but by no means necessary—condition for $X \leq_2 Y$ is that X be itself a co-existential image of Y. Another—and this speaks more to the choice of terminology—is that every universal-existential (Π_2^0) sentence holding for the closed-set lattice of Y also holds for the closed-set lattice of X.

A *continuum* is a connected compactum; by an *arc* (resp., a *simple closed curve*) we mean a homeomorphic copy of the usual closed unit interval in the real line (resp., the unit circle in the euclidean plane). Arcs are characterized (Theorem 6.17 in [12]) as being those metrizable continua that have exactly two noncut points, the minimum number allowed (Theorem 6.6 in [12]) for any nondegenerate continuum. These two points are the "end" points of the arc, in the sense of either of the two natural separation orders. Simple closed curves are characterized (Theorem 9.31 in [12]) as being those metrizable continua that become disconnected upon the removal of any two points. We use the adjective *Peano* in order to add the conditions of local connectedness and metrizability to whatever other properties are being considered. A *dendrite* is a Peano continuum that contains no simple closed curve as a subcontinuum; a *topological graph* is a Peano continuum that may be expressed as the finite union of arcs that are pairwise disjoint, except for the possible sharing of noncut points; a *topological tree* is a graph that is also a dendrite. (Because the words *graph* and *tree* mean so many different—but related—things in mathematics, we have been at pains to use the modifier *topological* when stating the definitions above. However we may safely drop this modifier in the sequel, as our meaning is constant throughout this paper.)

Our chief interest here concerns inheritance of topological properties when one compactum is 2-dominated by another. For example (see Remark 3.5 in [6]), if $X \leq_2 Y$ and Y is totally disconnected (resp., an indecomposable continuum, a hereditarily indecomposable continuum, a compactum of covering dimension $\leq n$, a continuum of multicoherence degree $\leq n$), then so is X. These properties are then said to be *2-inherited*.

One of the main results of [6] is Theorem 3.6, significantly improving on Theorem 0.6 in [2], is a *relative* 2-inheritance statement, saying that any Peano compactum 2-dominated by an arc (resp., a simple closed curve) is itself an arc (resp., a simple closed curve). (Both assumptions inherent in the word *Peano* are necessary.) In the present paper we go on to prove the following:

(1) Any Peano compactum 2-dominated by a dendrite (resp., graph, tree) is itself a dendrite (resp., graph, tree).
(2) The graphs that 2-dominate an arc (resp., a simple closed curve) are precisely those with cut points that have order two (resp., those that are not trees).

2. MAIN RESULTS

Given a Hausdorff space X and a point $a \in X$, we define $\text{ord}(a, X)$, the *order of a in X*, to be the least cardinal number α such that a has a neighborhood base of open sets with boundaries of cardinality at most α. Clearly, when X is locally connected, $\text{ord}(a, X) = 0$ if and only if a is an isolated point; a point $a \in X$ is called an *end point* (resp., a *branch point*) if $\text{ord}(a, X) = 1$ (resp., $\text{ord}(a, X) \geq 3$).

A tree T is called a *simple n-od*, $1 \leq n < \omega$, if it is the cone over an n-point discrete space; i.e., if it is the union of n arcs, pairwise disjoint except for one common end point a, called the *vertex*. (A simple 3-od is also called a *simple triod*.) The end points of T (aside from a, in case $n = 1$) are the other end points of the contributing arcs; $\text{ord}(a, T) = n$. $\text{ord}(b, T) \leq 2$ for all $b \in T \setminus \{a\}$.

A key ingredient in the proof of Theorem 3.6 in [6] (as well as its predecessor in [2]) is the following classic 1920s result of R. L. Moore.

Theorem 2.1 ([10], also Exercise 8.40 in [12])**.** *A Peano continuum that contains no simple triod is either an arc or a simple closed curve.*

Clearly, if a is the vertex of a simple n-od in the Hausdorff space X, then $\text{ord}(a, X) \geq n$. What is a far less trivial observation is the following theorem of K. Menger (one form of his *n-Beinsatz*, see, e.g., [9, 13], which will be useful in proving Theorem 2.11 below).

Theorem 2.2. *Let X be a locally connected locally compact metrizable space, with $1 \leq n < \omega$. If $a \in X$ has order $\geq n$ in X, then X contains a simple n-od with vertex a.*

An open set U in a topological space X is called an *open free arc in X* if there is an arc $A \subseteq X$, with end points b and c, such that $U = A \setminus \{b, c\}$. Clearly, if U is an open free arc in X and $a \in U$, then $\mathrm{ord}(a, X) = 2$.

Theorem 2.3. *Let U be an open free arc in a Peano continuum X. Then the compactum $X \setminus U$ has either one component or two. In the first (resp., second) case, there is a co-existential map from X onto a simple closed curve (resp., an arc).*

Proof. Let X be a Peano continuum with $U \subseteq X$ an open free arc. We fix an arc $A \subseteq X$, with end points b and c, such that $U = A \setminus \{b, c\}$. Because the closure of U in X is A, $X \setminus U$ cannot be the union of three pairwise disjoint closed nonempty subsets, without disconnecting X. Thus $X \setminus U$ has at most two components.

Suppose first there are two components, say $X \setminus U = B \cup C$, where B and C are disjoint nonempty subcontinua. The end points of A must clearly lie in different components; say $b \in B$ and $c \in C$.

We need to know that B and C are Peano continua; and for this it suffices to check local connectedness at the points b and c (since $B \setminus \{b\}$ and $C \setminus \{c\}$ are X-open sets). So, given an X-open neighborhood V of b, we need to find an X-open neighborhood W of b such that $W \cap B \subseteq V \cap B$, and $W \cap B$ is connected. Using local connectedness of X, find connected X-open neighborhood W of b contained in V. Then $(W \cap (A \cup C)) \setminus \{b\}$ and $(W \cap B) \setminus \{b\}$ are complementary clopen subsets of $W \setminus \{b\}$. Thus, because both W and $\{b\}$ are connected, we infer by a standard argument (see Proposition 6.3 in [12]) that $W \cap B = ((W \cap B) \setminus \{b\}) \cup \{b\}$ is connected too.

The argument for the local connectedness of C being identical to that just given, we now have X decomposed into a union $B \cup A \cup C$ of three Peano subcontinua, where A is an arc, $B \cap A$ and $C \cap A$ are the two end points of A ($\{b\}$ and $\{c\}$, respectively), and $B \cap C = \emptyset$.

We may as well assume both B and C are nondegenerate; otherwise we need do nothing. Then, since they are both arcwise connected, we may fix arcs $A_B \subseteq B$ and $A_C \subseteq C$ so that b (resp., c) is an end point of A_B (resp., A_C). Then

$A_B \cup A \cup A_C$ is an arc. By the Hahn-Mazurkiewicz theorem (Theorem 8.14 in [12]), there is a continuous surjection $f : A_B \cup A \cup A_C \to X$ that fixes the points of U; and maps A_B onto B and A_C onto C, in such a way that both b and c are fixed.

We now define $g : X \to A$ to be the map that fixes the points of A and collapses the points of B (resp., C) to b (resp., c). Then clearly $g \circ f : A_B \cup A \cup A_C \to A$ is a continuous surjection from one arc to another that is monotone (i.e., point-inverses are connected); hence (Proposition 2.7 in [3]) $g \circ f$ is a co-existential map. By Proposition 2.2 in [4], g is a co-existential map as well.

In the event $B = X \setminus U$ is connected, an argument similar to that above shows B to be a Peano continuum. It is safe to assume B is nondegenerate; hence we may fix an arc A_B in B with end points b and c. Then $A_B \cup A$ is a simple closed curve. Again using the Hahn-Mazurkiewicz theorem, there is a continuous surjection $f : A_B \cup A \to X$ that fixes the points of U; and maps A_B onto B, in such a way that both b and c are fixed.

Let Y be the simple closed curve that results from X when the points of B are identified to a single point, and let $g : X \to Y$ be the identification mapping. Then $g \circ f : A_B \cup A \to Y$ is a monotone continuous surjection from one simple closed curve to another; hence (again by Proposition 2.7 in [3]) $g \circ f$ is a co-existential map. Thus g is co-existential too. $\square$

Consequently, every nondegenerate graph 2-dominates either an arc or a simple closed curve (or both). We will be able to sharpen this statement in the sequel (see Corollaries 2.17 and 2.18).

Recall that a continuum is *unicoherent* if it is incapable of decomposition into two subcontinua with disconnected intersection. (This means having *multicoherence degree zero*, in the sense if S. Eilenberg [12].) Continuum X is *hereditarily unicoherent* if each subcontinuum of X is unicoherent.

Theorem 2.4 (Theorem 10.35 in [12])**.** *A Peano continuum is a dendrite if and only if it is hereditarily unicoherent.*

In [5] we proved (Corollary 5.4) that co-existential maps cannot raise multicoherence degree; hence a co-existential image of a unicoherent continuum is unicoherent. This, together with Theorem 5.1 of [5] (which implies that multicoherence degree is preserved by the taking of ultracopowers), tells us that unicoherence is a 2-inherited property. *Almost* the same story can be told for hereditary unicoherence, but first we need to recall a result that has repeatedly proven to be of key importance in the study of co-existential maps.

Lemma 2.5 (Theorem 2.4 in [4])**.** *Let $f : X \to Y$ be a co-existential map between compacta. Then there is a $\cup$-semilattice homomorphism f^* from the subcompacta of Y to the subcompacta of X such that for each subcompactum K of Y:*

(i) *the restriction $f|f^*(K)$ is a co-existential map from $f^*(K)$ to K;*
(ii) *$f^{-1}[U]$ is contained in $f^*(K)$ whenever U is a Y-open set contained in K; and*
(iii) *$f^*(K)$ shares any property of K that is preserved by ultracopowers and continuous images (e.g., finiteness, connectedness).*

Corollary 2.6. *Co-existential maps preserve hereditary unicoherence in continua.*

Proof. We use Lemma 2.5, plus the fact that co-existential maps preserve unicoherence. For if $f : X \to Y$ is co-existential and X is a hereditarily unicoherent continuum, let $K \subseteq Y$ be any subcontinuum. Then $f^*(K)$ is a subcontinuum of X, and is hence unicoherent. Since $f|f^*(K)$ is also a co-existential map, we infer that K is unicoherent. □

Remarks 2.7. (1) From Corollary 2.6 and Theorem 2.4, co-existential maps preserve the property of being a dendrite. But this is already well known (Exercise 10.52 in [12]) because co-existential maps in this setting are monotone (Theorem 2.7 in [4]).

(2) This brings us to an explanation of why we used the word *almost* after the statement of Theorem 2.4 above. The obvious question is whether hereditary unicoherence is a 2-inherited property, and we do not as yet have an answer that does not involve local connectedness (see Theorem 2.8 below). The answer would be affirmative if we could show that hereditary unicoherence is preserved under the taking of ultracopowers, but we do not presently have a way to decide this. We could show preservation to *fail* if it were true that hereditary unicoherence is not preserved by limits of inverse systems with surjective bonding maps, in light of Corollary 2.6, as well as Corollary 1.3 in [5]. However, this latter preservation does hold: Closure of unicoherence under inverse limits follows from the work of S. B. Nadler, Jr. [11]; closure of hereditary unicoherence then follows from the easily-proved fact that if K is a subcontinuum of the inverse limit, then K is the inverse limit of the canonical images of K in the various factor continua.

Before we can prove the next theorem, we need to introduce some notation to facilitate a discussion of the anatomy of topological ultracopowers. From what we know of the Stone-Čech compactification, points of $\beta(Y)$, when Y is a normal topological space, may be viewed as ultrafilters in the closed-set lattice of Y. Thus if X is a compactum and I is a set (considered as a discrete space), then $X \times I$ is normal; and we may view (see, e.g., [1]) the points of the ultracopower $XI\backslash\mathcal{D}$ as those ultrafilters $P \in \beta(X \times I)$ such that $X \times J \in P$ for each $J \in \mathcal{D}$. If $\langle K_i : i \in I\rangle$ is an I-indexed family of closed subsets of X, then $\sum_{\mathcal{D}} K_i$ is notation for the set of $P \in XI\backslash\mathcal{D}$ such that $\bigcup_{i\in J}(K_i \times \{i\}) \in P$ for some $J \in \mathcal{D}$.

Theorem 2.8. *Any locally connected compactum 2-dominated by a hereditarily unicoherent continuum is itself a hereditarily unicoherent continuum.*

Proof. Suppose Y is a continuum, X is a locally connected compactum, and $X \leq_2 Y$. Then X is a continuum; assume it is not hereditarily unicoherent. Then we have subcontinua H and K of X such that $H \cap K = A_0 \cup A_1$, a union of two disjoint nonempty subcompacta of X. Since both $H \setminus K$ and $K \setminus H$ are nonempty, we fix points a in the first and b in the second. Next, using local connectedness, we fix a connected open neighborhood U_a of a such that the closure $\overline{U_a}$ misses K. Let $H' := H \cup \overline{U_a}$. Again using local connectedness, we fix a connected open neighborhood U_b of b such that $\overline{U_b}$ misses H', and let $K' := K \cup \overline{U_b}$. Then H' and K' are subcontinua of X, $H' \cap K' = A_0 \cup A_1$ (so the intersection is disconnected), and both $H' \setminus K'$ and $K' \setminus H'$ have nonempty interiors in X.

Since $X \leq_2 Y$, we may fix an ultrafilter $\mathcal{D}$ on an index set I and a co-existential map $f : YI\backslash\mathcal{D} \to X$. By Theorem 2.7 in [4], f is monotone; hence the taking of inverse images along f allows us to say the following about $YI\backslash\mathcal{D}$ (recycling notation, because we are done with X): There exist subcontinua H and K such that $H \cap K = A_0 \cup A_1$, a union of two disjoint nonempey subcompacta, and both $H \setminus K$ and $K \setminus H$ have nonempty interiors. The rest of the proof proceeds in a manner similar to that of Theorem 5.1 in [5].

Given $\langle y_i : i \in I\rangle$ in the cartesian power Y^I, the set $\sum_{\mathcal{D}}\{y_i\}$ is a singleton set, whose sole element we denote by $\sum_{\mathcal{D}} y_i$. Then, by basic results in [1], the set of such points is dense in $YI\backslash\mathcal{D}$. In light of this, we fix $\sum_{\mathcal{D}} x_i \in H \setminus K$ and $\sum_{\mathcal{D}} y_i \in K \setminus H$.

For each $k = 0, 1$, choose an open neighborhood U_k of A_k in such a way that the closures $\overline{U_k}$ are disjoint and miss both points $\sum_{\mathcal{D}} x_i$ and $\sum_{\mathcal{D}} y_i$. Let $R := H \setminus (U_0 \cup U_1)$ and $S := K \setminus (U_0 \cup U_1)$. Then $\sum_{\mathcal{D}} x_i \in R$, $\sum_{\mathcal{D}} y_i \in S$, and both R and S are subcompacta of $YI\backslash\mathcal{D}$. Moreover, R and S are disjoint because $R \cap S \subseteq (H \setminus K) \cap (K \setminus H)$.

For each $i \in I$, pick subcompacta $R_i, S_i \subseteq Y$ such that $R \subseteq \sum_{\mathcal{D}} R_i$, $S \subseteq \sum_{\mathcal{D}} S_i$, and $\sum_{\mathcal{D}} R_i \cap \sum_{\mathcal{D}} S_i = \emptyset$. R_i and S_i may be chosen disjoint for each $i \in I$;

so, in like fashion, we may choose disjoint subcompacta $A_{i,0}, A_{i,1} \subseteq Y$ such that $\overline{U_k} \subseteq \sum_{\mathcal{D}} A_{i,k}$ for $k = 0, 1$. Let $R_i^* := R_i \cup (A_{i,0} \cup A_{i,1})$ and $S_i^* := S_i \cup (A_{i,0} \cup A_{i,1})$. Then clearly $H \subseteq \sum_{\mathcal{D}} R_i^*$ and $K \subseteq \sum_{\mathcal{D}} S_i^*$. For each $i \in I$, let C_i (resp., D_i) be the component of R_i^* (resp., S_i^*) containing x_i (resp., y_i). Because components may be separated from disjoint subcompacta via clopen sets, one can prove easily that "ultracoproducts of components are components of the ultracoproduct;" i.e., that $\sum_{\mathcal{D}} C_i$ (resp., $\sum_{\mathcal{D}} D_i$) is the component of $\sum_{\mathcal{D}} R_i^*$ (resp., $\sum_{\mathcal{D}} S_i^*$) containing $\sum_{\mathcal{D}} x_i$ (resp., $\sum_{\mathcal{D}} y_i$).

Thus we have $H \subseteq \sum_{\mathcal{D}} C_i$ and $K \subseteq \sum_{\mathcal{D}} D_i$; therefore $A_0 \cup A_1 = H \cap K \subseteq \sum_{\mathcal{D}} C_i \cap \sum_{\mathcal{D}} D_i \subseteq \sum_{\mathcal{D}} R_i^* \cap \sum_{\mathcal{D}} S_i^* = \sum_{\mathcal{D}} A_{i,0} \cup \sum_{\mathcal{D}} A_{i,1}$. For $i \in I$, $k = 0, 1$, let $B_{i,k} := C_i \cap D_i \cap A_{i,k}$. Then $A_k \subseteq \sum_{\mathcal{D}} B_{i,k}$, so each $\sum_{\mathcal{D}} B_{i,k}$ is nonempty. Also $\sum_{\mathcal{D}} B_{i,0} \cup \sum_{\mathcal{D}} B_{i,1} = \sum_{\mathcal{D}} C_i \cap \sum_{\mathcal{D}} D_i \cap (\sum_{\mathcal{D}} A_{i,0} \cup \sum_{\mathcal{D}} A_{i,1}) = \sum_{\mathcal{D}} C_i \cap \sum_{\mathcal{D}} D_i$. From this it is immediate that

$$\{i \in I : C_i \text{ and } D_i \text{ are subcontinua of } Y \text{ and } C_i \cap D_i \text{ is disconnected}\} \in \mathcal{D};$$

therefore Y is not hereditarily unicoherent. $\square$

Putting Theorems 2.4 and 2.8 together immediately yields the following.

Corollary 2.9. *Any Peano compactum 2-dominated by a dendrite is itself a dendrite.*

Our next goal is an analogue of Corollary 2.9 for graphs and trees; first we need the following well-known characterization of graphs.

Theorem 2.10 (Theorem 9.10 in [12])**.** *A metrizable continuum is a graph if and only if all of its points have finite order, and only finitely many of them are branch points.*

We are now ready to prove the advertised analogue.

Theorem 2.11. *Any Peano compactum 2-dominated by a graph is itself a graph.*

Proof. Suppose G is a graph, X is a Peano compactum, and $X \leq_2 G$. Then X is a metrizable continuum; so (Theorem 2.10) what we need to show is that all the points of X have finite order, and only finitely many of them are branch points.

First suppose $\text{ord}(a, X)$ is infinite for some $a \in X$. Then for any fixed positive whole number n, there exists (Theorem 2.2) a simple n-od $T \subseteq X$ with vertex a. Using local connectedness in X, then, we can create a sequence $\langle H, S_1, \ldots, S_n \rangle$ of subcontinua of X such that:

(i) the *hub* H intersects each *spoke* S_i, $1 \leq i \leq n$;

(ii) no two spokes intersect;
(iii) each spoke has interior points not contained in the hub; and
(iv) there are interior points of the hub that are not contained in any spoke.

We refer to $\langle H, S_1, \ldots, S_n \rangle$ as a *fat n-wheel* in X (the word *fat* referring to the various nonempty interiors). Using an argument similar to (but easier than) that given in the proof of Theorem 2.8, we may use this fat n-wheel in X to create a fat n-wheel $\langle H', S'_1, \ldots, S'_n \rangle$ in G. Let $G' := H' \cup \bigcup_{i=1}^{n} S'_i$. Then G', as well as its hub and spokes, is a subcontinuum of a graph; hence (Theorem 2.10) it is itself a graph. Thus (Theorem 2.1) any spoke that does not contain a branch point must be either an arc or a simple closed curve. Each new spoke that is an arc either contributes a new simple closed curve to G' (when both end points of the arc are in the hub) or contributes a new end point to G'.

Let $b(\ )$ (resp., $c(\ )$, $e(\ )$) denote the number of branch points (resp., simple closed curves, end points) of a graph. It is an immediate consequence of the definition that each of these numbers is finite.

Now $b(G') \leq b(G)$ and $c(G') \leq c(G)$; also there is a positive whole number k, depending on G alone, such that $e(K) \leq k$ for all subcontinua K of G. With this in mind, given any graph, there is a finite upper bound on the sum $b(K)+c(K)+e(K)$, where K ranges over subcontinua of the graph. But the spokes $S'_1, \ldots, S'_n$ are pairwise disjoint, so $b(G') + c(G') + e(G') \geq n$. Since n can be as large as we like, this is a contradiction. Hence each point of X has finite order in X.

Finally suppose $a_1, \ldots, a_n$ are n branch points of X. Then we may take pairwise disjoint connected open sets $U_1, \ldots, U_n$, where $a_i \in U_i$, $1 \leq i \leq n$. Each U_i is also locally connected and locally compact; so, using the full strength of Theorem 2.2, there is a simple triod $T_i \subseteq U_i$. Arguing as above, we are then able to construct n fat 3-wheels in G, in such a way that the subcontinua they cover are pairwise disjoint. Let $\langle H, S_1, S_2, S_3 \rangle$ be one such. Then $W := H \cup S_1 \cup S_2 \cup S_3$ is a Peano continuum; hence either an arc, a simple closed curve, or a continuum with a branch point. The first two cases are clearly impossible; hence we have shown that G must contain at least n branch points. Thus $n \leq b(G)$, and the proof is complete. □

Remark 2.12. An immediate corollary of Theorem 2.11 is that co-existential images of graphs are graphs. But of course (Theorem 2.7 of [4]) co-existential maps with locally connected images are monotone, and T. Maćkowiak [8] has shown that the Hausdorff image of a graph under a continuous map is also a graph if the map is weakly confluent; i.e., where subcontinua of the range are images of subcontinua of the domain.

Putting Corollary 2.9 together with Theorem 2.11 gives the following immediate consequence.

Corollary 2.13. *Any Peano compactum 2-dominated by a tree is itself a tree.*

Corollary 2.14. *A Peano compactum X is 2-dominated by a simple triod if and only if X is either a simple triod or an arc.*

Proof. If X is an arc, then X is 2-dominated by a simple triod, courtesy of Theorem 2.3. For the converse, suppose X is a Peano compactum, such that $X \leq_2 T$ for some simple triod T. By general results about 2-dominance, X is a nondegenerate continuum; by Corollary 2.13, X is a tree; by the proof of Theorem 2.11, X has at most one branch point. A nondegenerate tree with no branch points is an arc, so suppose $a \in X$ is the unique branch point, say $\mathrm{ord}(a, X) = n \geq 3$. Then X is a simple n-od, and it remains to show that n cannot be ≥ 4. But this follows quickly from the proof of Theorem 2.11. From a fat n-wheel covering X, we can obtain the same for the simple triod T. But then each spoke of that n-wheel, being a tree itself, must contribute at least one new end point to T. Since T has only three end points to begin with, we conclude that $n = 3$. □

Define two compacta to be *2-equivalent* if each 2-dominates the other. The same reasoning that went into Corollary 2.14 immediately gives the following categoricity result.

Corollary 2.15. *Fix $1 \leq n < \omega$, and suppose X is a Peano compactum that is 2-equivalent to a simple n-od. Then X is a simple n-od.*

Remark 2.16. Which Peano compacta are categorical in the sense of Corollary 2.15? Can the result of Corollary 2.15 be extended to arbitrary graphs, for example? [We have recently been able to show [7] that any Peano compactum co-elementarily equivalent to a graph is homeomorphic to that graph. This result may easily be strengthened to assume only 2-equivalence.]

We finish with two characterization results.

Corollary 2.17. *Let X be a graph. The following are equivalent:*

(i) *There is a co-existential map from X to a simple closed curve.*
(ii) *X 2-dominates a simple closed curve.*
(iii) *X is not a tree.*

Proof. ((i) $\Longrightarrow$ (ii)): This is immediate from the definition.
((ii) $\Longrightarrow$ (iii)): This is immediate from Corollary 2.13.

((iii) $\Longrightarrow$ (i)): If X is a graph that is not a tree, then X contains a simple closed curve. Consequently X contains an open free arc whose complement in X is connected. Apply Theorem 2.3. $\square$

Corollary 2.18. *Let X be a graph. The following are equivalent:*

(i) *There is a co-existential map from X to an arc.*
(ii) *X 2-dominates an arc.*
(iii) *X has a cut point that has order two.*

Proof. ((i) $\Longrightarrow$ (ii)): This is, again, immediate from the definition.
((ii) $\Longrightarrow$ (iii)): Suppose X is a graph, A is an arc, and $A \leq_2 X$. Fix a positive integer n and choose a sequence $\langle A_1, \dots, A_n \rangle$ of subcontinua of A such that: $A = A_1 \cup \cdots \cup A_n$, $A_i \setminus \bigcup_{j \neq i} A_j \neq \emptyset$ for all $1 \leq i \leq n$, and $A_i \cap A_j \neq \emptyset$ if and only if $|i - j| \leq 1$. For convenience, let us call such a sequence a *fat chain cover by subcontinua, of length n* for A. Again, using an argument similar to that given in the proof of Theorem 2.8, we may obtain a fat chain cover $\langle X_1, \dots, X_n \rangle$ by subcontinua, of length n for X. By Theorem 2.11, each X_i is a nondegenerate graph.

Now a graph contains only finitely many branch points and only finitely many simple closed curves. Since no branch point or simple closed curve in X can lie in more than two of the subcontinua X_i, we may choose n large enough so that at least one X_i contains no branch point or simple closed curve in X. This X_i, being a nondegenerate graph with no branch points or simple closed curves, must be an arc. Any point of order two in X_i and not in any X_j for $j \neq i$, guaranteed to exist, will then be a cut point of order two in X.
((iii) $\Longrightarrow$ (i)): If X has a cut point that has order two, then X contains an open free arc whose complement in X has two components. Apply Theorem 2.3. $\square$

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Received October 23, 2006
Revised version received October 24, 2007

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