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Gholamhossein Hamedani
Marquette University, gholamhoss.hamedani@marquette.edu

Z. Javanshiri
Ferdowsi University of Mashhad

Mehdi Maadooliat
Marquette University, mehdi.maadooliat@marquette.edu

A. Yazdani
University of Texas - Houston

Remarks on Characterizations of Malinowska & Szynal

G.G. Hamedani[a], Z. Javanshiri[b], M. Maadooliat[a,*], A. Yazdani[c]

[a]*Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI, USA*
[b]*Department of Statistics, Ferdowsi University, Mashhad, Iran*
[c]*School of Public Health, University of Texas-Houston, Houston, TX, USA*

Abstract

The problem of characterizing a distribution is an important problem which has recently attracted the attention of many researchers. Thus, various characterizations have been established in many different directions. An investigator will be vitally interested to know if their model fits the requirements of a particular distribution. To this end, one will depend on the characterizations of this distribution which provide conditions under which the underlying distribution is indeed that particular distribution. In this work, several characterizations of Malinowska and Szynal (2008) for certain general classes of distributions are revisited and simpler proofs of them are presented. These characterizations are not based on conditional expectation of the kth lower record values (as in Malinowska and Szynal), they are based on: (i) simple truncated moments of the random variable, (ii) hazard function.

Keywords: Characterizations, Hazard function, Truncated moments

1. Introduction

Characterizations of distributions are important to many researchers in the applied fields. An investigator will be vitally interested to know if their model fits

*Corresponding Author

Email addresses: g.hamedani@marquette.edu (G.G. Hamedani), zoh_javan@yahoo.com (Z. Javanshiri), mehdi@mscs.mu.edu (M. Maadooliat), ay.alamuti@gmail.com (A. Yazdani)

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the requirements of a particular distribution. To this end, one will depend on the characterizations of this distribution which provide conditions under which the underlying distribution is indeed that particular distribution. Various characterizations of distributions have been established in many different directions. In this work, several characterizations of Malinowska and Szynal (2008) for certain general classes of distributions are revisited and simpler proofs of them are presented. These characterizations are not based on conditional expectation of the kth lower record values (as in Malinowska and Szynal), they are based on: (i) simple truncated moments of the random variable, (ii) hazard function.

Let $X_1, X_2, ...X_n$ be $i.i.d.$ (independent and identically distributed) continuous random variables with common *cdf* (cumulative distribution function) F and corresponding *pdf* (probability density function) f . We denote their order statistics with $X_{j:n},\ j = 1, 2, ..., n$. The kth lower record value of $X_j's$ is defined by $Z_n^{(k)} = X_{k:L_k(n)+k-1}$, where $L_k(1) = 1$, $L_k(n+1) = \min\{j > L_k(n) : X_{k:L_k(n)+k-1} > X_{k:j+k-1}\}$, $n \geq 1$. The kth upper record value of $X_j's$ is defined by $Y_n^{(k)} = X_{U_k(n):U_k(n)+k-1}$, where $U_k(1) = 1$, $U_k(n+1) = \min\{j > U_k(n) : X_{k:j+k-1} > X_{U_k(n):U_k(n)+k-1}\}$, $n \geq 1$.

Malinowska and Szynal ((2008), page 339) assume that the common random variable X is an absolutely continuous random variable concentrated on the interval (α, β), with $F(x) < 1$ for $x \in (\alpha, \beta)$, $F(\alpha) = 0$ and $F(\beta) = 1$. For a given monotonic and differentiable function ϕ on (α, β), they write

$$\mu_{m+1|m}^{(k)} = E\left[\phi\left(Z_{m+1}^{(k)}\right)|Z_m^{(k)} = x\right], \tag{1.1}$$
$$\text{and} \quad \mu_{m|m+1}^{(k)} = E\left[\phi\left(Z_m^{(k)}\right)|Z_{m+1}^{(k)} = y\right],$$

and

$$\overline{\mu}_{m+1|m}^{(k)} = E\left[\phi\left(Y_{m+1}^{(k)}\right)|Y_m^{(k)} = x\right],$$
$$\text{and} \quad \overline{\mu}_{m|m+1}^{(k)} = E\left[\phi\left(Y_m^{(k)}\right)|Y_{m+1}^{(k)} = y\right],$$

and prove the following theorems.

Theorem 1.1. *Suppose that k is a positive integer. Then, referring to* (1.1)

$$F(x) = [a\phi(x) + b]^c, \tag{1.2}$$

if and only if

$$\mu^{(k)}_{m+1|m} = \frac{1}{kc+1}\left[kc\phi(x) - \frac{b}{a}\right], \tag{1.3}$$

where $a \neq 0$, $b, c > 0$ are finite constants.

Theorem 1.2. *Suppose that k is a positive integer. Then, referring to* (1.1)

$$F(x) = a + be^{-c\phi(x)}, \tag{1.4}$$

if and only if

$$\mu^{(k)}_{m+1|m} = \phi(x) + \frac{a^k[\phi(\alpha) - \phi(x)]}{[a + be^{-c\phi(x)}]^k} + \frac{1}{c[a + be^{-c\phi(x)}]^k} \times \sum_{i=0}^{k-1} \binom{k}{i} \frac{a^i \left[(be^{-c\phi(x)})^{k-i} - (be^{-c\phi(\alpha)})^{k-i}\right]}{k-i}, \tag{1.5}$$

where $c \neq 0$.

Theorem 1.3. *If k is a positive integer,*

$$1 - F(x) = [a\phi(x) + b]^c, \tag{1.6}$$

if and only if

$$\overline{\mu}^{(k)}_{m+1|m} = \frac{1}{kc+1}\left[kc\phi(x) - \frac{b}{a}\right], \tag{1.7}$$

where $a \neq 0$, $b, c > 0$ are finite constants.

Theorem 1.4. *Suppose that k is a positive integer. Then*

$$1 - F(x) = a + be^{-c\phi(x)}, \tag{1.8}$$

if and only if

$$\overline{\mu}^{(k)}_{m+1|m} = \phi(x) + \frac{a^k[\phi(\beta) - \phi(x)]}{[a + be^{-c\phi(x)}]^k} + \frac{1}{c[a + be^{-c\phi(x)}]^k} \times \sum_{i=0}^{k-1} \binom{k}{i} \frac{a^i \left[(be^{-c\phi(x)})^{k-i} - (be^{-c\phi(\beta)})^{k-i}\right]}{k-i}, \tag{1.9}$$

where $c \neq 0$.

Theorem 1.5. *Suppose that k is a positive integer. Then*

$$1 - F(x) = [a\phi(x) + b]^c,$$

if and only if

$$\overline{\mu}^{(k)}_{m+1|m+2} = -\phi(y)\frac{c(m+1)}{\overline{H}(y)} + \frac{c(m+1)}{\overline{H}(y)}\overline{\mu}^{(k)}_{m|m+1} - \frac{b}{a}, \tag{1.10}$$

and

$$\overline{\mu}^{(k)}_{1|2} = \frac{c[\phi(\alpha) - \phi(y)]}{\overline{H}(y)} - \frac{b}{a}, \tag{1.11}$$

where $\overline{H}(y) = -\log[1 - F(y)]$ and $a \neq 0$, $b, c \neq 0$ are finite constants.

Theorem 1.6. *Suppose that k is a positive integer. Then*

$$1 - F(x) = a + be^{-c\phi(x)},$$

if and only if

$$\overline{\mu}^{(k)}_{m|m+1} = \phi(y) - \frac{1}{c(m+1)}\overline{H}(y) - \frac{a\overline{H}(y)}{bc(m+1)}E[e^{c\phi(X)}|Y^{(k)}_{m+1} = y], \tag{1.12}$$

where $c \neq 0$ and $X < y$.

2. Characterizations

2.1. Characterizations based on truncated moments of certain functions of the random variable

In this subsection, characterizations similar to Theorems 1.1-1.4 are presented. These characterizations are based on truncated moments of the given function $\phi(x)$.

Theorem 2.1.1. $F(x) = [a\phi(x) + b]^c$ *if and only if*

$$E[\phi(X)|X \leq x] = \frac{1}{c+1}\left[c\phi(x) - \frac{b}{a}\right], \quad \alpha < x < \beta, \tag{2.1.1}$$

where $a \neq 0$, $b, c > 0$ are finite constants.

Proof. If $F(x) = [a\phi(x) + b]^c$, then

$$E[\phi(X)|X \le x] = [a\phi(x) + b]^{-c} \int_{\alpha}^{x} \phi(u) f(u) du.$$

Integration by parts on the RHS of the above equation results in

$$\int_{\alpha}^{x} \phi(u) f(u) du = \frac{1}{c+1} [a\phi(x) + b]^c \left[c\phi(x) - \frac{b}{a} \right],$$

from which (2.1.1) is obtained.

Conversely, if (2.1.1) holds, then

$$\int_{\alpha}^{x} \phi(u) f(u) du = F(x) \left\{ \frac{1}{c+1} \left[c\phi(x) - \frac{b}{a} \right] \right\}. \tag{2.1.2}$$

Taking derivative with respect to x from both sides of (2.1.2), and after some algebra, we arrive at

$$\frac{f(x)}{F(x)} = \frac{ac\phi'(x)}{a\phi(x) + b}. \tag{2.1.3}$$

Integrating both sides of (2.1.3) from x to β, we have

$$F(x) = [a\phi(x) + b]^c, \quad \alpha \le x \le \beta.$$

■

Remark 2.1.1. *The RHS of equation* (2.1.1) *is the same as that of equation* (1.3) *for* $k = 1$.

Example 2.1.1. *Let* $X \sim EE(\Theta)$ *denote the exponentiated generalized distribution (Cordeiro et al., 2013) with cdf*

$$F^G(x) = [1 - \{1 - G(x)\}^{\alpha}]^{\beta}, \qquad \alpha > 0, \quad \beta > 0,$$

where $G(x)$ *is a cdf of a continuous random variable. Then* $F(x)$ *is given by* (1.2) *with* $a = -1,\ b = 1,\ c = \beta,$ *and* $\phi(x) = \{1 - G(x)\}^{\alpha}$. *Some special cases for* $G(x)$ *are:*

- *exponentiated Kumaraswamy (Lemonte et al., 2013) distribution, denoted by* $EK(\gamma, \alpha, \beta)$*, where* $G(x) = x^{\gamma}$*, with* $0 < x < 1$*; and* $\gamma > 0$.

- *log-exponentiated Kumaraswamy (Lemonte et al., 2013) distribution, denoted by logEK*(γ, α, β)*, where* $G(x) = (1 - e^{-x})^{\gamma}$*, with* $x > 0$*; and* $\gamma > 0$*.*

Example 2.1.2. *Let* $X \sim \exp -G(\Theta)$ *(Barreto-Souza et al., 2013) with cdf*

$$F_{\lambda}^{G}(x) = \frac{1 - e^{-\lambda G(x;\theta)}}{1 - e^{-\lambda}}, \qquad \lambda \in \mathbb{R}\backslash\{0\},$$

where $\Theta = (\lambda, \theta)^T$*, and* $G(x;\theta)$ *is a cdf of a continuous random variable. Then* $F_{\lambda}^{G}(x)$ *is given by* (1.2) *with* $a = \dfrac{-1}{1 - e^{-\lambda}}$*,* $b = \dfrac{1}{1 - e^{-\lambda}}$*,* $c = 1$*, and* $\phi(x) = e^{-\lambda G(x;\theta)}$*. Some special cases of* $G(x)$ *are:*

- $\exp -$*Weibull distribution, where* $G(x;\alpha,\beta) = 1 - \exp\left[-\left(x/\beta\right)^{\alpha}\right]$*.*
- $\exp -$*beta distribution, where* $G(x;\alpha,\beta) = B(\alpha,\beta)^{-1} \int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt$*.*

Theorem 2.1.2. $F(x) = a + be^{-c\phi(x)}$ *if and only if*

$$E[\phi(X)|X \le x] = \frac{1}{c} + \phi(x) - a[\phi(x) - \phi(\alpha)][a + be^{-c\phi(x)}]^{-1}, \quad c \neq 0, \alpha < x < \beta. \tag{2.1.4}$$

Proof. If $F(x) = a + be^{-c\phi(x)}$,then

$$E[\phi(X)|X \le x] = [a + be^{-c\phi(x)}]^{-1} \int_{\alpha}^{x} \phi(u)f(u)du.$$

Integration by parts on the RHS of the above equation results in

$$\int_{\alpha}^{x} \phi(u)f(u)du = \left(\frac{1}{c} + \phi(x)\right)[a + be^{-c\phi(x)}] - a[\phi(x) - \phi(\alpha)],$$

from which (2.1.4) is obtained.

If (2.1.4) holds, then

$$\int_{\alpha}^{x} \phi(u)f(u)du = F(x)\left\{\frac{1}{c} + \phi(x) - a[\phi(x) - \phi(\alpha)][a + be^{-c\phi(x)}]^{-1}\right\}. \tag{2.1.5}$$

Taking derivatives with respect to x from both sides of (2.1.5), we have

$$\phi(x)f(x) = f(x)\left\{\frac{1}{c} + \phi(x) - a[\phi(x) - \phi(\alpha)][a + be^{-c\phi(x)}]^{-1}\right\} + \\ F(x)\{\phi'(x) - a\phi'(x)[a + be^{-c\phi(x)}]^{-1} - abc[\phi(x) - \phi(\alpha)]\phi'(x)e^{-c\phi(x)}[a + be^{-c\phi(x)}]^{-2}\},$$

from which we arrive at

$$\frac{f(x)}{F(x)} = \frac{-\{\phi'(x) - a\phi'(x)[a+be^{-c\phi(x)}]^{-1} - abc[\phi(x)-\phi(\alpha)]\phi'(x)e^{-c\phi(x)}[a+be^{-c\phi(x)}]^{-2}\}}{\left\{\frac{1}{c} - a[\phi(x)-\phi(\alpha)][a+be^{-c\phi(x)}]^{-1}\right\}} \tag{2.1.6}$$

Simplifying the RHS of (2.1.6), we obtain

$$\frac{f(x)}{F(x)} = \frac{-bc\phi'(x)e^{-c\phi(x)}}{[a+be^{-c\phi(x)}]}.$$

Integrating both sides of the last equation from x to β , we have

$$F(x) = a + be^{-c\phi(x)}, \quad \alpha \le x \le \beta.$$

■

Remark 2.1.2. *The RHS of equation* (2.1.4) *is the same as that of equation* (1.5) *for* $k = 1$.

Example 2.1.3. *Let* $X \sim \exp -G(\Theta)$ *with cdf*

$$F_\lambda^G(x) = \frac{1-e^{-\lambda G(x;\theta)}}{1-e^{-\lambda}}, \qquad \lambda \in \mathbb{R}\backslash\{0\},$$

where $\Theta = (\lambda,\theta)^T$, *and* $G(x;\theta)$ *is a cdf of a continuous random variable. Then* $F_\lambda^G(x)$ *is given by* (1.4) *with* $a = \frac{1}{1-e^{-\lambda}}$, $b = \frac{-1}{1-e^{-\lambda}}$, $c = \lambda$, *and* $\phi(x) = G(x;\theta)$.

Example 2.1.4. *Let* $X \sim EE(\Theta)$ *with cdf*

$$F^G(x) = [1 - \{1-G(x)\}^\alpha]^\beta, \qquad \alpha > 0, \quad \beta > 0,$$

where $G(x)$ *is a cdf of a continuous random variable. Then* $F(x)$ *is given by* (1.4) *with* $a = 0$, $b = 1$, $c = -\beta$, *and* $\phi(x) = \ln[1 - \{1-G(x)\}^\alpha]$.

Theorem 2.1.3. $1 - F(x) = [a\phi(x) + b]^c$ *if and only if*

$$E[\phi(X)|X \ge x] = \frac{1}{c+1}\left[c\phi(x) - \frac{b}{a}\right], \quad \alpha < x < \beta, \tag{2.1.7}$$

where $a \neq 0$, $b, c > 0$ *are finite constants.*

Proof. If $1 - F(x) = [a\phi(x) + b]^c$, then

$$E[\phi(X)|X \geq x] = [a\phi(x) + b]^{-c} \int_x^\beta \phi(u)\{-ac\phi'(u)[a\phi(u) + b]^{c-1}\}du.$$

Integration by parts on the RHS of the above equation results in

$$\int_x^\beta \phi(u)\{-ac\phi'(u)[a\phi(u) + b]^{c-1}\}du = \frac{1}{c+1}[a\phi(x) + b]^c \left[c\phi(x) - \frac{b}{a}\right],$$

from which (2.1.7) is obtained.

Conversely, if (2.1.7) holds, then

$$\int_x^\beta \phi(u)f(u)du = [1 - F(x)] \left\{\frac{1}{c+1}\left[c\phi(x) - \frac{b}{a}\right]\right\}. \qquad (2.1.8)$$

Taking derivative with respect x from both sides of (2.1.8), and after some algebra, we arrive at

$$\frac{-f(x)}{1 - F(x)} = \frac{ac\phi'(x)}{a\phi(x) + b}.$$

Integrating both sides of the last equation from α to x, we have

$$1 - F(x) = [a\phi(x) + b]^c, \quad \alpha \leq x \leq \beta.$$

■

Remark 2.1.3. *The RHS of equation* (2.1.7) *is the same as that of equation* (1.7) *for* $k = 1$.

Example 2.1.5. *Let* $X \sim$ *Kw-G*(Θ) *denote the Kumaraswamy generalized distribution (Cordeiro et al., 2011) with cdf*

$$F^G(x) = 1 - [1 - G(x)^\lambda]^\varphi, \qquad \lambda > 0, \quad \varphi > 0,$$

where $G(x)$ *is a cdf of a continuous random variable. Then* $F(x)$ *is given by* (1.6) *with* $a = -1,\ b = 1,\ c = \varphi,$ *and* $\phi(x) = G(x)^\lambda$. *Some special cases for* $G(x)$ *are:*

- *Kumaraswamy distribution,* $Kw(\lambda, \varphi)$*, where* $G(x) = x$.
- *Kw-Gumbel distribution, Kw-Gumbel*(λ, φ)*, where* $G(x) = 1 - \exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]$.

Example 2.1.6. *Let $X \sim EW(\alpha, \xi)$, extended Weibull family (Gurvich et al., 1997) with cdf*

$$F(x; \alpha, \xi) = 1 - e^{-\alpha H(x;\xi)}, \qquad \alpha > 0,$$

where $H(x;\xi)$ is a non-negative monotonically increasing function depending on the parameter vector ξ. Then $F(x;\alpha,\xi)$ is given by (1.6) with $a = 1$, $b = 0$, $c = -\alpha, \phi(x) = e^{H(x;\xi)}$. Some special cases for $G(x)$ are:

- *additive Weibull distribution (Xie and Lai, 1995), denoted by $AddW(\alpha, \beta, \theta, \gamma)$, where $H(x) = x^{\theta} + \frac{\beta}{\alpha} x^{\gamma}$, with $x \in \mathbb{R}^{+}$; α, β, θ, and γ are non-negative.*
- *new modified Weibull distribution (Almalki and Yuan, 2013), denoted by $NMW(\alpha, \beta, \theta, \gamma, \lambda)$, where $H(x) = x^{\theta} + \frac{\beta}{\alpha} x^{\gamma} e^{\lambda x}$, with $x \in \mathbb{R}^{+}$; α, β, θ, γ, and λ are non-negative.*

Theorem 2.1.4. $1 - F(x) = a + be^{-c\phi(x)}$ *if and only if*

$$E[\phi(X)|X \geq x] = \frac{1}{c} + \phi(x) - a[\phi(x) - \phi(\beta)][a + be^{-c\phi(x)}]^{-1}, c \neq 0, \quad \alpha < x < \beta. \tag{2.1.9}$$

Proof. If $1 - F(x) = a + be^{-c\phi(x)}$, then

$$E[\phi(X)|X \geq x] = [a + be^{-c\phi(x)}]^{-1} \int_x^{\beta} \phi(u)[bc\phi'(u)e^{-c\phi(u)}]du.$$

Integration by parts on the RHS of the above equation results in

$$\int_x^{\beta} \phi(u)[bc\phi'(u)e^{-c\phi(u)}]du = \left[\frac{1}{c} + \phi(x)\right][a + be^{-c\phi(x)}] - a[\phi(x) - \phi(\beta)],$$

from which (2.1.9) is obtained.

Conversely, if (2.1.9) holds, then

$$\int_x^{\beta} \phi(u)f(u)du = [1 - F(x)] \left\{\frac{1}{c} + \phi(x) - a[\phi(x) - \phi(\beta)][a + be^{-c\phi(x)}]^{-1}\right\}. \tag{2.1.10}$$

Taking derivative with respect to x from both sides of (2.1.10), and after some algebra, we arrive at

$$-\phi(x)f(x) = -f(x)\left\{\frac{1}{c} + \phi(x) - a[\phi(x) - \phi(\beta)][a + be^{-c\phi(x)}]^{-1}\right\} + (1 - F(x)) \times$$
$$\left\{\phi'(x) - a\phi'(x)[a + be^{-c\phi(x)}]^{-1} - abc[\phi(x) - \phi(\beta)]\phi'(x)e^{-c\phi(x)}[a + be^{-c\phi(x)}]^{-2}\right\}.$$

Simplifying the RHS of the above equation, we arrive at

$$\frac{f(x)}{1-F(x)} = \frac{bc\phi'(x)e^{-c\phi(x)}}{a+be^{-c\phi(x)}}.$$

Integrating both sides of the last equation from α to x , we have

$$1-F(x) = a+be^{-c\phi(x)}, \quad \alpha \leq x \leq \beta.$$

■

Remark 2.1.4. *The RHS of equation* (2.1.9) *is the same as that of equation* (1.9) *for* $k=1$*.*

Example 2.1.7. *Let* $X \sim EW(\alpha, \xi)$ *with cdf*

$$F(x;\alpha,\xi) = 1-e^{-\alpha H(x;\xi)}, \qquad \alpha > 0,$$

where $H(x;\xi)$ *is a non-negative monotonically increasing function depending on the parameter vector* ξ*. Then* $F(x;\alpha,\xi)$ *is given by* (1.8) *with* $a=0,\ b=1,\ c=\alpha, \phi(x)=H(x;\xi)$*.*

Example 2.1.8. *Let* $X \sim$ *Kw-G*(Θ) *with cdf*

$$F^G(x) = 1-[1-G(x)^\lambda]^\varphi, \qquad \lambda > 0, \quad \varphi > 0,$$

where $G(x)$ *is a cdf of a continuous random variable. Then* $F(x)$ *is given by* (1.8) *with* $a=0,\ b=1,\ c=-\varphi,$ *and* $\phi(x)=\ln\left[1-G(x)^\lambda\right]$*.*

Remark 2.1.5. *Our analogous Theorems 2.1.5 and 2.1.6 will be the same as our Theorems 2.1.3 and 2.1.4 respectively.*

2.2. Characterizations based on hazard function

For the sake of completeness, we state the following definition. In what follows, we assume that *cdf* is twice differentiable when needed.

Definition 2.2.1. *Let* F *be an absolutely continuous distribution with the corresponding pdf* f*. The hazard function corresponding to* F *is denoted by* h_F *and is defined by*

$$h_F(x) = \frac{f(x)}{1-F(x)}, \qquad x \in \mathit{Supp}\ F, \tag{2.2.1}$$

where Supp F *is the support of* F *.*

It is obvious that the hazard function of twice differentiable function satisfies the first order differential equation

$$\frac{h_F'(x)}{h_F(x)} - h_F(x) = q(x),$$

where $q(x)$ is an appropriate integrable function. Although this differential equation has an obvious form since

$$\frac{f'(x)}{f(x)} = \frac{h_F'(x)}{h_F(x)} - h_F(x), \tag{2.2.2}$$

for many univariate continuous distributions (2.2.2) seems to be the only differential equation in terms of the hazard function. The goal of the characterization based on hazard function is to establish a differential equation in terms of hazard function, which has as simple form as possible and is not of the trivial form (2.2.2). For some general families of distributions this may not be possible. Here, we present characterizations of two of the general classes of distributions discussed in this work based on a nontrivial differential equation in terms of the hazard function. In what follows we assume that $\phi(x)$ is twice differentiable on (α, β).

Proposition 2.2.1. *Let $X : \Omega \to (\alpha, \beta)$ be a continuous random variable. The cdf of X is* (1.2) *if and only if its hazard function $h_F(x)$ satisfies the differential equation*

$$h_F'(x) - h_F(x)\left[\frac{\phi''(x)}{\phi'(x)} - \frac{a\phi'(x)}{a\phi(x)+b}\right] = \frac{c^2a^2\phi'(x)^2[a\phi(x)+b]^{c-2}}{\{1-[a\phi(x)+b]^c\}^2}, \quad \alpha < x < \beta, \tag{2.2.3}$$

with initial condition $h_F(\alpha) = 0$.

Proof. If X has *cdf* (1.2), then clearly (2.2.3) holds. Now, if (2.2.3) holds, then

$$h_F'(x)\left[\frac{a\phi(x)+b}{\phi'(x)}\right] - h_F(x)\left\{\frac{[a\phi(x)+b]\phi''(x)}{\phi'(x)^2} - a\right\} = \frac{c^2a^2\phi'(x)[a\phi(x)+b]^{c-1}}{\{1-[a\phi(x)+b]^c\}^2},$$

from which we have

$$\frac{d}{dx}\left\{h_F(x)\left[\frac{a\phi(x)+b}{\phi'(x)}\right]\right\} = \frac{d}{dx}\left\{\frac{ca}{1-[a\phi(x)+b]^c}\right\}, \quad \alpha < x < \beta. \tag{2.2.4}$$

Integrating both sides of (2.2.4) from α to x, and using the fact $a\phi(\alpha)+b=0$, $h_F(\alpha)=0$ we arrive at

$$h_F(x)\left[\frac{a\phi(x)+b}{\phi'(x)}\right]=\frac{ca}{1-[a\phi(x)+b]^c}-ca,\quad \alpha<x<\beta.$$

Now, we obtain

$$h_F(x)=\frac{ca\phi'(x)[a\phi(x)+b]^{c-1}}{1-[a\phi(x)+b]^c},$$

or

$$\frac{f(x)}{1-F(x)}=\frac{ca\phi'(x)[a\phi(x)+b]^{c-1}}{1-[a\phi(x)+b]^c}.$$

Integrating both sides of the last equation from α to x and using the fact that $a\phi(\alpha)+b=0$ we arrive at

$$F(x)=[a\phi(x)+b]^c,\ \alpha\le x\le\beta.$$

■

Example 2.2.1. *Let* $a=b=1$, $c=\theta$, $\phi(x)=\dfrac{x-\nu}{\delta-x}$ *and the initial condition* $h_F(0)=0$. *By using* (1.2) *we have the following differential equation:*

$$h'_F(x)-h_F(x)\left(\delta-x\right)^{-1}=\frac{\theta^2\left(\dfrac{\delta-\nu}{\delta-x}\right)^\theta}{(\delta-x)^2\left\{1-\left(\dfrac{\delta-\nu}{\delta-x}\right)^\theta\right\}^2}.$$

Solving for $h_F(x)$, *we obtain:*

$$h_F(x)=\frac{f(x)}{1-F(x)}=\frac{\theta\left(\dfrac{\delta-\nu}{\delta-x}\right)^\theta}{(\delta-x)\left\{1-\left(\dfrac{\delta-\nu}{\delta-x}\right)^\theta\right\}},$$

from which we have:

$$F(x)=\left(\frac{\delta-\nu}{\delta-x}\right)^\theta.$$

Here F *is the cdf of negative Pareto distribution,* $NPar(\theta,\nu,\delta)$, *where* $x<\nu$; $\theta>0$, $\nu,\delta\in\mathbb{R}$, *and* $\nu<\delta$.

Proposition 2.2.2. *Let $X : \Omega \to (\alpha, \beta)$ be a continuous random variable. The cdf of X is* (1.4) *if and only if its hazard function $h_F(x)$ satisfies the differential equation*

$$\frac{h'_F(x)}{h_F(x)} = \frac{\phi''(x)}{\phi'(x)} - \frac{c(1-a)\phi'(x)}{1-[a+be^{-c\phi(x)}]}, \quad \alpha < x < \beta, \tag{2.2.5}$$

with initial condition $h_F(\alpha) = ca\phi'(\alpha)$.

Proof. If X has *cdf* (1.4), then clearly (2.2.5) holds. Now, if (2.2.5) holds, then

$$h'_F(x)\left[\frac{(1-a)e^{c\phi(x)}-b}{\phi'(x)}\right] - h_F(x)\left\{\frac{[(1-a)e^{c\phi(x)}-b]\phi''(x)}{\phi'(x)^2} - c(1-a)e^{c\phi(x)}\right\} = 0,$$

from which we have

$$\frac{d}{dx}\left\{h_F(x)\left[\frac{(1-a)e^{c\phi(x)}-b}{\phi'(x)}\right]\right\} = \frac{d}{dx}\text{const.}, \quad \alpha < x < \beta. \tag{2.2.6}$$

Integrating both sides of (2.2.6) from α to x, and using the fact $a + be^{-c\phi(\alpha)} = 0$, $h_F(\alpha) = ca\phi'(\alpha)$ we arrive at

$$h_F(x)\left[\frac{(1-a)e^{c\phi(x)}-b}{\phi'(x)}\right] - ca\phi'(\alpha)\left[\frac{(1-a)\left(\frac{-b}{a}\right)-b}{\phi'(\alpha)}\right] = 0, \ \alpha < x < \beta.$$

Now, we obtain

$$h_F(x) = \frac{-bc\phi'(x)}{(1-a)e^{c\phi(x)}-b},$$

or

$$\frac{f(x)}{1-F(x)} = \frac{-bc\phi'(x)e^{-c\phi(x)}}{1-[a+be^{-c\phi(x)}]}.$$

Integrating both sides of the last equation from α to x and using the fact that $a + be^{-c\phi(\alpha)} = 0$ we arrive at

$$F(x) = a + be^{-c\phi(x)}, \ \alpha \le x \le \beta.$$

■

Example 2.2.2. *Let* $a = 0$*,* $b = 1$*,* $c = \theta^\tau$*,* $\phi(x) = x^{-\tau}$ *and the initial condition* $h_F(0) = 0$*. By using* (1.4) *we have the following differential equation:*

$$\frac{h'_F(x)}{h_F(x)} = \frac{\exp\{(\theta/x)^\tau\}\{1+\tau-\tau(\theta/x)^\tau\} - (\tau+1)}{x\left[1-\exp\{(\theta/x)^\tau\}\right]}.$$

Solving for $h_F(x)$*, we obtain:*

$$h_F(x) = \frac{f(x)}{1-F(x)} = \frac{\tau(\theta/x)^\tau \exp\{-(\theta/x)^\tau\}}{x\left[1-\exp\{-(\theta/x)^\tau\}\right]},$$

from which we have:

$$F(x) = \exp\{-(\theta/x)^\tau\}.$$

Here F *is the cdf of inverse Weibull distribution (cf. Klugman et al., 1998), denoted by IWeib*(θ, τ)*, where* $x \in \mathbb{R}^+$*; and* $\theta, \tau > 0$*.*

Example 2.2.3.

(a) Let $a = 1$*,* $b = 0$*,* $c = 1$*,* $\phi(x) = \exp\{\lambda(x-\nu)\}$ *and the initial condition* $h_F(0) = 0$*. By using* (1.2) *we have the following differential equation:*

$$h'_F(x) = \frac{\lambda^2 \exp\{\lambda(x-\nu)\}}{\left[1-\exp\{\lambda(x-\nu)\}\right]^2}. \tag{2.2.7}$$

(b) Similarly, let $a = 0$*,* $b = 1$*,* $c = -1$*,* $\phi(x) = \lambda(x-\nu)$ *and the initial condition* $h_F(0) = 0$*. By using* (1.4) *we obtain the following differential equation:*

$$\frac{h'_F(x)}{h_F(x)} = \frac{\lambda}{1-\exp\{\lambda(x-\nu)\}} \tag{2.2.8}$$

The solutions of (2.2.7) *and* (2.2.8) *are equal and it has the form:*

$$h_F(x) = \frac{f(x)}{1-F(x)} = \frac{\lambda\exp\{\lambda(x-\nu)\}}{1-\exp\{\lambda(x-\nu)\}}.$$

Therefore both (a) and (b) have the following cdf:

$$F(x) = \exp\{\lambda(x-\nu)\}.$$

Here F *is the cdf of negative exponential distribution, NExp*(λ, ν)*, where* $x < \nu$*;* $\lambda > 0$ *and* $\nu \in \mathbb{R}$*.*

Proposition 2.2.3. *Let $X : \Omega \to (\alpha, \beta)$ be a continuous random variable. The cdf of X is* (1.6) *if and only if its hazard function $h_F(x)$ satisfies the differential equation*

$$\frac{h'_F(x)}{h_F(x)} = \frac{\phi''(x)}{\phi'(x)} - \frac{a\phi'(x)}{a\phi(x)+b}, \quad \alpha < x < \beta, \tag{2.2.9}$$

with initial condition $h_F(\alpha) = -ca\phi'(\alpha)$.

Proof. If X has *cdf* (1.6), then clearly (2.2.9) holds. Now, if (2.2.9) holds, then

$$h'_F(x)\left[\frac{a\phi(x)+b}{\phi'(x)}\right] - h_F(x)\left\{\frac{[a\phi(x)+b]\phi''(x)}{\phi'(x)^2} - a\right\} = 0,$$

from which we have

$$\frac{d}{dx}\left\{h_F(x)\left[\frac{a\phi(x)+b}{\phi'(x)}\right]\right\} = \frac{d}{dx}\text{const.}, \quad \alpha < x < \beta. \tag{2.2.10}$$

Integrating both sides of (2.2.10) from α to x, and using the fact $a\phi(\alpha) + b = 1$, $h_F(\alpha) = -ca\phi'(\alpha)$ we arrive at

$$h_F(x)\left[\frac{a\phi(x)+b}{\phi'(x)}\right] + ca\phi'(\alpha)\left[\frac{a\phi(\alpha)+b}{\phi'(\alpha)}\right] = 0, \ \alpha < x < \beta.$$

Now, we obtain

$$h_F(x) = \frac{-ca\phi'(x)}{a\phi(x)+b},$$

or

$$\frac{f(x)}{1-F(x)} = \frac{-ca\phi'(x)}{a\phi(x)+b}.$$

Integrating both sides of the last equation from α to x and using the fact that $a\phi(\alpha) + b = 1$ we arrive at

$$1 - F(x) = [a\phi(x) + b]^c, \ \alpha \le x \le \beta.$$

■

Example 2.2.4. *Let $a = c = 1$, $b = 0$, $\phi(x) = \exp(-x^\alpha)$ and the initial condition $h_F(0) = 0$. By using* (1.6) *we have the following differential equation:*

$$\frac{h'_F(x)}{h_F(x)} = \frac{\alpha - 1}{x}.$$

Solving for $h_F(x)$, we obtain:

$$h_F(x) \; = \; \frac{f(x)}{1-F(x)} \; = \; \alpha x^{\alpha-1},$$

from which we have:

$$F(x) = 1 - \exp\left(-x^{\alpha}\right).$$

Here F is the cdf of Weibull distribution, where $x \in \mathbb{R}^+$; and $\alpha > 0$.

Proposition 2.2.4. *Let $X : \Omega \to (\alpha, \beta)$ be a continuous random variable. The cdf of X is* (1.8) *if and only if its hazard function $h_F(x)$ satisfies the differential equation*

$$\frac{h_F'(x)}{h_F(x)} = \frac{\phi''(x)}{\phi'(x)} - \frac{ca\phi'(x)}{a+be^{-c\phi(x)}}, \quad \alpha < x < \beta, \tag{2.2.11}$$

with initial condition $h_F(\alpha) = c(1-a)\phi'(\alpha)$.

Proof. If X has *cdf* (1.8), then clearly (2.2.11) holds. Now, if (2.2.11) holds, then

$$h_F'(x)\left[\frac{ae^{c\phi(x)}+b}{\phi'(x)}\right] - h_F(x)\left\{\frac{[ae^{c\phi(x)}+b]\phi''(x)}{\phi'(x)^2} - cae^{c\phi(x)}\right\} = 0,$$

from which we have

$$\frac{d}{dx}\left\{h_F(x)\left[\frac{ae^{c\phi(x)}+b}{\phi'(x)}\right]\right\} = \frac{d}{dx}\text{const.}, \quad \alpha < x < \beta. \tag{2.2.12}$$

Integrating both sides of (2.2.12) from α to x, and using the fact $a + be^{-c\phi(\alpha)} = 1$, $h_F(\alpha) = c(1-a)\phi'(\alpha)$ we arrive at

$$h_F(x)\left[\frac{ae^{c\phi(x)}+b}{\phi'(x)}\right] - c(1-a)\phi'(\alpha)\left[\frac{a\left(\frac{b}{1-a}\right)+b}{\phi'(\alpha)}\right] = 0, \;\; \alpha < x < \beta.$$

Now, we obtain

$$h_F(x) = \frac{bc\phi'(x)}{ae^{c\phi(x)}+b},$$

or

$$\frac{f(x)}{1-F(x)} = \frac{bc\phi'(x)e^{-c\phi(x)}}{a+be^{-c\phi(x)}}.$$

Integrating both sides of the last equation from α to x and using the fact that $a + be^{-c\phi(\alpha)} = 1$ we arrive at

$$1 - F(x) = a + be^{-c\phi(x)}, \ \alpha \leq x \leq \beta.$$

■

Example 2.2.5. *Let* $a = 0$, $b = c = 1$, $\phi(x) = \alpha x + \frac{\theta}{2}x^2 + \frac{\beta}{3}x^3$ *and the initial condition* $h_F(0) = \alpha$. *By using* (1.8) *we have the following differential equation:*

$$\frac{h'_F(x)}{h_F(x)} = \frac{\theta + 2\beta x}{\alpha + \theta x + \beta x^2}.$$

Solving for $h_F(x)$, *we obtain:*

$$h_F(x) = \frac{f(x)}{1 - F(x)} = \alpha + \theta x + \beta x^2,$$

from which we have:

$$F(x) = 1 - \exp\left\{-\left(\alpha x + \frac{\theta}{2}x^2 + \frac{\beta}{3}x^3\right)\right\}.$$

Here F *is the cdf of quadratic hazard rate distribution,* QHRD(α, β, θ), *where* $x \in \mathbb{R}^+$; *and* $\alpha, \beta > 0$, $\theta \geq -2\sqrt{\alpha\beta}$.

Example 2.2.6.

(a) *Let* $a = \exp(1)$, $b = 0$, $c = 1$, $\phi(x) = \exp\{-(1 + \lambda x)^\alpha\}$ *and the initial condition* $h_F(0) = \alpha\lambda$. *By using* (1.6) *we have the following differential equation:*

$$\frac{h'_F(x)}{h_F(x)} = \frac{\lambda(\alpha - 1)}{1 + \lambda x}. \tag{2.2.13}$$

(b) *Similarly, let* $a = 0$, $b = \exp(1)$, $c = 1$, $\phi(x) = (1 + \lambda x)^\alpha$ *and the initial condition* $h_F(0) = \alpha\lambda$. *By using* (1.8) *we obtain the same differential equation as* (2.2.13).

Now, solving for $h_F(x)$, *we have:*

$$h_F(x) = \frac{f(x)}{1 - F(x)} = \alpha\lambda(1 + \lambda x)^{\alpha - 1},$$

from which we obtain:

$$F(x) = 1 - \exp\left\{1 - (1 + \lambda x)^{\alpha}\right\}.$$

Here F is the cdf of extension of the exponential distribution (Nadarajah et al., 2011), $exp - NH(\alpha, \lambda)$, where $x \in \mathbb{R}^{+}$; and $\alpha, \lambda > 0$.

Remark 2.2.1. *The following holds:*

(a) *Let $X : \Omega \to (\alpha, \beta)$ be a continuous random variable. The cdf of X is (1.4) if and only if its hazard function $h_F(x)$ satisfies one of the following differential equations:*

$$h_F'(x) - h_F(x)\frac{\phi''(x)}{\phi'(x)} = \frac{-(a-1)bc^2\phi'(x)^2 e^{-c\phi(x)}}{\{1 - [a + be^{-c\phi(x)}]\}^2}, \quad \alpha < x < \beta,$$

$$h_F'(x) + h_F(x)\frac{c(1-a)\phi'(x)}{1 - [a + be^{-c\phi(x)}]} = \frac{-bc\phi''(x)e^{-c\phi(x)}}{1 - [a + be^{-c\phi(x)}]}, \quad \alpha < x < \beta,$$

with initial condition $h_F(\alpha) = ca\phi'(\alpha)$.

(b) *Let $X : \Omega \to (\alpha, \beta)$ be a continuous random variable. The cdf of X is (1.6) if and only if its hazard function $h_F(x)$ satisfies one of the following differential equations:*

$$h_F'(x) - h_F(x)\frac{\phi''(x)}{\phi'(x)} = \frac{ca^2\phi'(x)^2}{[a\phi(x) + b]^2}, \quad \alpha < x < \beta,$$

$$h_F'(x) + h_F(x)\frac{a\phi'(x)}{a\phi(x) + b} = \frac{-ca\phi''(x)}{a\phi(x) + b}, \quad \alpha < x < \beta,$$

with initial condition $h_F(\alpha) = -ca\phi'(\alpha)$.

(c) *Let $X : \Omega \to (\alpha, \beta)$ be a continuous random variable. The cdf of X is (1.8) if and only if its hazard function $h_F(x)$ satisfies one of the following differential equations:*

$$h_F'(x) - h_F(x)\frac{\phi''(x)}{\phi'(x)} = \frac{-abc^2\phi'(x)^2 e^{-c\phi(x)}}{[a + be^{-c\phi(x)}]^2}, \quad \alpha < x < \beta,$$

$$h_F'(x) + h_F(x)\frac{ca\phi'(x)}{a + be^{-c\phi(x)}} = \frac{bc\phi''(x)e^{-c\phi(x)}}{a + be^{-c\phi(x)}}, \quad \alpha < x < \beta,$$

with initial condition $h_F(\alpha) = c(1-a)\phi'(\alpha)$.

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