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Aristotle and Aquinas on the Freedom of the Mathematician

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ARISTOTLE AND AQUINAS ON THE FREEDOM OF THE MATHEMATICIAN

IT IS NOT unusual to find contemporary mathematicians who claim to have an unlimited degree of freedom in their discipline. Some even maintain that they can study (at least symbolically) anything and everything. The mathematician, they say, simply posits any definitions he pleases concerning any group of symbols and relations among them, defines the operations thereupon, and then proceeds logically. Needless to say, these mathematicians do not consider themselves bound in any way to treat entities which resemble real physical things. (Indeed, they not infrequently give the impression that they have little or no concern as to whether their mathematical considerations have any application to physical reality.) Nor do they consider mathematics to be a science of abstracted quantity in the traditional sense, fearing that to assert this would needlessly restrict the range of their science.

The purpose of this essay is not to pass judgment on the claims of today's mathematicians regarding freedom in their science. I intend rather to investigate the philosophies of mathematics of two much earlier men, Aristotle and Thomas Aquinas, both of whom considered mathematics to be a science of quantity, in order to determine the degree of freedom each allowed the mathematician in his science. Specifically, I will show that the medieval theologian's doctrines contain significant advances in this area over those of his Greek predecessor. Moreover, it will be suggested that to designate mathematics as a science of quantity, as these two thinkers do, still allows for a tremendous degree of freedom on the part of the mathematician—though it is not claimed that either man envisioned, or would agree with, the degree of freedom claimed by some mathematicians today.

I. The Question

Let us begin by returning to a point just mentioned, that for both Aristotle and Aquinas mathematics is considered to be a science of quantity. Let us hasten to add, however, that the quantity studied in mathematics is, according to both thinkers, a quantity not found as such in real things but a quantity abstracted from such things. As is well known, this abstraction involves mentally setting aside all the nonquantitative attributes of things and retaining only their quantitative ones.

In his famous text of the *Metaphysics*, a text which Thomas repeats with approval in his *Commentary*, the Stagirite speaks of the mathematician " stripping away " all features of things but their quantitative attributes,

> . . . the mathematician investigates abstractions (for before beginning his investigation he strips off all the sensible qualities, e. g., weight and lightness, hardness and its contrary, and also heat and cold and other sensible contrarieties, and leaves only the quantitative and continuous, sometimes in one, sometimes in two, sometimes in three dimensions, and the attributes of these *qua* quantitative and continuous, and does not consider them in any other respect, . . .[1]

Of course, it is precisely because of this mental abstraction, or subtraction, that the quantities studied in mathematics are said by both men to acquire their specific features as immobile, nonsensible, free from time and place and from sensible matter, and often possess less than three dimensions.

And yet, though the features of abstract mathematical quantities and quantified things are radically different, this does not mean that these quantities are totally dissimilar; indeed, both philosophers stress that it *is* in fact the quantities of physical things that the mathematician studies. However, they add—it is not *as quantities of physical things* that they are studied. One text of Aristotle's which makes this clear is the following:

[1] *Metaphysics*, XI, 3, 1061a 29-36. Thomas's commentary is *In XI Metaphysics*, L. 3, 2202.

Obviously physical bodies contain surfaces and volumes, lines and points, and these are the subject-matter of mathematics. . . . Now the mathematician, though he too treats of these things, nevertheless does not treat of them as the limits of a physical body, nor does he consider the attributes indicated as the attributes of such bodies. That is why he separates them; for in thought they are separable. . . .[2]

Thomas Aquinas makes exactly the same point in his commentary on this passage. He affirms that the mathematician and the natural philosopher both treat the *same* things, but not in the same way.

The mathematician and the natural philosopher treat the same things, i. e., points, and lines, and surfaces, and things of this sort, but not in the same way. For the mathematician does not treat these things insofar as each of them is a boundary of a natural body, nor does he consider those things which belong to them insofar as they are the boundaries of a natural body. But this is the way in which natural science treats them. . . . Because the mathematician does not consider lines and points, and surfaces, and things of this sort, quantities and their accidents, insofar as they are the boundaries of a natural body, he is said to abstract from sensible and natural matter.[3]

Clearly then for both men the mathematician does treat real quantities but not as real.

And this brings us to the heart of the question of this study. If mathematical quantities are nothing more than abstracted real quantities; if they are gained simply by " stripping away " all nonquantitative attributes of things, does this mean that for Aquinas and Aristotle the mathematician is limited *in his science to treating objects which in their quantitative features* resemble the quantitative attributes of physical things? It is true that both men give as examples of geometrical objects rather elementary figures, circles, triangles, angles, etc., which could easily be gained by abstraction from similarly figured sensible things.[4] But does this mean that they believe that

[2] *Physics*, II, 2, 193b 23-24; 32-34.

[3] *In II Physics*, L. 3, 160-61.

[4] Heath points out both in *A History of Greek Mathematics* (Oxford: Clarendon

mathematics is limited to just such quantities, quantities which bear almost a one-to-one relation to real quantities? If this is the case, then clearly the freedom of the mathematician is severely restricted.

In order to answer this crucial question we will turn to a more detailed consideration of what psychologically is actually involved in mathematical abstraction according to both men. This will aid us in determining just how free each considers the mathematician to be in his act of abstraction. First, Aristotle.

II. The Freedom of the Mathematician According to Aristotle

In the famous text of his *Posterior Analytics* where he describes the general procedure of obtaining the universal from sense experience [5] Aristotle refers to the presence of what he there calls "memory." Animals which have memory, he says, are able to retain sense impressions and so provide for themselves some stability in the changeable data of sense experience. Actually what Aristotle there calls memory he will later more precisely designate imagination.[6] Thus the role of imagination in all abstraction (using this term now in a wider application meaning the mental act of obtaining the universal from the sensible particular) is evident. This would mean, of course, that imagination is present in mathematical abstraction, too, for it also begins with perception of changing sensible particulars. However, and this is a point which should be emphasized, Aristotle never refers to imagination as having a particular or special part in mathematics or mathematical abstraction.[7]

Press, 1960), I, 341 and *Mathematics in Aristotle* (Oxford: Clarendon Press, 1949), p. 1, that Aristotle refers only to the most elementary geometrical figures. As for Aquinas, I can only state that in my reading of him I have found nothing that would invalidate this same conclusion.

[5] *Posterior Analytics*, II, 19, 99b 36-100b 1.

[6] *De Anima*, III, 3.

[7] Some authors, particularly those inclined to read Aristotle through the eyes of St. Thomas, ignore this fact. See for example, Mère St. Édouard, "La division

It may be that he simply did not develop this point, or, of course, it may be that he did not think imagination had any special role in mathematical abstraction. An elaboration of this second possibility is in order. According to the Stagirite, imagination "has the objects of sense for its object."[8] Imagination is said to be the act of a sense faculty (though just what sense faculty is not clear)[9] and is clearly distinguished from the acts of the mind, affirmation and negation, and the knowledge of incomposites.[10] But the objects of mathematics according to the Stagirite are not sensible for, as we noted, the mathematician leaves out the proper sensibles. Though his abstraction is based upon perception of the common sensibles, it is not these *qua* sensible which he studies. Since mathematical quantities are not sensible, it would apparently follow that they are not imaginable either, for, as was said, imagination is the act of a sense power, it has "the objects of sense for its objects." Mathematicals, then, would be knowable only by the mind. We might note in support of this last statement that Aristotle calls the matter of mathematicals "intelligible"; he never refers to it as "imaginable."[11]

And yet, from another point of view it would seem that this very notion of intelligible matter indicates that mathematical

aristotélicienne des sciences, selon le professeur A. Mansion," *Laval Théologique et Philosophique,* XV (1959), 228 and M-V. Leroy, "Le savoir spéculatif," *Revue Thomiste,* XLVIII (1948), 303 ff. Frère Augustin-Gabriel, "Matière intelligible et mathématique," *Laval Théologique et Philosophique,* XVII (1961), 187, admits Aristotle does not have the doctrine and says one must "read between the lines" to find it.

[8] *De Anima,* III, 3, 428b 13.

[9] In his *On Memory and Reminiscence,* Aristotle states that imagination is an "affection of the sensus communis." (1, 450a 12) In the *De Somnis,* on the other hand, he distinguishes between that power which is the controlling or judging sense faculty (apparently the *sensus communis*) and that which presents images (2, 460b 16-18; see also 3, 461 18-31). Furthermore, he explicitly identifies the imaginative faculty with the sensitive faculty *qua* imaginative, though he does not say what this sensitive faculty is. (1, 459a 15-16)

[10] *De Anima,* III, 8, 432a 9-14.

[11] For a discussion of Aristotle's notion of intelligible matter, consult my article "Intelligible Matter and the Objects of Mathematics in Aristotle," *The New Scholasticism,* XLIII (1969), 1-28.

quantities must be imaginable, for intelligible matter for Aristotle is viewed by him precisely as the principle of *individuation* of mathematical forms.[12] Since individuals are attained directly only by sense and not by mind which is directly of the universal,[13] individual mathematicals could be grasped directly only by a sense faculty. But since the quantities studied in mathematics are not possessed of any proper sensible features, they cannot be grasped by the exterior senses. Would it, then, be imagination which grasps them? To be sure, Aristotle does speak in the *Metaphysics* of individual mathematicals as known by " intuition."

But when we come to the composite thing, e. g., this circle, i. e., one of the singular circles, whether sensible or intelligible (I mean by intelligible circles the mathematical, and by sensible circles those of bronze, or of wood)—of these there is no definition, but they are known with the aid of intuition or of sensation; and when they pass out of this actual cognition it is not clear whether they are or not; but they are always expressed and known by the universal formula.[14]

But is this intuition imagination? Some have so interpreted it; [15] Aristotle himself does not say. This much is clear from his text; it is not an act of direct sensation, nor is it an act of mind, that which grasps the definition, the universal formula. In the absence of statements to the contrary, it is logical to presume that it is imagination which is meant.[16] Though exactly how such entities could be imaginable, in view of the fact that they lack sensible qualities, is still a question.

But if Aristotle never mentions it, why this stress on my part on imagination? The reason is, and admittedly we are

[12] *Metaphysics*, VII, 11, 1036b 35-1037a 4.

[13] *De Anima*, III, 4, distinguishes sense knowledge from intellectual. See explicitly 429b 10-33. Also see *Metaphysics*, VII, 10, 1036a 1-12 and *Posterior Analytics*, I, 31, 87b 36-40; II, 19, 100a 15-100b 1.

[14] *Metaphysics*, VII, 10, 1036a 1-8.

[15] St. Thomas Aquinas interprets this intuition as imagination in *In VII Metaphysics*, L. 9, 1494-95.

[16] Diego Pro, " Filosofía de la matemática en Aristóteles," *Sapientia*, XI (1956), 99, discusses Aristotle's obscurity on this point.

looking ahead to Thomas Aquinas, if individual mathematical objects have their locus in imagination, it would follow that there is a certain degree of freedom on the part of the mathematician in regard to his objects. The Stagirite himself refers in various places to the freedom men have in imagining.[17] If the locus of individual mathematicals were the imagination it would seem to follow that the mathematician would be free to deal with objects which do not closely correspond to anything found in the physical world. There would be no reason to limit him to simply studying abstracted quantities which resemble the quantities of things, but he could treat quantities which he himself had devised in imagination which have no one-to-one correspondence to any physical quantities. Indeed, an epistemological basis could be provided for the tremendous development in modern times of nonrepresentational mathematical systems such as the nonEuclidean geometries.

Now it is true, as we mentioned earlier, that the Stagirite always cites as examples of geometrical objects figures which could easily be gained by abstraction from similarly figured sensible things. But our question is, does Aristotle in his philosophy of mathematics hold that the mathematician *must* limit himself to such easily abstractable entities? In attempting to answer this question it might be helpful to realize that it is only the most general and basic elements of the genus quantity, e. g., lines, planes, etc., that he explicitly mentions as obtained by abstraction.[18] Apparently all other mathematical objects are to be constructed out of these basic abstracted entities. No science, Aristotle says, demonstrates the very existence of the subject with which it deals.[19] The mathematician, then, apparently at first posits the existence of these most basic ele-

[17] *De Anima,* III, 3, 427b 18-20; 11, 434a 9.

[18] Thomas Greenwood, "Aristotle on Mathematical Constructibility," *Thomist,* XVII (1954), 89 and 93. The fact that these elements are so general and hence so easily abstracted may well be the reason why Aristotle says that little experience is needed in order to become a mathematician (*Nicomachean Ethics,* VI, 8, 1142a 16-19).

[19] *Posterior Analytics,* I, 10, 76b 3-23.

ments of the genus quantity gained by abstraction [20] and then through construction using these elements goes on to " demonstrate the existence " and investigate the properties of all the other objects he deals with. (Aristotle does say that before the properties of a mathematical object can be investigated it must be *demonstrated* that that object exists.[21] The actual practice used at his time to " demonstrate " the existence of a particular mathematical quantity was to construct it.) [22] Our question is then is the mathematician free to use these basic elements to construct (and hence demonstrate the existence of) any figure he desires—any figure that is, whose very existence is not self-contradictory (like square circles) ? Certainly the most basic abstracted elements, those whose existence is simply posited, are so general as to be able to form any figure or number. And yet the Stagirite never states that the mathematician has the freedom to construct these elements into any non-self-contradictory objects he pleases. In fact, it is just the opposite as we have said, the only objects of geometry he cites are those which closely resemble physical magnitudes. Could this indicate that he never thought of allowing the mathematician freedom to construct and treat objects not resembling quantified physical things? [23] On the other hand, it might be suggested that Aristotle would never have intended such a limitation of mathematics since numbers by their very nature as more abstract than magnitudes are clearly not able to be closely bound to physical quantities.

[20] *Ibid.*, 76b 3-7.

[21] *Ibid.*, 76b 8-10.

[22] This is pointed out by Heath, . . . *Greek Mathematics*, I, 337 and 377; Greenwood, ". . . Mathematical Constructibility," 89-93; H. G. Apostle, *Aristotle's Philosophy of Mathematics* (Chicago: University of Chicago Press, 1952), p. 42. Euclid, for example, always constructed a particular mathematical entity before making use of it in a demonstration; for example, only after he had constructed a square did he go on to study it; only after he had constructed a perpendicular to a straight line did he use lines at right angles to one another. Though Aristotle does not explicitly say what he means by the demonstration of the existence of a mathematical, it seems most reasonable to conclude that the Stagirite has in mind the common Greek practice of construction.

[23] Greenwood, ". . . Mathematical Constructibility," 93-94 and " The Characters of the Aristotelian Logic," *Thomist*, IV (1942), 244, seems to hold this position.

In reply to this last point we must bring out some interesting features concerning the way the Greek mathematicians of Aristotle's time tended to look upon their science. In the first place, it should be pointed out that among the Greeks arithmetic *was* closely tied to geometry and to actual physical magnitudes. In general number theory was treated by them in the framework of geometry.[24] From the time of the Pythagoreans on, numbers were often represented geometrically.[25] Euclid, for example, (about a generation after Aristotle) represents numbers by straight lines, planes, squares, cubes, etc.[26] This is especially true of irrational numbers, e. g., the square root of two which could not be assigned a definite numerical value but could be represented by magnitudes.[27] Furthermore, the Greeks had no notion of imaginary numbers or of negative numbers, numbers which could hardly be said to correspond to numerical aspects of physical things. Instead, the only numbers they used were the ordinary whole numbers and ratios, 1, 2, 3, $\frac{1}{2}$, $\frac{1}{4}$, etc. Interestingly enough, it is not until Diophantes (late third century A. D.) that we find any mathematical equations used which involve numbers raised to any power above three, the cube.[28] Apparently, because there is no physical

[24] Heath says, " With rare exceptions . . . the theory of numbers was only treated in connexion with geometry, and for that reason only the geometrical form of proof was used, whether the figures took the form of dots marking out squares, triangles, gnomons, etc. (as with the early Pythagoreans), or of straight lines (as in Euclid VII-IX). . . ." (. . . *Greek Mathematics*, I, 16) Heath also points out that even problems which we would call algebraic were only solved geometrically by the Greeks. (*Mathematics in Aristotle*, p. 223, also . . . *Greek Mathematics*, I, 379 ff. See also his explanation of " geometrical algebra," pp. 150-154.)

See also M. R. Cohen and I. E. Drabkin, *A Source Book in Greek Science* (Cambridge, 1958), p. 1 and p. 14, n. 1.

[25] Heath, . . . *Greek Mathematics*, I, 76 ff.

[26] Heath, *ibid.*, I, 16, 98 and 379 ff.; *Mathematics in Aristotle*, p. 222.

[27] The square root of two would be represented simply by drawing a square of sides one and one whose diagonal would then be the square root of two. Many authorities feel that it was the discovery of the irrational that turned the Greeks in the direction of geometry and accounted for the " geometrizing " of number. See, for example, Marshall Clagett, *Greek Science in Antiquity* (New York: Abelard-Schuman, Inc., 1955), p. 57 and Cohen and Drabkin, *op. cit.*

[28] Cohen and Drabkin, *A Source Book . . .*, p. 25.

magnitude which has more than three dimensions, the Greeks felt any higher power would be meaningless. The very terms they used in arithmetic, some of which are still in use today, probably show more than anything else the geometrical framework in which this study was carried on. Our terms like square (a number is squared when it is multiplied by itself once) and cube (a number is cubed when it is multiplied by itself once and this in turn multiplied by the given number) clearly indicate their geometrical origin. (Plato even refers to square and cube numbers as planes and solids respectively.) [29] Indeed, numbers were referred to by the Greek mathematicians as cubes, squares, as oblong, triangular, polygonal, diagonal, as sides, as rectilinear, scalene, spherical, circular—all fundamentally *geometrical* terms.[30] A certain kind of proportion between numbers was called a geometrical proportion.[31] Various quadratic equations were solved geometrically using the construction of figures.[32] Clearly, as we said, Greek arithmetic was closely tied to geometry and then to physical magnitudes.

Since Aristotle, too, uses some of these geometrical terms in reference to numbers,[33] this could indicate that he shares the views of his countrymen that arithmetic is closely related to geometry and thus that numbers somehow relate to physical magnitudes. Thus, the arithmetician also may be considered by the Stagirite to be restricted to constructing and hence treating objects in some way corresponding to physical things.[34]

[29] The reference to Plato is in Heath, . . . *Greek Mathematics,* I, 89.

[30] All these expressions can be found between pages 76 and 117 in Heath, . . . *Greek Mathematics,* I.

[31] Heath, *ibid.*, I, 85.

[32] Heath, *Mathematics in Aristotle,* p. 223; . . . *Greek Mathematics,* I, 379 ff. B. L. van der Waerden, *Science Awakening* (New York: Science Editions, 1964), pp. 118-126.

[33] *Physics,* III, 4, 203a 13-15; *Posterior Analytics,* I, 12, 78a 4; *Nicomachean Ethics,* V, 3, 1131b 12-15. A particularly significant text is in the *Metaphysics,* V, 14, 1020b 3-6, where he refers to number in one or more *dimensions,* ". . . numbers which are composite and not of one dimension only, viz. those of which the plane and the solid *are copies,*" [italics mine] and of other similar features of numbers which he calls their " qualities."

[34] I do not mean to imply by this that Aristotle denies the specific distinction

True, he in no place explicitly states that there is this restriction, on either geometry or arithmetic. Yet neither does he give an indication that he feels that the mathematician, either geometer or arithmetician, is free to construct or consider objects which do not in some way correspond to physical quantities. And most important, though there is nothing in his philosophy of mathematics which positively precludes this freedom, compared to St. Thomas, there is precious little that could form the epistemological basis for such freedom. It seems reasonable to conclude, then, in the absence of statements to the contrary, that Aristotle in this respect is a man of his time, i.e., he considers the objects of mathematics to be idealized representations of actual physical quantities and the mathematician to be restricted to such objects.

In concluding this section we should note the one text that some claim gives some indication (though I believe it to be extremely slight) that the Stagirite has some recognition of the freedom of the mathematician.[35] Aristotle refers to the necessity present in mathematical science as of a hypothetical type. He states specifically that " It is impossible, for instance, on a certain hypothesis that the triangle should have its angles equal to two right angles. . . ." [36] On a different hypothesis, if a straight line, for example, is defined in a different way, the value of the interior angles will be two right angles. Does this imply that either hypothesis is permissible? To generalize, does this mean that the mathematician is free to construct and define his figure any way he pleases? Note clearly that Aristotle never

between arithmetic and geometry, between number and magnitude. Just the opposite. For instance, he criticizes the Pythagoreans for turning units into magnitudes. Nevertheless, even though he does assert the specific difference between the objects of arithmetic and of geometry, there is no indication that this leads him to disagree with his contemporaries who consider number in a geometrical context as representative of magnitudes. Numbers certainly are not magnitudes; they cannot be reduced to magnitudes; but still they can *represent* (Aristotle calls them *copies* in text of previous footnote) magnitudes.

[35] Two who make this claim are Greenwood, ". . . Mathematical Constructibility," 91-93, and Heath, *Mathematics in Aristotle*, p. 101.

[36] *De Caelo*, I, 12, 281b 5-6.

says this. Indeed, it seems impossible to say he is even implying that either hypothesis is permissible. He is saying simply that if a different hypothesis were chosen different conclusions would follow. He never says that either one *can be* chosen. At best the passage shows that he does recognize that different conclusions follow from different premises, but nowhere does he really say that the premises are a matter of free choice. Indeed, in the light of all we have already seen, viz., that the only geometrical objects he mentions are those resembling real quantities, and that numbers, too, at his time corresponded to physical things and their quantitative features, the indication that he broke with the prevalent view of his time that mathematical objects are limited to representation of physical quantities seems very slight.

Let us now consider the philosophy of mathematics of Thomas Aquinas with a view toward seeing if he has any more explicit recognition of or epistemological basis for the freedom of the mathematician.

III. The Freedom of the Mathematician According to Aquinas

We should remind ourselves at the very beginning of the areas of agreement of Thomas and Aristotle. For Thomas, like his predecessor, mathematics is a science of quantity abstracted from physical things, i. e., of real quantity not considered *qua* real. Does this mean that he limits mathematics to quantities closely resembling real things? We must reply that it is only such quantities that he, like Aristotle, explicitly mentions. And yet there are doctrines of his, doctrines not explicitly expressed by the Stagirite, that seem to provide the basis for a greater freedom on the part of the mathematician.

One such doctrine has to do with mathematical abstraction itself and the objects which are its result. In one text, Thomas describes these objects in a manner that indicates that he is much more aware than Aristotle of their great independence from (even though they are based upon) physical things.

Aristotle, of course, clearly affirmed that mathematical quantities exist as such (i. e., with their peculiar mathematical characteristics) only in the mind of the mathematician. Aquinas not only agrees with this but goes on to describe the objects of mathematics in terms which he uses to describe beings of reason.[37] He explains that, like the logical notions of genus, species, etc., a mathematical is *not* simply a likeness of realities existing outside the mind but instead is a consequence of man's way of knowing some things outside the mind. Things of this type, he says, are intentions which our intellect devises (*adinvenit*) because of its knowledge of extramental things. And he adds, significantly, the *proximate* foundation for such intentions is not "in things, but in the intellect, however the remote foundation is the thing itself."[38] The expressions used here by Aquinas to describe mathematical entities are the same as those he uses in other places to describe beings of reason.[39] This is not to say that mathematical quantities are simply created by man's intellect, for the intellect's act is of course rooted in physical things. But this is to say that that which immediately gives mathematicals their reality, that which is their proximate foundation, is the activity of the mind itself. (This is not, of course, the case with the beings studied in either physics or metaphysics. They exist in their own right apart from any act of a human intellect.)

I would like to suggest a contrast, or at least a difference in emphasis, between Aquinas and Aristotle on this point. The difference as I see it is that, compared to St. Thomas, Aristotle tends to view the mathematician as more passive in his act

[37] *In I Sententiarum*, d. 2, q. 1, a. 3 c (Parma edition, VI, p. 23). (Incidentally, this passage was written by Aquinas late in his life and inserted in his *Commentary*. It should, therefore, give his mature position on the subject. On this point, see A. Maurer, "A Neglected Thomistic Text on the Foundation of Mathematics," *Medieval Studies*, XXI (1959), 187.)

[38] *In I Sent., loc. cit.*

[39] In *In IV Metaphysics*, L. 4, 574, for example, St. Thomas states that in contrast to a natural being an *ens rationis* is strictly speaking an intention which reason devises from the objects it considers, an intention which is not found in the nature of things but is a consequence of the consideration of reason.

of abstraction. To be sure, he " strips away " all the nonquantitative features of physical things—and this " stripping " itself is an activity on his part. Yet when it comes to the actual grasping of physical quantity the connotation is that the mathematician simply grasps what remains after all nonquantitative features are removed. He simply " liberates," so to speak, the real quantities of things from their sensible, mobile, material existence, and proceeds to study them—real quantities but not *qua* real. Now it is certainly true that Aquinas in many places, especially his *Commentaries*, speaks of the mathematician's abstraction in the same terms that his Greek predecessor uses. (See, for example, texts cited in my first Section.) Nevertheless, in the passage discussed in the previous paragraph he shows, I believe, more recognition of the activity of the intellect in the actual production of mathematicals. The mathematician does not just grasp real quantity stripped clean, he does not simply study a likeness of real quantities, rather his object is directly a product of his intellect's own activity—granted that the activity has its *remote* foundation in the experience of physical quantities.

Now in putting stress on the intellect as the proximate foundation of mathematicals, in stressing therefore that these entities are not mere likenesses of physical things, in describing mathematicals as similar to beings of reason, it seems to me that St. Thomas indicates much more clearly than did the Stagirite that he recognizes that the mathematician's activity of abstraction, and hence the object of his science, is not simply a replication of real physical quantities. And there are other doctrines of Aquinas which also have as their result the freeing of the mathematician from strict dependence on physical quantities, doctrines which also bring more precision into Thomas's statement that " the intellect " is the proximate foundation of mathematicals. Of great significance is his teaching on the role of imagination in mathematics. We will first discuss that role in general and then its specific relevance to the question of freedom in mathematics.

As is well known, the imagination for Aquinas plays a vital role in all knowledge, for he believes there can be no intellectual knowledge without the phantasms it supplies.[40] Of particular import to our topic, however, is the special role it has in the science of mathematics. Unlike Aristotle, Thomas leaves no doubt that he holds that mathematicals, some at least, are imaginable. His texts which assert this are numerous; I will cite only one.

When sensible characteristics are removed there remains something which is apprehended by the imagination. . . . Now mathematicals are of this sort.[41]

Of course, in speaking of mathematicals being grasped by the imagination, Thomas is referring to individual mathematicals, not to mathematical essences which are grasped only by the intellect. We noted in the previous section that in one place Aristotle spoke of individual mathematicals as grasped by " intuition," and distinguished this from mathematical essences which are grasped by the mind. We noted also that he defined this intuition no further. St. Thomas clearly refers this intuition to imagination.[42] Individual mathematicals as such are not attained by external senses, nor as individual are they present in the intellect which is directly of the universal. Yet as individual they must be grasped by a sense power—the imagination.[43]

And yet, to say that individual mathematicals are imaginable presents problems of its own. We noted in the previous section that Aristotle never asserts that mathematicals are imaginable,

[40] *Summa Theologiae,* I, q. 85, a. 1 c; *In III De Anima,* lect. 12, 781.

[41] *De Trinitate,* q. 6, a. 2 c. Other texts which affirm that mathematicals are imaginable are: *De Trinitate,* q. 6, a. 1 c and a. 2 c; *De Veritate,* q. 15, a. 2 c; *In VII Metaphysics,* lect. 10, 1495; *In III De Anima,* lect. 8, 715-6; *Summa Theol.,* I, q. 7, a. 3 c; *In III Physics,* lect. 7, 341; and *In VI Nic. Eth.,* lect. 7, 1210, 1214.

[42] *In VII Metaphysics,* lect. 10, 1494-5.

[43] In the following passage Thomas clearly distinguishes the individual mathematicals which the imagination grasps from the essence of these mathematicals which is grasped by the intellect.

" In the case of mathematics it can be shown that that which knows the essence, i. e., the intellect, is distinct from what apprehends mathematical objects themselves, i. e., the imagination." (*In III De Anima,* lect. 8, 715) and we suggested why. The imagination is a sense power, but in his abstraction the mathematician leaves aside sensible qualities. How then can nonsensible mathematicals be grasped by the imagination?

Because of this difficulty, some commentators have suggested that the mathematicals which Thomas designates as imaginable are not really the individual, quality-less, non-three-dimensional objects of mathematics but only individual sensible objects which come very close to being like them, e. g., a colored line made up of very small dimensions but not actually colorless or unidimensional.[44] But this interpretation is contrary to too many explicit statements by Aquinas. In no uncertain terms he asserts that mathematical objects (in other words, the qualityless, uni- and bidimensional entities) are in the imagination. For example, in the *De Trinitate* he asserts:

> Mathematicals themselves come under the senses and are objects of imagination, such as figures, lines, numbers and the like.[45]

And there are countless places where he makes the same assertion.[46] In fact, mathematics is the most certain science, he says, precisely because its objects are free from sensible matter and yet imaginable.[47] The problem, therefore, remains—how can objects lacking sensible qualities be apprehended by a sense power?

The solution must lie in showing that Aquinas believes mathematicals to be sensible; in other words, in showing that mathematical abstraction does not leave aside all the sensible attributes of quantified physical things. Bear in mind that quantity is a common sensible and that the common sensibles,

[44] Some who hold this view are Bernard Lonergan, "Note on geometrical possibility," *Modern Schoolman*, XXVII (1950), 127; É. Winance, "Note sur l'abstraction mathématique selon saint Thomas," *Revue Philosophique de Louvain*, LIII (1955), 509; F. Collingwood, "Intelligible Matter in Contemporary Science," *Proceedings of the American Catholic Philosophical Association*, XXXVIII (1964), 110.

[45] *De Trinit.*, q. 6, aa. 1, 2 c.

[46] See the texts cited in footnote 41.

[47] *De Trinitate*, q. 6, aa. 1, 2 c.

like the proper sensibles, are directly, not incidentally, sensed.[48] Would it not be possible then for the imagination, which is able to combine and divide imaginary forms and so end up with images "even of things not perceived by the senses,"[49] to present an image of an originally apprehended physical thing which image would be of only part of that thing, viz., of some or all of its dimensions minus all of its proper sensible qualities? This ability of the imagination would explain how Thomas can say in reference to mathematicals that "even when sensible characteristics are removed there remains something which is apprehended by the imagination."[50] At least some mathematicals are sensible, and hence imaginable, because they are the abstracted dimensional quantitative features of physical things.[51] However, these imagined dimensions are mathematical and not physical because by the power of imagination they have been separated from the other sensible characteristics of physical things and may have even been reduced in dimension from the physical three dimensions. What I am suggesting in effect is that the imagination itself performs an abstraction on the common sensibles; after all, it is not only the intellect which abstracts according to Aquinas.[52]

[48] *Summa Theol.*, I, q. 78, a. 3, ad 2.

[49] *Ibid.*, a. 4 c.

[50] *De Trinitate*, q. 6, a. 2 c.

[51] I say "at least some" (not all) mathematicals are sensible and hence imaginable. In *In III De Anima*, lect. 7, 758, Thomas, following Aristotle, apparently says that points, which are dimensionless units having position, and units, which are both dimensionless and positionless, precisely because they lack all dimension cannot be grasped by any sense power but are only known mentally by negation. It would follow that a number, which is a plurality of units, would not be imaginable, though some symbol representing it could be.

[52] To abstract, St. Thomas says, is to consider one entity without another when they are actually together in reality. (*De Trinitate*, q. 5, a. 3 c) Since each sense power considers only what is proper to it and omits all other features of the material thing, it can truly be said to abstract. Cf. *Summa Theol.*, I, q. 85, a. 3, ad 2.

One should not identify this abstraction of the imagination with the second degree of abstraction, else he will end up with the difficulty Winance has, "Note sur l'abstraction mathématique . . . ," 507 ff. He clearly sees that merely eliminating sensible qualities by the imagination does not result in an object of a different

Incidentally, the fact that St. Thomas continually refers to mathematicals as " nonsensible " does not contradict this conclusion. For in making such statements it seems clear that the sensible features from which he considers the mathematician to abstract are the accidents which follow *after* the accident of quantity. " Accidents," he says,

> befall substance in a definite order. Quantity comes first, then quality, then passions and action. So quantity can be considered in substance before the sensible qualities, in virtue of which matter is called sensible, are understood in it.[53]

Clearly, the sense qualities he is talking about, those which follow quantity, are only the proper sensibles. Since the mathematician does not abstract from the accidents of quantity neither does he abstract from all sensible features, for quantity is a common sensible. The dimensional figures studied by mathematicians are not sensible inasmuch as they lack all proper sensible qualities. Since it is " the sensible qualities [which follow after quantity] in virtue of which matter is called sensible," the mathematicals can be called nonsensible. They are sensible, and hence imaginable, however, inasmuch as they are abstracted dimensions, for dimensions are sensible.[54]

degree of intelligibility, or indeed in any intelligibility at all. Therefore, because he has identified this abstraction of the imagination with the second degree of abstraction, he denies it any validity as a means of distinguishing the intelligible objects of the sciences, 510. The degrees of abstraction for St. Thomas refer to abstraction by the *intellect* from matter and motion, *De Trinitate,* q. 5, a. 1 c.

[53] *De Trinitate,* q. 5, a. 3 c. See also *Summa Theol.,* I, q. 85, a. 1, ad 2.

[54] In the previous section in order to emphasize the fact that Aristotle never refers to mathematicals as imaginable, we pointed to his use of the term intelligible, rather than imaginable, to designate the special kind of matter found in mathematicals. Some have in fact suggested that since some mathematicals are imaginable according to Aquinas he should have designated their matter as imaginable, rather than retaining the Aristotelian designation of it as intelligible. (Winance, " Note sur l'abstraction mathématique . . . ," 508-510) However, such a change of terminology is unnecessary, since in its most fundamental sense intelligible matter designates for Aquinas substance as the substrate of only the accident of quantity. But he notes, " the sense powers do not reach a comprehension of substance," (*De Trinitate,* q. 5, a. 3 c) only the intellect does. Therefore, substance as the substrate of quantity is properly termed "intelligible " matter. On this point, see my article,

It follows from all of this that the imagination has an especially important role in mathematics for Aquinas—a role which, as we have said, is never mentioned by the Stagirite. For, in addition to providing a stable image from which the universal can be abstracted (this it does in all abstraction),[55] in mathematics it furnishes to the intellect perfectly appropriate individual mathematicals, which simply cannot be found in nature, individuals from which the mathematical essence can then be abstracted. The direct senses are able to supply an appropriate object for the abstraction of physical essences; for the intellect's abstraction the imagination simply provides a stability in the changing objects grasped by sense. But the direct senses themselves cannot provide a perfectly appropriate object for abstraction of mathematical essences, for mathematical objects as such are not attainable by these senses. Rather the imagination, through its abstraction discussed above, provides the proper object, the suitable individual mathematical quantity, from which the mathematical essence can be abstracted.

By locating individual mathematicals in imagination, Thomas has served to further liberate the objects the mathematician studies from a close dependence on physical quantities. This freedom is even more clearly brought to the fore by his assertion that the judgments of mathematics need only *terminate* in the imagination. In a passage of the *De Trinitate* Aquinas distinguishes between the origin and the termination of man's knowledge.[56] "Now the beginning of all our knowledge," he writes, "is in the senses"; however, the termination of knowledge is different in each of the three general kinds of science,

"Intelligible Matter and the Objects of Mathematics in Aquinas," *The New Scholasticism*, LXIII (1969), 555-576, in which I distinguish the various meanings of intelligible matter in Aquinas.

[55] *De Trinitate*, q. 6, a. 2. On this point one might profitably consult the articles by C. De Koninck, "Abstraction from Matter: Notes on St. Thomas's Prologue to the *Physics*," *Laval Theologique et Philosophique*, XIII (1957), 140-1 and W. Gerhard, "Natural Science and the Imagination," *Thomist*, XVI (1953), 190-216.

[56] *De Trinitate*, q. 6, a. 2 c.

metaphysics, mathematics, and physics. " Judgment in mathematics," he asserts, " must terminate in the imagination." I take this to mean that these judgments are true of, refer to, imaginable entities. Thomas explains that, if a judgment is true of realities which are only intelligible, it must stop, " terminate," in the intellect, as do metaphysical judgments; it could not refer to imaginable or sensible realities and still be true of purely intelligible entities *qua* intelligible. Thus, if a judgment is true of imaginable entities which are not sensible, it must stop, " terminate " in the imagination:

> ... because, when sensible characteristics are removed there remains something which is apprehensible by the imagination, we must judge about such things according to what the imagination reveals.[57]

Finally, a judgment true of sensible realities, as in physics, must stop in the senses. To repeat, since mathematicals according to St. Thomas are neither sensible things of nature, nor purely intelligible realities, but (some at least) are imaginable, a judgment about these objects cannot terminate in the senses, nor simply in the intellect, but rather must do so in the imagination. In order to be true, judgments dealing with imaginable objects must refer to what the imagination presents.

There is another way of looking at this notion that judgments about mathematicals terminate in the imagination. According to Aquinas, in the mental act of judging we grasp the existence of an object, we grasp an entity as it *is*. This is distinguished from the act of apprehension which only grasps the nature of a thing and not its act of existence.[58] Now since some individual mathematicals exist as such by and in the imagination, it stands to reason that the act of judgment must refer to, terminate in, that which the imagination presents. In this connection, we mentioned in the previous section that Aristotle maintains that before a mathematical entity can be examined it must be " demonstrated " that it exists. Though he never said exactly how demonstrations of existence take place, judging from the common practice of his time he is referring to

[57] *Ibid.*

[58] *Ibid.*, q. 5, a. 3 c.

the construction of these objects. Thomas also speaks of demonstrations of existence in mathematics, and he designates them as " operational " since they are by construction.[59] Now since this construction can only be of *individual* mathematicals (it makes no sense to speak of " constructing " a mathematical essence), it must take place in a sense power. But the only sense power which grasps individual mathematicals as such is the imagination. Hence, the locus of the construction of individual mathematicals must be this power. In other words, it is in the imagination that mathematicals are shown to exist, and this, of course, squares with the previously mentioned point that judgments of their existence must terminate in and only in the imagination.

Of course, as we have pointed out, if the mathematician's judgments need only refer to imagined entities, this makes the mathematician very free in his choice of objects and the operations he performs on them. While in physics and metaphysics the intellect must conform itself to sensible being and intelligible being respectively as they are in reality, in mathematics the intellect need only conform to beings which exist in the mathematician's imagination.

Both this position and the earlier one which stressed the intellect's activity as the proximate foundation of the objects of mathematics clearly show that Aquinas considers the mathematician to be free from treating only objects which resemble physical things. Yet how free? Is the mathematician free to construct any mathematical he can and then go on to investigate its properties? Perhaps it would be of some help to look more closely at the passage in which Aquinas speaks of mathematical demonstrations of existence—for this passage also sets forth clearly his analysis of the general procedure of the mathematician in his science. (One will note that it is the same general procedure Aristotle recognized.)

[59] *In I Posterior Analytics*, lect. 2, 5. Thomas also refers to construction in mathematics as the means of demonstrating the existence of mathematicals in *In II Posterior Analytics*, lect. 6, 4.

There is supposed in these [mathematical] sciences those things which are first in the genus of quantity such as unity and line and surface and other such. These being presupposed, certain other things are sought by demonstration, such as the quadrilateral triangle, the square in geometry, and other such things. These demonstrations are said to be, so to speak, operational, as is: On a given straight line to construct an equilateral triangle. This having been proved, certain further passions are proved, as that its angles are equal or some other such thing. . . .[60]

The mathematician supposes that those entities " which are first in the genus quantity " exist (in imagination) and using these entities goes on to construct, to demonstrate " operationally " certain figures or numbers composed of them. These constructions show that these composite objects do exist, and he then proceeds to prove the properties of these figures or numbers. As for the freedom of the mathematician in his demonstrations of existence, it would seem that he is at liberty to construct in imagination any mathematicals he can, and this would apparently mean any quantities whose existence is not self-contradictory. As far as the most basic quantities are concerned, these seem to present no limitation either. Certainly, as St. Thomas says, these elements—units, points, lines, and surfaces—are ultimate in the genus quantity. Nothing more basic could be abstracted and " supposed " by the mathematician—and indeed, since they are the most basic quantities, how could the mathematician do anything else but " suppose " them? [61] These certainly contain no built-in limitation as to what the mathematician can study, for they are able to make up any mathematical object in the imagination. They present no limitation other than that the mathematician must deal with quantity.

[60] *In I Posterior Analytics*, lect. 2, 5.

[61] We might point out here that it is not up to the mathematician as such to investigate the real foundation of those elements whose existence he assumes. He simply takes them and goes to work from there. It would seem to be the province of the philosopher of nature to show the basis in reality of these quantitative elements and hence to show that they are not mere mental fictions.

IV. Conclusion

We have stressed the fact that, because individual mathematicals are located by Aquinas in the imagination and hence mathematical constructions of existence and scientific judgments need refer to only such entities, the mathematician is radically free in his choice of objects, and more specifically he need not consider himself limited to dealing with mathematical quantities which closely correspond to and/or resemble physical quantities. We have also suggested that this freedom is indicated by Thomas's teaching that it is the intellect's activity, not things, which is the proximate foundation of mathematical objects and, following from this, his description of mathematicals as similar to beings of reason (though one might quarrel with Aquinas and propose that it would be more accurate to say rather that the imagination's activity under the direction of the intellect is the proximate foundation of individual mathematicals).

Though these doctrines provide an epistemological foundation for the freedom of the mathematician from physical things as far as his object is concerned, it remains the case that, like Aristotle, Thomas also refers only to mathematical quantities which in fact resemble physical quantities. The only geometrical figures and solids he mentions are those of Euclidean geometry. He too refers only to *real* numbers (not negative or imaginary), and he refers to them in terms which may indicate that they are still being viewed as related to physical magnitudes. For example, he refers to numbers as surfaces, as solids, as two and three-dimensional, as squares, cubes, etc. (though he clearly recognizes that such words are used metaphorically),[62] and he never refers to a number raised to any power higher than three, the cube.

It is true, of course, that by the thirteenth century mathematical objects were not considered to be simply idealized representations of actual physical quantities, at least not to

[62] *In V Metaphysics*, lect. 14, 974; lect. 16, 989-991.

the degree that they were in Aristotle's day. For one thing, the algebra had been introduced by the Arabs and put into Latin by some of the earliest translators.[63] According to historians of mathematics, the most prominent mathematics book in Latin during Aquinas's time was probably the *Liber Abaci* by Leonardo Fibonacci (Leonardo of Pisa), published in 1202, and it was devoted to arithmetic and elementary algebra. Though it contained no recognition of negative or imaginary numbers,[64] it did have, in addition to the algebra, the use of the zero and of fractions and operations upon them.[65] Furthermore, during Aquinas's day symbols were more and more being used to represent quantities; in fact, one who pioneered this was a friar, Jordanus de Nemore, who in 1222 became general of the Dominican Order. Certainly, the use of symbols, instead of figures or numbers related to figures, to stand for quantities, implies a view of mathematics which sees its objects removed from direct correspondence to physical quantities. In fact, the use of the zero alone indicates this, for it has no physical counterpart, and, indeed, for this reason it was looked upon by many as suspect.

It is difficult to believe that Thomas Aquinas, who in other areas was so keenly cognizant of the newly introduced knowledge of his time, would not at least have been aware of these developments in the mathematics of his day. Indeed, one author speculates that St. Thomas as a student used in his

[63] Maurer, "A Neglected Thomistic Text . . . ," 185.

[64] First used by Raffael Bombelli, 1550. (D. Struik, *A Concise History of Mathematics* [New York, 1948], p. 114)

[65] For information on this book, its author, and the general state of mathematics in the thirteenth century, consult F. Cajori, *A History of Mathematics* (New York: The Macmillan Company, 1951), pp. 117-125; H. Eves, *An Introduction to the History of Mathematics* (New York, 1961), pp. 209 ff. See also T. Greenwood, *Études sur La Connaissance Mathématique* (Ottawa: Ottawa University Press, 1942), pp. 66 ff.

[66] Greenwood, *Études sur . . .* , p. 65. However, Vernon Bourke, in his more recent work, *Aquinas' Search for Wisdom* (Milwaukee, 1965), says that the quadrivium was no longer followed in the thirteenth century because masters proficient in the mathematical sciences were scarce, p. 22. And he gives nothing to support the view that Thomas was taught the "new mathematics."

studies the *Liber Abaci*, for it was a commonly used text in the quadrivium.[66] Be that as it may, I know of no place in Aquinas's writings where he explicitly refers either to the algebra or to the zero or to the use of symbolism in mathematics. He, like Aristotle, refers only to figures and numbers which correspond to physical quantities.

Nevertheless, in spite of this, it seems clear to me that the aforementioned epistemological doctrines of Aquinas go much further than Aristotle's toward allowing great freedom to the mathematician. It may well be that Thomas himself was barely aware of the consequence of his own position. But it still remains that his teachings which emphasize that it is man's intellectual activity not physical things which is the proximate foundation of mathematical objects, and in particular his stress on the role of the imagination as that in which individual mathematicals are demonstrated to exist and in which mathematical judgments terminate, are at best only implied in Aristotle. And it is these doctrines which serve to liberate mathematics from any requirement of dealing with quantities which match real quantities.

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