1-1-2010

Local Out-Tournaments with Upset Tournament Strong Components I: Full and Equal \{0,1\}-Matrix Ranks

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Published version. *Ars Combinatoria, Volume 95, No. 1 (2010).* Publisher link. © 2010 Charles Babbage Research Center. Used with permission.
Local Out-Tournaments with Upset Tournament
Strong Components I: Full and Equal
\{0,1\}-Matrix Ranks

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Abstract
A digraph \( D \) is a local out-tournament if the outset of every vertex is a tournament. Here, we use local out-tournaments, whose strong components are upset tournaments, to explore the corresponding ranks of the adjacency matrices. Of specific interest is the out-tournament whose adjacency matrix has boolean, nonnegative integer, term, and real rank all equal to the number of vertices, \( n \). Corresponding results for biclique covers and partitions of the digraph are provided.

1 Introduction
The topics of local tournaments, \( \{0,1\} \)-matrix ranks, upset tournaments, and digraph biclique cover and partition numbers have been the foundation of many papers in the area of graph theory. Work in the area of local tournaments originates with Bang-Jensen [1]. Further work includes Bang-Jensen et al. [3], Bang-Jensen, Hell, and Huang ([4], [16]), and Huang [17], with the introduction of local in- and out-tournament digraphs by Bang-Jensen et al. [5].

Biclique cover and partition numbers of bipartite graphs and digraphs, as well as the related matrix ranks of the corresponding adjacency matrices, have been popular research topics during the past twenty-five years. As the answer to the interesting question of what digraphs have adjacency matrices with equal semiring ranks remains elusive, many have partially answered the question by considering certain classes of digraphs. The following list represents only a portion of the research that has been generated by this interest. See Brualdi et al. [7], Barefoot et al. [6], deCaen [9], Doherty et al. [10], Gregory et al. [11], Heffner (Factor) et al. ([12], [13], [14],[15]), Lundgren and Stiewert ([18], [19], [20]), Maybee and Pullman [21], Monson et al. [22],
We further this research by bringing together concepts from these areas, and beginning the exploration of matrix ranks of the adjacency matrices of local out-tournaments. This is done through the use of upset tournaments that serve as the building blocks of the local out-tournaments. In this paper, we are interested in isolating the digraph structures that have adjacency matrices with full real rank, which is equal to the boolean, nonnegative integer, and term ranks.

The structure of the local out-tournament is determined in the first part of this paper following the definitions and preliminary results. Additionally, upset tournaments are defined and then used as the strong components of local out-tournaments. The resulting adjacency matrices are examined to determine which of these digraphs have corresponding adjacency matrices, \( A \), where \( r(A) = r_B(A) = r_{2+}(A) = r_t(A) = n \). Similar results follow for the associated biclique cover and partition numbers of the out-tournaments. Finally, open questions are discussed.

2 Terminology and Preliminaries

Many notational conventions are adopted from Bang-Jensen and Gutin [2]. A digraph \( D = (V, A) \), where \( V(D) \) is the nonempty vertex set of \( D \) and \( A(D) \) is the arc set of \( D \). For any arc \((u, v) \in A(D)\), we say that \( u \) dominates (or beats) \( v \), and write \( u \rightarrow v \). The outset of a vertex \( v \), \( O^+(v) \), is the set of all vertices that \( v \) dominates, and \( \{O^+(v)\} = d^+(v) \). Similarly, the in-set of a vertex \( v \), \( O^-(v) \), is the set of all vertices that dominate \( v \), and \( \{O^-(v)\} = d^-(v) \). In this paper, all digraphs are considered to be loopless. If we condense \( D \) by replacing each strong component with a vertex, the strong component digraph, \( SC(D) \), is obtained. A digraph \( D \) is connected if its underlying graph is connected.

A tournament is a digraph where each pair of vertices defines exactly one arc. A local out-tournament (respectively, local in-tournament) is a digraph where the outset of every vertex is a tournament (respectively, the in-set of every vertex). For ease in notation, these digraphs will often be referred to as out-tournaments and in-tournaments. A local tournament is a digraph where both the in-set and out-set of every vertex is a tournament. Local tournaments are also referred to more generally as locally semi-complete digraphs. To use the language of the majority of the research done on biclique covers and partitions and the associated matrix ranks, the authors will use the more specialized terms of local, in- and out-tournaments. Related to the results on in- and out-tournaments is out-branching and in-branching. A subdigraph \( T \) of \( D \) is an out-branching if \( T \) is a spanning, oriented tree of \( D \) and \( T \) has only one vertex \( v \) of indegree zero. An in-branching is defined analogously with only one vertex of outdegree zero.

The relationship of domination is an important one in defining the structure of the out-tournament. Therefore, it is necessary to use notation that models certain nuances in the domination relationships. Let \( D_1 \) and \( D_2 \) be vertex disjoint digraphs. The notation \( D_2 \Rightarrow D_2 \) means that there is no arc from \( V(D_2) \) to \( V(D_1) \). If every vertex in \( D_1 \) dominates every vertex in \( D_2 \), then we use \( D_1 \rightarrow D_2 \). Since we will be using tournaments as strong components, it will be the case that if arcs go one direction between the strong components, then there will not be any going in the other direction. Therefore, we need to use \( D_1 \rightarrow D_2 \), which means that \( V(D_1) \) dominates \( V(D_2) \) and there is no arc from \( V(D_2) \) to \( V(D_1) \).

Specifically in this paper, we will be constructing out-tournaments using upset tournaments as strong components. An upset tournament is a tournament on \( n \geq 3 \) vertices with score-list \( \{1, 2, 3, \ldots, n-3, n-2, n-3\} \). The score-list of a tournament is the multiset of the outdegrees of its vertices.

The adjacency matrix of a digraph \( D \) on \( n \) vertices is the \( n \times n \) matrix \( A = [a_{ij}] \) where \( a_{ij} = 1 \) if \((v_i, v_j) \in A \) and equals 0 otherwise. Ranks corresponding to the \( \{0, 1\} \)-matrix are the real rank, \( r(A) \), the boolean rank, \( r_B(A) \), and the nonnegative integer rank, \( r_{2+}(A) \). The boolean rank of an \( m \times n \) \( \{0, 1\} \)-matrix is the smallest \( k \) for which there exist an \( m \times k \{0, 1\} \)-matrix \( B \) and a \( k \times n \{0, 1\} \)-matrix \( C \) such that \( A = BC \) when boolean arithmetic is used \((1 + 1 = 1) \). Similarly, the nonnegative integer rank is the smallest \( k \) for which there exist \( m \times k \) and \( k \times n \) matrices \( B \) and \( C \) respectively such that \( A = BC \), where the entries of \( B \) and \( C \) are nonnegative integers. If \( A \) is a \( \{0, 1\} \)-matrix, then both \( B \) and \( C \) are \( \{0, 1\} \)-matrices. The relationship between the boolean and nonnegative integer ranks is \( r_B(A) \leq r_{2+}(A) \) for any \( \{0, 1\} \)-matrix \( A \). Since real rank can be defined similarly to nonnegative integer rank, only over all the real numbers, we have \( r(A) \leq r_{2+}(A) \). There is no standard relationship between \( r(A) \) and \( r_B(A) \). Finally, the term rank of a matrix, \( r_t(A) \), is the smallest number of rows and columns containing all of the nonzero entries of \( A \). When \( A \) is a \( \{0, 1\} \)-matrix, \( r_B(A) \leq r_{2+}(A) \leq r_t(A) \). The relationship between the real rank and nonnegative integer rank also gives us \( r(A) \leq r_{2+}(A) \leq r_t(A) \).

The real, nonnegative integer, and term ranks for the matrices of \( n \)-tournaments, on \( n \) vertices, were bounded by deCaen [9].

Theorem 2.1 [9] If \( A \) is an \( n \)-tournament matrix, then \( r(A) \geq n - 1 \).

Corollary 2.2 [9] If \( A \) is an \( n \)-tournament matrix, then \( (n-1) \leq r(A) \leq r_{2+}(A) \leq r_t(A) \leq n \).

These results indicate that if any tournament has equal ranks, then the ranks must be \( n - 1 \) or \( n \). In general, \( r_B(A) \) is very difficult to obtain. Thus, when looking for corresponding matrices with equal ranks, knowing the bounds on the remaining three ranks forces the search for tournament matrices where \( r_B(A) = n - 1 \) or \( r_B(A) = n \). Additionally, we know that \( r_B(A) \leq r_t(A) \), so the term rank serves as an upper bound on the boolean rank in general \( \{0, 1\} \)-matrices.

In this paper, we use the fact that it is known which upset tournaments have \( r_B(A) = r_{2+}(A) \), and use it to help us characterize a class of local out-tournaments where all ranks equal \( n \). Gregory et al. [11] linked the boolean and nonnegative integer ranks of \( \{0, 1\} \)-matrices to biclique cover and partition numbers of bipartite graphs. The biclique cover number of a graph \( G \), \( bc(G) \), is the smallest number of complete bipartite subgraphs that cover the edges of \( G \). The biclique partition number of a graph \( G \), \( bp(G) \), is...
defined similarly using a partitioning of the edges of \( G \). By labeling the rows of the adjacency matrix of a digraph \( D \) with a set of numbers and the columns with a disjoint set of numbers, the adjacency matrix of \( D \) also represents the adjacency matrix of a bipartite graph \( B \). Using this common matrix, the following result is obtained.

**Lemma 2.3** [11] If \( D \) is a digraph, then \( r_B(A) = bc(D) \) and \( r_{2+}(A) = bp(D) \).

The bicliques of \( B \) correspond to directed bicliques of \( D \). In this paper, we use this relationship to extend the results obtained for the matrix ranks to include the biclique cover and partition numbers of the out-tournaments.

### 3 Local Out-Tournaments and Upset Tournaments

#### 3.1 Out-Tournaments

Before examining the \( \{0,1\} \)-matrix ranks of the local out-tournaments, it is important to understand the structure of the digraphs. It is this that will determine which out-tournaments have adjacency matrices with full and equal ranks.

Bang-Jensen [1] shows that local tournaments have a structure that resembles that of tournaments. If \( D \) is a local tournament, then every strong component is a tournament. In addition, if two strong components are adjacent in \( D \), then one completely dominates the other. For an out-tournament, however, not all of this structure is necessary. Since only the out of each vertex need be a tournament, the constraints on the structure of the inset are relaxed. Thus, every strong component of an out-tournament is not necessarily a tournament, and complete domination is not required.

In-tournament digraphs were examined in depth by Bang-Jensen et al. [5], and much of the underlying structure identified. The following lemma and theorem are results for in-tournaments that are of specific interest in this paper in defining the structure of the out-tournaments. The corollaries following each result are the out-tournament results for in-tournaments that are of specific interest in this paper in defining the structure of the out-tournaments.

**Lemma 3.1** [5] Every connected in-tournament has an out-branching.

**Corollary 3.2** Every connected out-tournament has an in-branching.

**Theorem 3.3** [5] Let \( D \) be an in-tournament.

(a) Let \( A \) and \( B \) be distinct strong components of \( D \). If a vertex \( a \in A \) dominates some vertex in \( B \), then \( a \rightarrow b \). Furthermore, \( A \cap O^- \) (b) induces a tournament for each \( a \in A \).

(b) If \( D \) is connected, then \( SC(D) \) has an out-branching. Furthermore, if \( R \) is the root and \( A \) is any other component, there is a path from \( R \) to \( A \) containing all the components that can reach \( A \).

#### 3.2 Upset Tournaments

When implementing a structure where tournaments are the strong components, it is helpful for our purpose to use tournaments for which information exists as to the boolean, nonnegative integer, and term ranks of the tournament matrices. For this paper, we restrict our exploration to out-tournaments whose strong components are upset tournaments. To this end, we first describe the standard form that is used to represent the upset matrices, then verify that they are, indeed, strong tournaments.

Figure 1 shows an upset tournament in standard form by representing its upset path. All other arcs are directed in the opposite direction. The arcs \((v_1,v_2)\) and \((v_{n-1},v_n)\) are in every upset path. The arc \((v_i,v_j)\) can only be on the upset path when \( i < j \). Vertices are presented in the order \( v_1,v_2,\ldots,v_n \).

As stated by Poet and Shader [24], every upset tournament is isomorphic to exactly
Lemma 3.7

where

two upset tournaments are dominant.

Proof. Let D be an upset tournament in standard form. Then D has a unique path from vertex v1 to vertex vn, and this path consists of the upset arcs of D.

A result of Lemma 3.5 is that we know that an upset tournament is strongly connected.

Proposition 3.6 If T is an upset tournament, then T is strongly connected.

Proof. Let T be an upset tournament in standard form. By Lemma 3.5, there is a unique path from v1 to vn. Vertex vn dominates all vertices except vertex vn−1, and reaches vn−1 using arc (vn, v1) and the upset path. If v is a vertex on the upset path other than vn, then v reaches vn−1 and vn. It reaches all other vertices through vn. If v is a vertex that is not on the upset path, then v dominates v1, and reaches all vertices on the upset path through v1. It then reaches all vertices not on the upset path from vn. Thus, T is strongly connected.

Because the upset tournaments are strong, they can be used as the strong components of a local out-tournament D. Corollary 3.4 guides the placement of arcs between upset tournaments T1 and T2. Additionally, the structure of SC(D) is acyclic, but the underlying graph is not necessarily a tree. The second part of the following lemma addresses the structure when two upset tournaments are dominated by a third upset tournament.

Lemma 3.8 Let D be a local out-tournament with strong components T1, Tj, and Tk where v1 ∈ V(T1), vj ∈ V(Tj), and vk ∈ V(Tk).

(a) If T1 and Tj are upset tournaments where (v1, vj) is an arc in D, then T1 → vj.

(b) If Tj, Tk, and Tk are upset tournaments where Tj → Tk and Tj → Tk, then Tk → Tk or Tk → Tj.

Proof. Since upset tournaments are strong, part (a) follows directly from part (a) of Corollary 3.4. For (b), we can use part (b) from Corollary 3.4, but will prove it from the definition to support the further understanding of the tournament structure. Given that upset tournaments T1, Tj, and Tk are strong, if T1 → Tj and Tk → Tj, then there exists vj ∈ V(Tj) and vk ∈ V(Tk) such that T1 → vj and Tj → vk. By definition of an out-tournament, both vj and vk are in a tournament, so they must be adjacent. Thus, vj → vk or vk → vj. From part (a), we extend this to Tj → vk or Tk → vj. So, Tj → Tk or Tk → Tj.

4 Matrices and Matrix Ranks

Now that the structure of the out-tournaments with upset tournament strong components has been described, we direct our attention to finding which of these digraphs have adjacency matrices with r(A) = rB(A) = n. To do so, we use results on the matrix ranks of upset tournament matrices.

First, consider the basic structure of the adjacency matrix A of upset tournament D with strong components T1. Let A1 be the adjacency matrix of the upset tournaments T1. Thus SC(D) has vertices T1. We will carefully order the vertices of SC(D) based upon the following proposition.

Proposition 4.1 Every acyclic digraph has an acyclic ordering of its vertices.

Since SC(D) is only guaranteed an in-branching, there may be more than one vertex in SC(D) with indegree of zero. Thus, we cannot state that there is a path including every vertex. However, Proposition 4.1 states that there is an acyclic ordering of the T1. We will assume this ordering of the T1. This gives the adjacency matrix structure for SC(D) shown in Figure 3. The ordering places each component along the diagonal.

Figure 3: General adjacency matrix structure of SC(D), where D is a local tournament.

Keeping the same acyclic labeling of the T1 above, we obtain the adjacency matrix structure for the adjacency matrix of D shown in Figure 4. The two structures are the same only because the T1 are strong components in D. Note that the upper triangle regions of both matrices are not labeled with values. That is because these values can vary, while the values shown are set.

Consider how the structure of D dictates the placement of 1's in the upper triangular region of A. According to part (a) of Lemma 3.7, if (v1, vj) is an arc in D, then all of
the vertices in $T_i$ dominate $v_j$. This translates into a column of 1’s to the right of $A_i$ and above $A_j$. There is a 1 in every row of $A_j$ in the column corresponding to vertex $v_j$.

In the matrix, it may become necessary to discuss particular rows, columns, and regions. To help in the identification process, the following notation will be used. Let $n_t$ be the number of vertices in $T_i$. So, $\sum_{i=1}^{k} n_t = n$. Further, let the vertices of $T_i$ be labeled $v_{i1}, v_{i2}, ..., v_{in_t}$. If this is extended to the labeling of columns and rows in $A$, then vertex $v_{jn_t}$ would be represented by column and row $n_1 + ... + n_{j-1} + m$.

To further identify the structure of these matrices, consider part (b) of Lemma 3.7 in conjunction with the acyclic labeling that has been adopted. With the acyclic labeling, if $T_i$ and $T_j$ are adjacent, then $T_i \Rightarrow T_j$ if and only if $i < j$. Additionally, if $T_i \Rightarrow T_j$ and $T_i \Rightarrow T_k$, we will have $T_j \Rightarrow T_k$ whenever $j < k$. Since the digraphs are isomorphic within labeling, we will assume the alpha ordering of $i < j < k$ for these indices. In $A$, if there is a submatrix of 1’s in the rows of $A_j$ that includes some columns of $A_j$ and $A_k$, then there will be a submatrix of 1’s in the rows of $A_j$ in the same columns of $A_k$.

For an example, consider the out-tournament $D$ consisting of upset tournament components $T_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$, $T_2 = \{(v_4, v_5), (v_5, v_6), (v_6, v_4)\}$ and $T_3 = \{(v_7, v_8), (v_8, v_9), (v_9, v_7)\}$. If $T_1 \Rightarrow \{v_4, v_5\}$ and $T_1 \Rightarrow \{v_8, v_9\}$, then $T_2 \Rightarrow \{v_8, v_9\}$. All of $T_2$ must dominate $v_8$ and $v_9$ in order to satisfy Lemma 3.7. The vertices in $T_2$ could also dominate more than $\{v_8, v_9\}$. It is the minimum set that must be dominated. The adjacency matrix $A(D)$ is shown in Figure 5.

In an upset tournament, every vertex has an outdegree greater than zero. So, every row in the adjacency matrix of an upset tournament contains a 1. Visually, a directed biclique of a digraph is a submatrix which forms a block of 1’s in the digraph’s adjacency matrix. In a biclique partition, these submatrices must be disjoint. In a biclique cover, they may overlap. Given the structure of the adjacency matrices here, every biclique in $A_1$ can be expanded to cover any 1’s to the right of $A_i$ in a biclique cover. This relationship is important in determining what upset tournaments can be used as strong components in out-tournaments where $bc(D) = bp(D) = n$.

Consider the matrix in Figure 6 representing an upset tournament with vertices $v_1, ..., v_6$. A minimum biclique cover is given, where $B_i$ are the bicliques. Each of the $B_i$ can be expanded to cover any 1’s in a column to the right or the left of this submatrix. Figure 7 shows the same bicliques expanded to cover 1’s representing the vertices in the original matrix dominating vertices $v_8$ and $v_9$.

**Lemma 4.2** Let $D$ be an out-tournament with $k$ upset tournament strong components.
Lemma 4.3 Let $A$ be the adjacency matrix of an out-tournament with $k$ upset tournament strong components, where $A_i$ is the adjacency matrix of strong component $T_i$. Then $r_B(A) \leq \sum_{i=1}^{k} r_B(A_i)$.

Thus to find the matrices with full and equal ranks, the submatrices, $A_i$, must have $r_B(A_i) = n_i$. So we look for upset tournaments where $bc(T_i) = n_i$. Since $bc(T_i) \leq bp(T_i)$, the upset tournaments must have $bc(T_i) = bp(T_i) = n_i$.

Theorem 4.4 [18] Let $T$ be an upset tournament in standard form on $n \geq 6$ vertices. Then $bc(T) = n$ if and only if the upset path does not contain any arcs of the form $(v_i, v_{i+1})$ for $3 \leq i \leq n - 3$.

When we have $n_i \geq 6$, Theorem 4.4 gives us the structure that must be used for the upset tournament strong components of the out-tournament. What about upset tournaments on $3$, $4$ or $5$ vertices? To answer this question, we use results from Gregory, et al. [11]. A set $S$ of independent 1’s of a $0, 1$ -matrix is said to be isolated if no two 1’s are in a $2 \times 2$ submatrix of 1’s.

Lemma 4.5 [11] If the adjacency matrix $A$ of a digraph $D$ has an isolated set of $r$ 1’s, then $r_B(A) = bc(D) \geq r$.

We use this result in the proof of the following lemma where we establish the boolean and nonnegative integer ranks of upset tournaments on 3, 4 or 5 vertices.

Lemma 4.6 If $T$ is an upset tournament on $n = 3, 4$, or 5 vertices with adjacency matrix $A$, then $bc(T) = bp(T) = n$, and $r_B(A) = r_{2^+}(A) = n$.

Proof. When $n = 3$, there is exactly one upset tournament on $n$ vertices in standard form, and it has upset path $(v_1, v_2)$, $(v_2, v_3)$. Entries $a_{12}$, $a_{23}$ and $a_{31}$ of the adjacency matrix are isolated 1’s. When $n = 4$, there is exactly one upset tournament on $n$ vertices in standard form, and it has upset path $(v_1, v_2)$, $(v_2, v_3)$ and $(v_3, v_4)$. Entries $a_{12}$, $a_{23}$, $a_{34}$ and $a_{41}$ of the adjacency matrix are isolated 1’s. When $n = 5$, there are two distinct upset tournaments on $n$ vertices in standard form. One upset path is two distinct upset tournaments on $n$ vertices in standard form. One has upset path $(v_1, v_2)$, $(v_2, v_3)$, $(v_3, v_4)$ and $(v_4, v_5)$. The other $(v_1, v_2)$, $(v_2, v_3)$, $(v_3, v_4)$, and $(v_4, v_5)$, and isolated 1’s $a_{12}$, $a_{23}$, $a_{34}$, $a_{42}$. The other $(v_1, v_2)$, $(v_2, v_3)$, $(v_3, v_4)$, and $(v_4, v_5)$, and isolated 1’s $a_{12}$, $a_{23}$, $a_{34}$, $a_{45}$ and $a_{51}$. The other $(v_1, v_2)$, $(v_2, v_3)$, $(v_3, v_4)$, and $(v_4, v_5)$, and isolated 1’s $a_{12}$, $a_{23}$, $a_{34}$, $a_{45}$ and $a_{51}$. Theorem 4.7 [25] Let $A$ be an adjacency matrix corresponding to an upset tournament. Then $r(A) = r_{2^+}(A)$.

This translates to $r(A_i) = r_{2^+}(A_i)$ for the upset tournament strong components. Since it is possible for the real rank to be less than both the boolean and nonnegative integer ranks in general, it remains to show that $r(A) = n$ in the matrices we have discussed where $r_B(A) = r_{2^+}(A) = n$. That will be done in the proof of the following theorem, which characterizes the out-tournaments with upset tournament strong components with full and equal ranks.

Theorem 4.8 Let $D$ be an out-tournament with $k$ upset tournament strong components, $T_i$, and adjacency matrix $A$. For each $T_i$, either $T_i$ is on 3, 4 or 5 vertices or it does not contain any arcs of the form $(v_j, v_{j+1})$ for $3 \leq j \leq n - 3$ for $n_i \geq 6$ if and only if $r_B(A) = r_{2^+}(A) = r_e(A) = r(A) = n$.

Proof. ($\Rightarrow$) If $n = 3$, 4 or 5, the calculated real rank of the adjacency matrices for any upset tournament in standard form is 3, 4 and 5 respectively. This combined with Lemma 4.6 gives us $r_B(A) = r_{2^+}(A) = r(A) = n$. If $n \geq 6$ and there are no arcs of the form $(v_j, v_{j+1})$ for $3 \leq j \leq n - 3$, we know from Theorem 4.4 that $r_B(A_i) = r_{2^+}(A_i) = n_i$. Also, from Corollary 4.3, $r_B(A) \leq \sum_{i=1}^{k} r_B(A_i)$. Note that $r_B(A) = n$, we will show that $bc(D) = n$. Consider minimum biclique covers of $T_i$ and $T_j$. Because $V(T_i) \cap V(T_j) = \emptyset$, no fewer bicliques can be used to cover $A(T_i)$ and $A(T_j)$ if arcs are created from $T_i$ to $T_j$. So, $bc(D) \geq \sum_{i=1}^{k} bc(T_i)$. In the proof of Lemma 4.2, we know that $bc(D) \leq \sum_{i=1}^{k} bc(T_i)$. Therefore, $bc(D) = \sum_{i=1}^{k} bc(T_i) = n$, so $r_B(A) = n$. Since $r_B(A) \leq r_{2^+}(A)$, we have $r_{2^+}(A) = n$.

Finally, we examine the real rank of $A$. The rows of each $A_i$ are linearly independent. We use the results of the following theorem, which characterizes the out-tournaments with upset tournaments strong components with full and equal ranks.
Figuratively, \( r_B(A) = n \) implies that \( \sum_{i=1}^{k} r_B(A_i) = n \). So, each \( A_i \) must have full boolean rank, \( r_B(A_i) = n_i \). Since \( r_B(A) \leq r_{Z^+}(A), r_{Z^+}(A_i) = n_i \) for each \( i = 1, \ldots, k \). Thus, \( r_B(A_i) = r_{Z^+}(A_i) = n_i \) for \( 1 \leq i \leq k \). This only occurs when \( T_i \) is on 3, 4, or 5 vertices, or by Theorem 4.4 when \( n \geq 6 \) and there are no arcs of the form \((v_j, v_{j+1})\) for \( 3 \leq j \leq n_i - 3 \). \( \square \)

Although this paper concentrates on local out-tournaments, the same results hold for local in-tournaments.

5 Miles to Go

A characteristic of upset tournaments that makes them interesting is that it is also known when \( r_B(A) = r_{Z^+}(A) = n - 1 \). If we use these upset tournaments as strong components in an out-tournament \( D \), the singular matrices \( A_i \) make for a variety of rank values.

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Figure 8: Adjacency matrix of an out-tournament where \( T_i \) are upset tournaments on 6 vertices, and \( r_B(A) \neq r_{Z^+}(A) \).

To illustrate this, consider the matrix in Figure 8. The upset tournament components \( T_1 \) and \( T_2 \) have six vertices each with upset paths isomorphic to \((v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5)\). Each has \( bc(T_i) = bp(T_i) = n - 1 \). All vertices of \( T_1 \) dominate the first labeled vertex of \( T_2 \) to form a local out-tournament. As shown in Section 4, the bicliques of \( T_1 \) can be expanded to cover the arcs from \( T_1 \) to \( T_2 \). So, \( r_B(A) = (n_1 - 1) + (n_2 - 1) = 10 \). However, the partitions cannot be expanded in this way. Nor can the partitions in \( T_2 \) be expanded upward to cover the 1's. While this in itself is not enough to show that \( r_{Z^+} > 10 \), it must be at least 11 since \( r(A) = 11 \).

What if, instead of dominating the first labeled vertex of \( T_2 \), the vertices of \( T_1 \) dominate the second? Figure 8 shows this slightly different adjacency matrix. Here, we have the same boolean rank as in Figure 8, but a biclique in the partition cover of \( T_2 \) can be extended to cover the column of 1's above it. Therefore, \( r_B(A) = r_{Z^+}(A) = 10 \). As a bonus, \( r(A) = 10 \) as well. This shows that for a local out-tournament, \( r(A) < n - 1 \) is possible, unlike the case for tournaments.

Thus, the question now becomes, how can local out-tournaments with upset tournament strong components be constructed where \( r_B(A) = r_{Z^+}(A) < n \) and \( r_B(A) = r_{Z^+}(A) = r_+(A) = r(A) < n? \) Additionally, what local out-tournaments have adjacency matrices with equality for some subsets of these ranks, and what are the subsets?


