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Abstract: The calculus of variations is used to develop the necessary theory and derive the optimality conditions for a spacecraft to transfer between a set of initial and final conditions, while minimizing a combination of fuel consumption and a function of the estimation error covariance matrix associated with the spacecraft state. The theory is developed in a general manner that allows for multiple observers, moving observers, covariance associated with an arbitrary frame, a wide variety of observation types, multiple gravity bodies, and uncertainties in the spacecraft equations of motion based on the thrusting status of the engine. A series of example trajectories from low Earth orbit (LEO) to a near geosynchronous Earth orbit (GEO) shows that either the trace of the covariance at the final time or the integral of the trace of the covariance matrix associated with the error in the Cartesian position and velocity can be reduced significantly with a small increase in the fuel consumption. An additional example illustrates the covariance associated with the semimajor axis can be significantly reduced for a transfer from Earth orbit to lunar orbit. This example illustrates multiple, moving observers as well as a transfer in a multi-body gravitational field.

SECTION I.

Nomenclature

a	Semimajor axis
$\mathbf{A}$	Derivative of $\mathbf{y}$ state with respect to $\mathbf{x}$ state
$\mathbf{B}$	relates covariance before and after a discrete measurement
e	Eccentricity
$\mathbf{F}$	Derivative of time derivative of Cartesian spacecraft position and velocity with respect to Cartesian spacecraft position and velocity
$\mathbf{g}$	Gravitational acceleration acting on spacecraft
$\mathbf{G}$	Derivative of time derivative of $\mathbf{y}$ frame state with respect to $\mathbf{y}$ frame state
$\mathbf{H}$	Derivative of observation with respect to spacecraft position and velocity
H_a	Hamiltonian function
J	One-half of the integral of spacecraft thrust acceleration squared
$\mathbf{L}$	Derivative of measurement constraint with respect to covariance
m	Spacecraft mass
n	Number of discrete times a function of the spacecraft covariance is included in the cost function
$\mathbf{n}$	Random vector of errors in observation
p	Number of discrete measurements
$\mathbf{p}$	Vector from central body to observer
P	Power
$\mathbf{P}$	Covariance matrix for spacecraft position and velocity
q	Integral of a function of the spacecraft covariance
$\mathbf{Q}$	Spectral density matrix for errors in equations of motion
r	Radial position
$\mathbf{r}$	Spacecraft position
$\mathbf{R}$	Spectral density matrix for observation noise
$\tilde{\mathbf{R}}$	Covariance matrix for observation noise
s	Term that appears in the $\mathbf{L}$ matrix
t	Time
T_1	Error in thrust acceleration
u	Thrust acceleration magnitude
$\mathbf{u}$	Thrust direction unit vector
$\mathbf{v}$	Velocity
$\mathbf{IV}$	Random vector of errors in equations of motion
W	Cost function
w_1^2	Constant term in spectral density matrix for equations of motion
x_s	x component of vector from observer to spacecraft

x	Cartesian frame
$\mathbf{X}$	Spacecraft Cartesian state including only position and velocity
$\tilde{\mathbf{X}}$	Spacecraft state including covariance
$y_s y$	component of vector from observer to spacecraft
y	Arbitrary frame
$\mathbf{Y}$	Spacecraft state in nonCartesian frame
z	Observation
α	Thrust direction
α_1	Error in thrust direction
α	Free variables in planar orbit to orbit transfer
γ	Lagrange multiplier
Γ	Thrust acceleration magnitude
δ	Dirac delta function
η	Lagrange multiplier
θ	Polar angle
θ	Constraints on the spacecraft state at the initial time
κ	True anomaly
λ_r	Spacecraft radial position costate
λ_r	Spacecraft position costate
λ_r	Spacecraft radial velocity costate
λ_v	Spacecraft velocity costate
λ_j	Thrust acceleration costate
λ_θ	Spacecraft polar position costate
$\lambda_{\dot{\theta}}$	Spacecraft polar velocity costate
λ_{1-21}	Spacecraft covariance costate
μ	Gravitational parameter of central body
v	Lagrange multiplier
ξ	Constant
ρ	Parameter vector used in constraints
σ	Constraint relating covariance before and after a discrete measurement
φ	Scalar weight
χ	Function of the covariance added to the cost function
ψ	Constraints on the spacecraft state at the final time
ω	Longitude of perigee.

I. Subscripts

- Initial
- f Final
- Integrator
- Maximum
- Nominal
- r Radial
- s Prescribed
- t Cross-track
- x Associated with the **X** frame
- Y** Associated with the **Y** frame.

I. Superscripts

- * True value
- Targeted value
- $\sim$ Augmented state including the covariance terms
- $\wedge$ Function equivalent in different frame
- Covariance terms or covariance costates associated with y frame.

SECTION II.

Introduction

The objective of this investigation is to demonstrate a technique that determines trajectories which minimize a combination of fuel consumption and spacecraft state uncertainty. Most current research first seeks a mass or time optimal transfer, followed by a second, independent study to determine if the optimal trajectory can be navigated. One must determine the number, quality, frequency, and types of measurements of the spacecraft state which must be taken in order to navigate the spacecraft with the required precision. Because no effort is made to improve observability during the optimization process, observability along the optimal trajectory may be significantly worse than the observability along another trajectory that has only a slightly greater fuel or time cost.

Betts provides an excellent summary of the techniques available for determining mass and time optimal trajectories.¹ This investigation relied on indirect trajectory optimization techniques.^{2,3} Previous efforts to simultaneously improve the observability along a trajectory have been limited. In 1968 Vander Stoep presented a method of transferring from a specified initial state to a specified final state in a fixed period of time.⁴

During this transfer both the continuous observations and the equations of motion were corrupted by white noise. The cost function for this study was an integral of a function of the control, state, covariance, and time, combined with a weighted sum of elements of the Cartesian covariance matrix at the final time. In 1970 Schmidt extended Vander Stoep's method to allow free final time as well as both free and linearly constrained final states.⁵ This dissertation focused on determining the optimal controls for a closed-loop feedback control system instead of designing a trajectory to minimize open-loop estimation errors.

Between 1984 and 1990 Speyer et. al. and Hull et. al. presented a method of improving the observability in the homing missile problem by maximizing the trace of the Fisher information matrix.^{6-7,8} They showed that the likelihood of missile intercept in the homing missile problem could be increased by altering the intercept trajectory away from proportional navigation. In 1999 Oshman and Davidson presented a method of optimizing the trajectory of an observer who is attempting to determine the location of a fixed target by taking observations of the angle between the local horizontal and the target.⁹ The cost function in this study was the determinant of the Fisher information matrix plus a penalty function that was based on the observer location. In addition constraints were placed on the observer location. In this study the control was discretized, and an approximate optimal control was determined.

Between 1996 and 2004 Seywald investigated a technique to determine trajectories that were less sensitive to uncertainties in the initial spacecraft state.^{10-11,12} Seywald used the state transition matrix to estimate the first-order error in the final state caused by errors in the initial spacecraft state. He included these errors in the cost function, along with a measure of the fuel efficiency of the transfer. This technique would have provided the same results as the current study if the equations of motion were modeled without noise and if no measurements were employed. In 2005 Mishne applied the same technique as Seywald to optimally reconfigure a satellite formation when there is uncertainty in the initial state of the satellites.¹³

This study develops the optimality conditions for minimizing a combination of spacecraft state uncertainty and fuel consumption for an open-loop transfer between an arbitrary set of initial and final conditions. The third section develops the theory in a general manner that allows for multiple observation types, arbitrary gravity fields, discrete measurements, periods without measurements, and dynamic errors based on the thrust. The limited previous research into incorporating observability into trajectory optimization has all sought to minimize uncertainty associated with the spacecraft state in the integration frame.^{4-5,6,7,8,9,10,11,12,13} Because most trajectory optimization is done in a

Cartesian frame, these techniques provide a method to minimize a combination of the propellant mass and the sum of the expected mean square errors in the spacecraft Cartesian position and velocity. Instead of, or in addition to, minimizing the covariance associated with the spacecraft Cartesian position and velocity, this work demonstrates a technique to minimize the covariance associated with nonCartesian frames, such as orbital elements or flight path angle.

The fourth section of this paper contains a series of example problems. The first four example problems are planar orbit to orbit transfer problems that illustrate the differences between continuous scalar measurements, alternating periods of continuous measurements and no measurements, discrete measurements, and dynamic errors based on the spacecraft thrusting status. The final example involves a transfer from near geosynchronous Earth orbit (GEO) to a low lunar orbit, where the covariance associated with the semimajor axis in the target orbit is minimized. Minimizing the covariance associated with the semimajor axis is especially interesting because the error in estimating the spacecraft state at some time after the spacecraft is inserted into the target orbit is dominated by the error in the spacecraft semimajor axis when it is inserted into the target orbit.¹⁴ This example demonstrates that the covariance associated with the semimajor axis can be reduced significantly with only a slight increase in fuel consumption.

SECTION III.

General Theory

A. Problem formulation

In order to use the calculus of variations to optimize a trajectory, a scalar cost function must be defined to measure the trajectory's merit.¹⁵ Because the goal is to maximize a combination of the observability along a trajectory and the trajectory's fuel efficiency, the cost function must include both mass and observability terms. Improving the observability along a trajectory is important because the ability to deliver the spacecraft to the target orbit or constraint is dependent on how well the spacecraft state can be estimated. for a mass-optimal solution, the choice of cost function is straightforward, with the mass of the spacecraft at the final time providing one obvious choice. Determining a scalar representation of the observability along a trajectory is more difficult. Bishop et. al. presented a method based on the inverse function theorem to estimate the state observability along a trajectory using the determinant of the observability matrix.¹⁶ This measure was rejected for this study because its effectiveness in measuring the

observability along a trajectory is relatively unknown, and it yields an abstruse function that must be incorporated in the cost function.

The observability term for this study is a function of the estimation error covariance matrix associated with the spacecraft position and velocity because it provides an estimate of the mean square errors in the spacecraft position and velocity as well as the correlation between the errors in different components of the position and velocity vectors. A general cost-function-incorporating covariance is given by (1), where ε_i is an arbitrary function of the covariance, position, and velocity at discrete times, t_i and where χ is an arbitrary function of covariance, position, and velocity. The choice of ε_i and t_i allow one to minimize a given function of the covariance at any discrete time(s), such as immediately before an impulsive maneuver, a gravity assist, or an atmospheric entry. The function χ allows one to minimize the integral of a function of the covariance along the trajectory. Note that the covariance matrix included in the cost function is the covariance associated with Cartesian position and velocity. It is possible to select the functions ε and χ so the covariance associated with another coordinate frame, such as orbital elements, is included in the cost function.

$$W = J(t_f) + \sum_{i=1}^n \varepsilon_i[\mathbf{r}(t_i), \mathbf{v}(t_j), \mathbf{P}(t_i)] + \int_{t_0}^{t_f} \chi[\mathbf{r}(t), \mathbf{v}(t), \mathbf{P}(t)] dt .$$

(1)

The mass term in the cost function J is defined to be one-half the integral of the thrust acceleration squared as specified by (2). for a power limited propulsion (PLP) engine, minimizing J is equivalent to maximizing the final mass since J and the fuel required are ordered and one-to-one.¹⁷ The mass at any time can be determined from the initial mass and the power available to the engine by using (3). Utilizing J instead of mass, allows one to solve optimization problems when the initial spacecraft mass or the power available to the engine is not known. One can make this PLP formulation mimic a constant specific impulse formulation by constraining either the engine's thrust or specific impulse. The variable q is defined to be the integral of χ as shown in (4).

$$\begin{aligned}
 J(t) &= \int_{t_0}^t \frac{\Gamma^2}{2} dt \\
 m(t) &= \frac{m(t_0)P_{max}}{P_{max} + m(t_0)J(t)} \\
 q(t) &= \int_{t_0}^t \chi[\mathbf{r}(t), \mathbf{v}(t), \mathbf{P}(t)] dt.
 \end{aligned}$$

(2)(3)(4)

In order to make the cost function depend only on the state, time, and constant parameters, define the augmented state $\tilde{\mathbf{X}}$ as in (5), where P_i are defined in (6). The covariance terms are only associated with the position and velocity states and are defined in (6) for the three dimensional case. Because the covariance matrix is symmetric, there are only 21 unique elements for the three dimensional case.

$$\begin{aligned}
 \tilde{\mathbf{X}}^T &= (\mathbf{r}^T \mathbf{v}^T J P_1 \dots P_{21} q). \\
 \mathbf{P}_X &\equiv \begin{pmatrix} P_{xx} & P_{xy} & P_{xz} & P_{\cdot xx} & P_{\cdot xy} & P_{\cdot xz} \\ P_{xy} & P_{yy} & P_{yz} & P_{\cdot yx} & P_{\cdot yy} & P_{\cdot yz} \\ P_{xz} & P_{yz} & P_{zz} & p_{\cdot zx} & p_{\cdot zy} & p_{\cdot zz} \\ P_{\cdot xx} & P_{\cdot yx} & P_{\cdot zx} & P_{\cdot \cdot xx} & P_{\cdot \cdot xy} & P_{\cdot \cdot xz} \\ P_{\cdot xy} & P_{\cdot yy} & P_{\cdot zy} & P_{\cdot \cdot xy} & P_{\cdot \cdot yy} & P_{\cdot \cdot yz} \\ P_{\cdot xz} & P_{\cdot yz} & P_{\cdot zz} & P_{\cdot \cdot xz} & P_{\cdot \cdot yz} & P_{\cdot \cdot zz} \end{pmatrix} \\
 &\equiv \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\ p_2 & p_7 & p_8 & p_9 & p_{10} & p_{11} \\ p_3 & p_8 & p_{12} & p_{13} & p_{14} & p_{15} \\ p_4 & p_9 & p_{13} & p_{16} & p_{17} & p_{18} \\ p_5 & p_{10} & p_{14} & p_{17} & p_{19} & p_{20} \\ p_6 & p_{11} & p_{15} & p_{18} & p_{20} & p_{21} \end{pmatrix}.
 \end{aligned}$$

(5)(6)

The time derivatives of the spacecraft position and velocity are given by (7) and (8). In order to find the time derivative of the covariance matrix, the linearized time derivative of the position and velocity, given by (9), is needed. The noise in the equations of motion is accounted for by $\mathbf{w}$, which is a zero-mean Gaussian white noise process with spectral density $\mathbf{Q}$. The model for the observation is given by (10). The noise in the observation measurement is accounted for by $\mathbf{n}$, which is a zero-mean Gaussian white noise process with spectral density $\mathbf{R}$. The time derivative of the covariance for periods of continuous measurements is given by the matrix Riccati equation given in (11).¹⁸ If no measurements are being taken, the time derivative of the covariance is given by (12). The relationship between the covariance after a discrete measurement and the covariance before the measurement is given by (13), where t_k is the time of the measurement.¹⁸ The covariance at any time can be determined from the initial value of the covariance and from (11) to (13).

$$\begin{aligned}
 \dot{\mathbf{r}} &= \mathbf{v} \\
 \dot{\mathbf{v}} &= \mathbf{g}(\mathbf{r}) + \Gamma \mathbf{u} \\
 \begin{pmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{pmatrix} &= \mathbf{F} \begin{pmatrix} \mathbf{r} \\ \mathbf{v} \end{pmatrix} + \mathbf{w}, \mathbf{F} = \frac{\partial \begin{pmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{pmatrix}}{\partial (\mathbf{r}\mathbf{v})} \Big|_{\mathbf{r}, \mathbf{v} = \mathbf{r}_{\text{nom}}, \mathbf{v}_{\text{nom}}} \\
 \mathbf{z} &= \mathbf{H} \begin{pmatrix} \mathbf{r} \\ \mathbf{v} \end{pmatrix} + \mathbf{n}, \mathbf{H} = \frac{\partial \mathbf{z}(\mathbf{r}, \mathbf{v})}{\partial (\mathbf{r}\mathbf{v})} \Big|_{\mathbf{r}, \mathbf{v} = \mathbf{r}_{\text{nom}}, \mathbf{v}_{\text{nom}}} \\
 \dot{\mathbf{P}} &= \mathbf{F}\mathbf{P} + \mathbf{P}\mathbf{F}^T + \mathbf{Q} - \mathbf{P}\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H}\mathbf{P} \\
 \dot{\mathbf{P}} &= \mathbf{F}\mathbf{P} + \mathbf{P}\mathbf{F}^T + \mathbf{Q} \\
 \sigma &\equiv \mathbf{P}(t_{k+}) - \mathbf{P}(t_{k-}) + \mathbf{P}(t_{k-})\mathbf{H}^T(t_k) \\
 &\quad [\mathbf{H}(t_k)\mathbf{P}(t_{k-})\mathbf{H}^T(t_k) + \tilde{\mathbf{R}}]^{-1} \mathbf{H}(t_k)\mathbf{P}(t_{k-}) \\
 &= \mathbf{0}.
 \end{aligned}$$

(7)(8)(9)(10)(11)(12)(13)

It is important to note that both $\mathbf{Q}$ and $\mathbf{R}$ may be functions of the time, spacecraft state, or controls. Although extended Kalman filter theory does not allow these spectral densities to be a function of the spacecraft state or control, empirical evidence suggests that this model is valid. The values of spectral density matrix $\mathbf{R}$ are based on the type of

measurements taken, sensors used, and other factors related to a specific mission. for a discrete measurement $\tilde{R}$ is the covariance associated with the errors in the observation measurements. The values of the spectral density matrix Q are determined by how well the dynamic equations of motion are modeled as well as how accurately the engine can produce the desired thrust level.

The spectral density matrices Q and R and the initial covariance $P(t_0)$ must be determined a priori. Only the relative value of these matrices is important because they can be scaled by the same factor without altering the solutions obtained from the optimization process. Because the covariance throughout the trajectory is scaled by that factor, the functions ϵ and χ should be divided by that same factor.

The time derivative of the full spacecraft state can easily be determined from (2), (4), (7), (8), (11), (12), and (13). The costates associated with the state defined by (5) are specified in (14). The Hamiltonian H_a can be determined using the equations for the time derivative of the state and (15).

$$\begin{aligned} \lambda^T &= (\lambda_r^T \lambda_v^T \lambda_J \lambda_1 \dots \lambda_{21} \lambda_q)^T \\ H_a &= \lambda^T \tilde{\mathbf{x}}. \end{aligned} \quad (14)(15)$$

The optimal control problem is to transfer the spacecraft from an initial state constraint, given by (16), to the final state constraint, given by (17). The constraints may or may not include parameters ρ . Note the constraint may specify a prescribed initial and final orbit, a prescribed initial and final state, or some other function of the initial and final states. These constraints are adjoined to the cost function to form the augmented performance index in (18).

$$\begin{aligned} \theta(t_0, x_0, \rho) &= \mathbf{0} \\ \psi(t_f, x_f, \rho) &= \mathbf{0} \\ W' &= W + v^T \theta + \eta^T \psi. \end{aligned} \quad (16)(17)(18)$$

The first variation of the augmented performance index is given by (19), where the $\sim$ above the δ symbolizes the time fixed variation. Requiring the first variation of W' to be

zero in order to minimize W' yields the conditions in (20) to (28). These equations, along with (16) and (17), form the constraints required to set up a two point boundary value problem (TPBVP). If the covariance associated with the error in the linearized dynamics and measurements is independent of the controls, then (21) and (22) take on the familiar form of (29) and (30).

$$\begin{aligned}
 \delta W' = & \left(\frac{\partial W'}{\partial \tilde{\mathbf{X}}_f} - \lambda_f^T \right) \delta \tilde{\mathbf{X}}_f + [H_a(t_f) + \frac{\partial W'}{\partial t_f}] \delta t_f \\
 & + \left(\frac{\partial W'}{\partial \tilde{\mathbf{X}}_0} + \lambda_0^T \right) \delta \tilde{\mathbf{X}}_0 + [H_a(t_0) - \frac{\partial W'}{\partial t_0}] \delta t_0 \\
 & + \frac{\partial W'}{\partial \rho} \delta \rho + \sum_{i=1}^n \left[\frac{\partial \varepsilon_i}{\partial \tilde{\mathbf{X}}(t_i)} - \lambda(t_{i-}) + \lambda(t_{i+}) \right] \\
 & \times \delta \tilde{\mathbf{X}}(t_i)_{t_i \neq t_f} + \int_{t_0}^{t_f} \left\{ \left(\frac{\partial H_a}{\partial \tilde{\mathbf{X}}} + \dot{\lambda}^T \right) \delta \tilde{\mathbf{X}} \right. \\
 & \left. + \frac{\partial H_a}{\partial \Gamma} \delta \Gamma + \frac{\partial H_a}{\partial \mathbf{u}} \delta \mathbf{u} \right\} dt \\
 H_a(t_0) = & \frac{\partial W'}{\partial t_0} \\
 \frac{\partial H_a}{\partial \Gamma} = & 0 \\
 \frac{\partial H_a}{\partial \mathbf{u}} = & 0 \\
 \frac{\partial W'}{\partial \tilde{\mathbf{X}}_f} = & \lambda_f^T \\
 H_a(t_f) = & - \frac{\partial W'}{\partial t_f} \\
 \lambda_0^T = & - \frac{\partial W'}{\partial \tilde{\mathbf{X}}_0} \\
 \frac{\partial W'}{\partial \rho} = & 0 \\
 \frac{\partial \varepsilon_i}{\partial \tilde{\mathbf{X}}(t_i)} - \lambda(t_{i-}) + \lambda(t_{i+}) = & 0 |_{i=1 \rightarrow n} \\
 \dot{\lambda}^T = & - \frac{\partial H_a}{\partial \tilde{\mathbf{X}}} \\
 \mathbf{u} = & \frac{\lambda_v}{\lambda_v} \\
 \Gamma = & - \frac{\lambda_v}{\lambda_j}.
 \end{aligned}$$

(19)(20)(21)(22)(23)(24)(25)(26)(27)(28)(29)(30)

B. Discrete Measurements

Because the optimality conditions derived with the calculus of variations require the covariance matrix to be included in the spacecraft state, each discrete observation provides a discontinuity in the spacecraft state. Consequently the constraint that relates the covariance after the measurement to the covariance before the measurement must be included in the cost function, as shown in (31). The summation indicates that the covariance must be constrained at each of the p measurements. This discontinuity in the spacecraft state causes the optimal costates to be discontinuous across each measurement as well. Because discrete observations each add either 10 constraints for a planar transfer or 21 constraints for a three dimensional transfer, the state will have 10 or 21 times the number of observations discontinuities. These constraints require that the unique elements of the covariance matrix satisfy (13). While these jumps do not provide theoretical difficulty, they do make it more difficult to numerically find solutions to the problem.

$$\begin{aligned}
 W' = & J(t_f) + \sum_{i=1}^n \varepsilon_i [\mathbf{x}(t_i), \mathbf{P}_x(t_i)] \\
 & + \int_{t_0}^{t_f} \chi[\mathbf{x}(t), \mathbf{P}_x(t)] dt + v^T \boldsymbol{\theta} + \boldsymbol{\eta}^T \boldsymbol{\psi} + \sum_{k=1}^p (\gamma^T \boldsymbol{\sigma})_k.
 \end{aligned}
 \tag{31}$$

The costates before and after each measurement must satisfy (32) to (36). Because of the formulation of the constraints in (13), where $\mathbf{P}(t_{k+})$ appears linearly, the covariance costates after the measurement must satisfy (37). Consequently one can formulate the constraint expressed in (38), where the $\mathbf{L}$ matrix is defined in (39). The covariance costates after each measurement can be uniquely determined from the covariance costates before the measurement and from the augmented spacecraft state $\tilde{\mathbf{X}}$ as long as the $\mathbf{L}$ matrix can be inverted. If the $\mathbf{L}$ matrix is singular, then there exist an infinite number of possible values for the covariance costates after the measurement that satisfy the first-order extremal conditions. As a result no unique solution exists for the TPBVP if $\mathbf{L}$ is singular.

$$\begin{aligned}
 \lambda_i(t_{k-}) &= \frac{\partial W}{\partial P_i(t_{k-})} \\
 \lambda_i(t_{k+}) &= -\frac{\partial W}{\partial P_i(t_{k+})} \\
 \lambda_r(t_{k+}) - \lambda_r(t_{k-}) + \frac{\partial W}{\partial \mathbf{r}(t_k)} &= \mathbf{0} \\
 \lambda_v(t_{k+}) - \lambda_v(t_{k-}) + \frac{\partial W}{\partial \mathbf{v}(t_k)} &= \mathbf{0} \\
 \lambda_J(t_{k+}) &= \lambda_J(t_{k-}) \\
 \lambda_i(t_{k+}) &= -\gamma_i \\
 \mathbf{L}\lambda_{1-21}(t_{k+}) &= \lambda_{1-21}(t_{k-}) \\
 \mathbf{L}(i, j) &= \frac{\partial \sigma_j}{\partial P_i}.
 \end{aligned}$$

(32)(33)(34)(35)(36)(37)(38)(39)

In order to verify the existence of $\mathbf{L}^{-1}$, one could, alternatively express the constraints from (13) as the constraints in (40), where the $\mathbf{B}$ matrix is defined in (41). If the $\mathbf{B}$ matrix is invertible, then the $\mathbf{L}$ matrix must also be invertible. Consider the case where the solution to the TPBVP is sought by integrating backwards in time. At the time just after the last measurement, the entire augmented state and costate are known. The position, velocity, and integral of thrust acceleration squared do not change during the measurement, and the covariance before the measurement can be determined by inverting $\mathbf{B}$ in (40). The covariance costates before the measurement can be determined from (38) and the other costates before the measurement can be determined from (34) to (36). This known, complete set of states and costates can then be integrated back in time until the time of the previous measurement, and the process can be repeated until the initial time is reached. The set of optimal costates are unique in this solution. Because the solution is unique when the TPBVP is solved by integrating backwards in time, the costates must be unique when integrating forward in time, and the inverse of $\mathbf{L}$ must exist.

$$\begin{aligned}
 \mathbf{P}(t_{k+}) &= \mathbf{B}(t_{k+})\mathbf{P}(t_{k-}) \\
 \mathbf{B} &\equiv \mathbf{I} - \mathbf{P}(t_{k+})\mathbf{H}^T(t_k)\tilde{\mathbf{R}}^{-1}\mathbf{H}(t_k).
 \end{aligned}$$

(40)(41)

C. Dynamic Errors Based on Thrust

Errors in the thrust magnitude and direction contribute significantly to the errors in the dynamic equations of motion. These engine thrust errors are more significant when the engine is providing more thrust. This application allows the spectral density of the errors in the dynamics w from (9) to be functions of the thrust magnitude and direction. The resulting optimal control depends on whether the controls are modeled as separate engines, each of which provides thrust acceleration in one of the coordinate directions, or as a single engine and two angles, which provide the overall three-dimensional thrust.

In the former case the errors in the thrust magnitudes for each engine are assumed to be independent. As a result w can be written as in (42), where ω_x , ω_y , and ω_z account for the errors in the dynamic equations due to gravitational terms, solar winds, and other forces which are not modeled. They have zero-mean, are independent, and have spectral density ω_1^2 . The errors in the thrust acceleration T_1 , T_2 , and T_3 are also zero-mean and independent of each other and ω . These errors are modeled as dependent on the magnitude of the thrust acceleration provided by each engine, with spectral density specified by (43) to (45), where E is the expectation operator. The spectral density Q can now be written as (46).

$$\begin{aligned}
 \mathbf{w}(t) &= \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} + \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} \\
 \frac{E[T_1(t)T_1(\tau)]}{\delta(t-\tau)} &= \frac{\Gamma_x^2}{\xi_1} \\
 \frac{E[T_2(t)T_2(\tau)]}{\delta(t-\tau)} &= \frac{\Gamma_y^2}{\xi_2} \\
 \frac{E[T_3(t)T_3(\tau)]}{\delta(t-\tau)} &= \frac{\Gamma_z^2}{\xi_3} \\
 \mathbf{Q}(t) &= \frac{E[\mathbf{w}(t)\mathbf{w}^T(\tau)]}{\delta(t-\tau)} \\
 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w_1^2 + \frac{\Gamma_x^2}{\xi_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & w_1^2 + \frac{\Gamma_y^2}{\xi_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & w_1^2 + \frac{\Gamma_z^2}{\xi_3} \end{pmatrix}.
 \end{aligned}$$

(42)(43)(44)(45)(46)

The first-order optimality conditions require that the derivative of the Hamiltonian with respect to Γ_x, Γ_y , and Γ_z be zero. The terms in the Hamiltonian involving Γ_x, Γ_y , and Γ_z are given by (47). The thrust acceleration along the optimal trajectory is then given by (48). Notice the similarity between (29), (30), and (48). If the error in the dynamic equations is not related to the magnitude of the thrust acceleration, then ξ_1, ξ_2 , and ξ_3 would be infinite and the thrust acceleration terms in (48) would be unchanged from the values in (29) and (30).

$$\begin{aligned}
 H_a &= \lambda_{vx}\Gamma_x + \lambda_{vy}\Gamma_y + \lambda_{vz}\Gamma_z + \lambda_J \frac{\Gamma_x^2 + \Gamma_y^2 + \Gamma_z^2}{2} \\
 &\quad + \lambda_{16} \frac{\Gamma_x^2}{\xi_1} + \lambda_{19} \frac{\Gamma_y^2}{\xi_2} + \lambda_{21} \frac{\Gamma_z^2}{\xi_3} + \text{non}\Gamma\text{terms} \\
 \Gamma_x &= \frac{-\lambda_{vx}}{\lambda_J + \frac{2\lambda_{16}}{\xi_1}} \Gamma_y = \frac{-\lambda_{vy}}{\lambda_J + \frac{2\lambda_{19}}{\xi_2}} \Gamma_z = \frac{-\lambda_{vz}}{\lambda_J + \frac{2\lambda_{21}}{\xi_3}}.
 \end{aligned}$$

(47)(48)

In most applications separate engines are not used to create thrust in the three coordinate axes. In these cases the errors in the dynamic equations of motion will depend on the difference between the actual thrust acceleration magnitude and the desired thrust acceleration magnitude as well as errors in the direction of the thrust acceleration. Consider a two dimensional example, where the controls are the magnitude of the thrust acceleration Γ and the thrust direction measured from the x-axis α . The noise in the equations of motion is specified by (49), where * indicates an actual value provided by the engine and where & indicates a prescribed value. The difference between the actual thrust acceleration magnitude and direction and the prescribed values is given by (50), where both T_1 and α_1 are zero-mean and independent, with spectral density given by (51). Because both T_1 and α_1 are small, terms involving T_1 multiplied by α_1 or α_1^2 are ignored. As a result (49) can be simplified as (52) by using a Taylor Series expansion. The spectral density Q can now be written as in (53). Note that the expected error in the thrust acceleration in the coordinate directions is coupled in this formulation unlike the previous formulation, that assumed separate engines in the component directions.

$$\begin{aligned}
 \mathbf{w}(t) &= \begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} + \Gamma^* \begin{bmatrix} \cos(\alpha^*) \\ \sin(\alpha^*) \end{bmatrix} - \Gamma^\& \begin{bmatrix} \cos(\alpha^\&) \\ \sin(\alpha^\&) \end{bmatrix} \\
 \Gamma^* &= \Gamma^\& + T_1, \alpha^* = \alpha^\& + \alpha_1 \\
 \frac{E[T_1(t)T_1(\tau)]}{\delta(t-\tau)} &= \frac{(\Gamma^\&)^2 E[\alpha_1(t)\alpha_1(\tau)]}{\xi_1 \delta(t-\tau)} = \frac{1}{\xi_2} \\
 \mathbf{w}(t) &= \begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} + \frac{T_1}{\Gamma^*} \begin{pmatrix} \Gamma_x \\ \Gamma_y \end{pmatrix} + \alpha_1 \begin{pmatrix} -\Gamma_y \\ \Gamma_x \end{pmatrix} \\
 \mathbf{Q}(t) &= \frac{E[\mathbf{w}(t)\mathbf{w}^T(\tau)]}{\delta(t-\tau)} \\
 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & w_1^2 + \frac{\Gamma_x^2}{\xi_1} + \frac{\Gamma_y^2}{\xi_2} & \Gamma_x \Gamma_y \left(\frac{1}{\xi_1} - \frac{1}{\xi_2} \right) \\ 0 & 0 & \Gamma_x \Gamma_y \left(\frac{1}{\xi_1} - \frac{1}{\xi_2} \right) & w_1^2 + \frac{\Gamma_x^2}{\xi_2} + \frac{\Gamma_y^2}{\xi_1} \end{pmatrix}.
 \end{aligned}$$

(49)(50)(51)(52)(53)

The terms in the Hamiltonian involving Γ_x and Γ_y are given by (54). The first-order optimality conditions again require that the derivative of the Hamiltonian, with respect to Γ_x and Γ_y , be zero. The optimal thrust acceleration is then given by (55). Again (55) reduces to (29) and (30), when the error in the thrust acceleration is independent of the thrust magnitude and direction.

$$\begin{aligned}
 H_a &= \lambda_{vx}\Gamma_x + \lambda_{vy}\Gamma_y + \lambda_J \frac{\Gamma_x^2 + \Gamma_y^2}{2} + \lambda_8 \left(\frac{\Gamma_x^2}{\xi_1} + \frac{\Gamma_y^2}{\xi_2} \right) \\
 &\quad + \lambda_9 \Gamma_x \Gamma_y \left(\frac{1}{\xi_1} - \frac{1}{\xi_2} \right) + \lambda_{10} \left(\frac{\Gamma_x^2}{\xi_2} + \frac{\Gamma_y^2}{\xi_1} \right) \\
 \Gamma_y &= - \frac{\lambda_{vy} - \lambda_9 \lambda_{vx} d \left(\frac{1}{\xi_1} - \frac{1}{\xi_2} \right)}{\lambda_J + 2 \left(\frac{\lambda_{10}}{\xi_1} + \frac{\lambda_8}{\xi_2} \right) - \lambda_9^2 d \left(\frac{1}{\xi_1} - \frac{1}{\xi_2} \right)^2} \\
 \Gamma_x &= -d \left[\lambda_{vx} + \lambda_9 \Gamma_y \left(\frac{1}{\xi_1} - \frac{1}{\xi_2} \right) \right] \\
 d &= \left[\lambda_J + 2 \left(\frac{\lambda_8}{\xi_1} + \frac{\lambda_{10}}{\xi_2} \right) \right]^{-1}.
 \end{aligned}$$

(54)(55)

D. Minimizing Covariance in Alternate Frames

Instead of minimizing the covariance associated with the spacecraft Cartesian position and velocity, defined to be the $\mathbf{x}$ frame, one may wish to minimize a function of the covariance associated with the spacecraft orbital elements, or any other nonCartesian frame, defined to be the $\mathbf{y}$ frame, that can be expressed as in (56). The cost function, given by (57), allows one to minimize a function of the covariance along the trajectory as well as at discrete times. One technique to employ the calculus of variations to determine an optimal trajectory with this cost function, which includes the covariance associated with the $\mathbf{y}$ frame, requires the entire problem be formulated in the $\mathbf{y}$ frame. This formulation requires that the spacecraft equations of motion, time derivative of the covariance matrix, time derivatives of the costates, and optimal values of the costates be derived in the $\mathbf{y}$ frame. Alternatively one could express the $\mathbf{y}$ frame covariance using (58), where $\mathbf{A}$ is the derivative of the $\mathbf{y}$ state with respect to the $\mathbf{x}$ state. This derivative must be continuous with a continuous inverse and must not be an explicit function of time in order to use the following technique. Because $\mathbf{A}$ and $\mathbf{y}$ can be written as functions of $\mathbf{x}$, the cost function from (57) can be expressed as (59). with (59) as the cost function, the entire problem can be solved in the $\mathbf{x}$ frame without deriving equations of motion or optimality conditions in the $\mathbf{y}$ frame. Although linear transformations are often used to approximate the covariance in another frame at a particular time,¹⁹ proves that the orbit obtained using the linear

transformation and cost function in (58) and (59) is the exact same orbit obtained by formulating the problem in the alternate frame, as long as the linear matrix Riccati equation is utilized for covariance propagation.

$$\mathbf{y} = \mathbf{s}(\mathbf{x})$$

$$W = J_f + \sum_{i=1}^n \varepsilon_i [\mathbf{y}(t_i), \mathbf{P}_y(t_i)] + \int_{t_0}^{t_f} \chi[\mathbf{y}(t), \mathbf{P}_y(t)] dt$$

$$\mathbf{P}_y = \mathbf{A} \mathbf{P}_x \mathbf{A}^T$$

$$W = J_f + \sum_{i=1}^n \varepsilon_i [\mathbf{x}(t_i), \mathbf{P}_x(t_i)] + \int_{t_0}^{t_f} \chi[\mathbf{x}(t), \mathbf{P}_x(t)] dt .$$

(56)(57)(58)(59)

E. Solution Process

The equations in the previous sections provide a general method to determine optimal trajectories between any initial and final constraints. The solution process sets up a TPBVP, where the unknowns are the parameters associated with the initial state and the initial costates. The constraints restrict the final state and costate. Note the term optimal means trajectories that maximize a combination of fuel efficiency and trajectory observability. By varying the relative values of the terms in the cost function, either fuel efficiency or observability can be emphasized.

A major difficulty associated with solving this type of TPBVP is the sensitivity of the solution process to the initial estimate of the costates. One might assume that including observability in the cost function makes determining an initial estimate quite difficult since including observability requires the estimate of 14 initial costates instead of 4 initial costates for the mass optimal solution in two dimensions or 27 costates instead of 6 in three dimensions. In fact it is no more difficult to estimate the costates with observability than without observability. The first step of the solution process can always be to solve the mass optimal problem and to obtain the value of the costates for that problem. This solution, with all of the covariance costates set to zero, is the solution if the observability terms in the cost function are multiplied by a scalar weight of zero. This solution, with the scalar weight of zero, is an excellent initial guess for a problem where the observability terms are multiplied by a small scalar weight. The solution with a small scalar weight can

then be utilized as the initial guess for a larger scalar weight. By slowly increasing these scalar weights for the observability terms in the cost function, one can 'walk' a mass optimal solution to a solution that optimizes for observability and fuel efficiency without needing to guess the initial costates for the covariance terms. Using the state transition matrix to compute derivatives instead of finite differences provides a faster, more robust method to 'walk' from solutions that ignore observability to ones that incorporate observability.^{20,21}

SECTION IV.

Example Problems

A. Planar Orbit to Orbit Transfer Optimality Conditions

The first four example applications are planar transfers through a central body gravity field specified by (60) from an initial orbit with eccentricity e_0 semimajor axis a_0 , and argument of perigee ω_0 to a target orbit with eccentricity e_f , semimajor axis a_f and argument of perigee ω_f . The initial time is specified to be t_0 , and the final time is specified to be t_f . The location of the spacecraft in the initial and final orbit is free because the initial and final true anomalies are free variables. The constraints requiring the spacecraft transfer between the initial and final orbit are specified by (61) and (62). The initial covariance is defined to be $\mathbf{P}_0$, and the spectral density of the uncertainty in the dynamic equations is given by the constant matrix $\mathbf{Q}$. The measurements are continuous scalar range measurements from a single observer to the spacecraft. The observer's location is specified by $\mathbf{p}(t)$, which is a vector from the center of the central body to the observer. The spectral density of the uncertainty in the measurement is given by the constant matrix $\mathbf{R}$, which is a scalar. By utilizing these conditions (5) to (10) and (14) can now be written as (63) to (66). The specific cost function for the examples in this paper is given by (67), where q is the integral of the trace of the covariance.

$$\begin{aligned}
 \mathbf{g}(\mathbf{r}) &= -\mu \frac{\mathbf{r}}{r^3} \\
 \theta &= \begin{pmatrix} \mathbf{r}(t_0) - \frac{a_0(1-e_0^2)}{1+e_0 \cos(\kappa_0)} \begin{pmatrix} \cos(\kappa_0 + \omega_0) \\ \sin(\kappa_0 + \omega_0) \end{pmatrix} \\ \mathbf{v}(t_0) + \sqrt{\frac{\mu}{(1-e_0^2)}} \begin{pmatrix} \sin(\kappa_0 + \omega_0) + e_0 \sin(\omega_0) \\ -\cos(\kappa_0 + \omega_0) - \cos(\omega_0) \end{pmatrix} \\ t_0 - t_{0s} \end{pmatrix} \\
 \psi &= \begin{pmatrix} \mathbf{r}(t_f) - \frac{a_f(1-e_f^2)}{1+e_f \cos(\kappa_f)} \begin{pmatrix} \cos(\kappa_f + \omega_f) \\ \sin(\kappa_f + \omega_f) \end{pmatrix} \\ \mathbf{v}(t_f) + \sqrt{\frac{\mu}{(1-e_f^2)}} \begin{pmatrix} \sin(\kappa_f + \omega_f) + e_0 \sin(\omega_f) \\ -\cos(\kappa_f + \omega_f) - \cos(\omega_f) \end{pmatrix} \\ t_f - t_{fs} \end{pmatrix} \\
 \begin{pmatrix} \dot{\mathbf{r}}(t) \\ \dot{\mathbf{v}}(t) \end{pmatrix} &= \begin{pmatrix} 0 & \mathbf{I} \\ \frac{\partial \mathbf{g}}{\partial \mathbf{r}} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ \mathbf{v} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ w_x \\ w_y \end{pmatrix} \\
 \mathbf{z}(t) &= \left(\frac{\mathbf{r}_s^T}{r_s} \mathbf{0}^T \right) \begin{pmatrix} \mathbf{r}(t) \\ \mathbf{v}(t) \end{pmatrix} + n_1 \\
 \tilde{\mathbf{X}}^T &= (\mathbf{r}^T \mathbf{v}^T J P_1 P_2 P_3 P_4 P_5 P_6 P_7 P_8 P_9 P_{10} q) \\
 \lambda^T &= (\lambda_{\mathbf{r}}^T \lambda_{\mathbf{v}}^T \lambda_J \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \lambda_6 \lambda_7 \lambda_8 \lambda_9 \lambda_{10} \lambda_q) \\
 W &= \varphi_1 P_{xx}(t_f) + \varphi_2 P_{yy}(t_f) + \varphi_3 P_{\ddot{x}x}(t_f) \\
 &\quad + \varphi_4 P_{\ddot{y}y}(t_f) + \varphi_5 J(t_f) + \varphi_6 q(t_f). \\
 (60)(61)(62)(63)(64)(65)(66)(67)
 \end{aligned}$$

The six scalars in the cost function, denoted by φ , determine the relative importance of each element of the trace of the covariance at the final time, the integral of the trace of the covariance, and the propellant. The mass optimal solution is obtained by setting all of

the φ_j to zero except for φ_5 . In order to place more emphasis on the trajectory's observability at the final time, φ_1 to φ_4 can be increased. Increasing φ_6 places more emphasis on minimizing the trace of the covariance along the entire trajectory. Because the magnitudes of the φ_i are irrelevant and because only the relative values of the φ_i are important, φ_5 is always set to one. If all the scalars are positive, the goal is to minimize the cost function; if the scalars are negative, the goal is to maximize the cost function.

The problem can now be solved as a TPBVP, with the 18 unknowns given by (68). The combination of α and (61) specifies the initial states and costates of the system given by (65) and (66). The initial position and velocity costates are determined using the adjoint control transformation.^{22,23} By utilizing (15) and (28), the time derivative of the costates along the optimal path can be written. The explicit expression is omitted here because of its length. The inclusion of (29) and (30) specifies the equations of motion for the entire system (states and costates).

$$\alpha^T = (\lambda_r^T(t_0)\lambda_v^T(t_0)\lambda_f(t_0)\lambda_1(t_0) \dots \lambda_{10}(t_0)\lambda_q\kappa_0\kappa_f).$$

(68)

The constraints that form the TPBVP are determined from (23), (26), and (62). Equation (23) provides the 12 constraints given by (69), and (62) provides four additional constraints. The final two constraints given by (70) are provided by (23) and (26). Any trajectory satisfying these equations is first-order extremal.

$$\begin{aligned}
 \lambda_J(t_f) &= \varphi_5 \\
 \lambda_1(t_f) &= \varphi_1 \\
 \lambda_5(t_f) &= \varphi_2 \\
 \lambda_8(t_f) &= \varphi_3 \\
 \lambda_{10}(t_f) &= \varphi_4 \\
 \lambda_q(t_f) &= \varphi_6 \\
 \lambda_2(t_f) &= \lambda_3(t_f) = \lambda_4(t_f) \text{ }^{(69)}\text{ }^{(70)} \\
 &= \lambda_6(t_f) = \lambda_7(t_f) \\
 &= \lambda_9(t_f) = 0 \\
 \lambda_{\mathbf{r}}^T(t_0) \frac{\partial \mathbf{r}(t_0)}{\partial \kappa_0} + \lambda_{\mathbf{v}}^T(t_0) \frac{\partial \mathbf{v}(t_0)}{\partial \kappa_0} &= 0 \\
 \lambda_{\mathbf{r}}^T(t_f) \frac{\partial \mathbf{r}(t_f)}{\partial \kappa_f} + \lambda_{\mathbf{v}}^T(t_f) \frac{\partial \mathbf{v}(t_f)}{\partial \kappa_f} &= 0.
 \end{aligned}$$

B. Continuous Scalar Range Measurements

The example used to illustrate this technique is a 20,000 second-planar transfer from an initial low Earth orbit (LEO) with eccentricity equal to 0.05, semimajor axis equal to 7,000 km, and argument of perigee equal to zero to a near GEO with eccentricity equal to 0.1, semimajor axis equal to 42,200 km, and argument of perigee equal to zero. Unless specified all distance units are kilometers, and all time units are seconds. The Earth is the central gravity body, and all third-body effects are ignored. The observer is located at the center of the Earth in this example. In all of the examples that follow, φ_1 , φ_2 , φ_3 , and φ_4 always have the same value, defined to be ℓ . Unless specified the spectral density matrices for all the examples and the initial value of the covariance are given by (71) to (73). These values are determined based on the sensors on the spacecraft and based the models of the equations of motion that are employed. Additionally $q(t)$ is always the integral of the equally weighted sum of the diagonal elements of the covariance matrix. Because these problems are solved with distance units of kilometers and time units of seconds, the magnitudes of the covariance elements associated with position are orders of magnitude larger than the magnitudes of the covariance elements associated with velocity. In effect this choice of scaling places a much greater emphasis on decreasing P_1 and P_5 than on decreasing P_8 and P_{10} . To emphasize all elements of the covariance matrix equally, one

could easily choose φ_3 and φ_4 to be orders of magnitude larger than φ_1 and φ_2 and could determine $q(t)$ from (74) instead of as the trace of the covariance.

$$\begin{aligned} \mathbf{P}(t_0) &= \begin{pmatrix} 10^{-5} & 0 & 0 & 0 \\ 0 & 10^{-5} & 0 & 0 \\ 0 & 0 & 10^{-8} & 0 \\ 0 & 0 & 0 & 10^{-8} \end{pmatrix} \\ \mathbf{Q} &= \begin{pmatrix} w_1^2 & 0 \\ 0 & w_1^2 \end{pmatrix} = \begin{pmatrix} 10^{-16} & 0 \\ 0 & 10^{-16} \end{pmatrix} \\ \mathbf{R} &= 10^{-1} \\ q(t) &= \int_{t_0}^t \left(P_{xx} + P_{yy} + 10^8 P_{xx} + 10^8 P_{yy} \right) dt. \end{aligned}$$

(71)(72)(73)(74)

In Figs. 1–7 φ is zero, and the value of φ_6 is increased in order to decrease the integral of the trace of the spacecraft's covariance. In Fig. 1 it is clear that the paths followed on the optimal trajectories obtained with the integral of trace of the covariance increasingly contributing to the cost function are quite similar. In fact at the scale of Fig. 1, the trajectories appear to be the same. In Fig. 2 one can observe that the trajectories that weight the integral of the trace of the covariance more heavily remain closer to the central body during most of the trajectory.

Fig. 1. Optimal trajectories.

Fig. 2. Magnified portion of optimal trajectories.

Fig. 3. Perturbation to thrust acceleration magnitude caused by including observability in cost function.

Fig. 4. Trace of covariance versus time.

Fig. 5. λ_1 versus time.

Fig. 6. Pareto front for optimal solutions.

Fig. 7. Trace of covariance versus time.

Fig. 3 shows the difference between the thrust acceleration magnitude on trajectories that include observability in the cost function and the trajectory that is mass optimal. This difference is on the order of ten percent of the thrust acceleration on the mass optimal transfer.

Fig. 4 shows the major difference between the optimal trajectories that incorporate covariance into the cost function. As the value of φ_6 is increased, the trace of the covariance is decreased significantly. Table I illustrates that a small increase in fuel consumption can significantly reduce the covariance. Between the mass optimal solution and the solution with φ_6 equal to 18×10^{-7} , the integral of the trace of the covariance is reduced by over 12 percent, while the integral of the thrust acceleration squared is only increased by about one-half of one percent. Notice the diminishing returns that are obtained by increasing the value of φ_6 . As φ_6 is increased from 0 to 6×10^{-7} , it takes less additional fuel to reduce the integral of the trace of the covariance than it does as φ_6 is increased from 6 to 12×10^{-7} .

Table I Observability and Fuel Usage for Trajectories Minimizing the Integral of the Trace of the Covariance

$\varphi_6 \times 10^{-7}$	Integral of Thrust Acc. Squared $\times 10^{-4}$	Change in Integral of Thrust Acc. Squared $\times 10^{-7}$	Integral of Trace of Cov.	Change in Integral of Trace of Cov.	Trace of Cov. at Final Time $\times 10^{-3}$	Change in Final Trace of Cov. $\times 10^{-4}$
0	16.9774	N/A	43.7312	N/A	1.8217	N/A
6	16.9910	13.6	41.3646	-2.3666	1.5806	-2.411
12	17.0230	45.6	39.5520	-4.1792	1.4122	-4.095
18	17.0662	88.8	38.1030	-5.6282	1.2876	-5.341

Fig. 5 shows a sampling of the behavior of the costates associated with the elements of the covariance matrix. As the value of φ_6 is increased, the graphs of the costate retain the same shape with the amplitude increasing. This observation is true for all of the costates, not just the sample illustrated in this figure.

In the previous example φ was set to zero, and the value of φ_6 was increased in order to incorporate observability into the optimization process. Alternatively one could set φ_6 to zero and increase the value of φ in order to minimize the trace of the covariance at the final time instead of the integral of the trace of the covariance. This technique provides similar results to the above example. In fact plots of the trajectories, controls, and costates are similar to those shown above, with one exception. As φ is increased the trajectories no longer remain closer to the central body during the majority of the trajectory as they do in the examples where φ_6 is increased. In fact there is no correlation between the distance the spacecraft is from the central body during the majority of the trajectory and the value of φ .

Table II provides a summary of the fuel consumption and observability along each trajectory. The trace of the covariance at the final time can be reduced by over 31 percent with less than a 0.6 percent increase in the integral of the thrust acceleration squared. for this specific transfer problem, either method (increasing φ or increasing φ_6) results in a trajectory where both the integral of the trace of the covariance and the trace of the covariance at the final time are reduced. As expected increasing φ more efficiently reduces the trace of the covariance at the final time, while increasing φ_6 allows the integral of the trace of the covariance to be reduced with a smaller fuel penalty. In other transfer problems increasing φ_6 may cause the trace of the covariance to be greater at the final time, even though it reduces the integral of the trace of the covariance. Similarly increasing φ may cause the integral of the trace of the covariance to increase, even though it reduces the trace of the covariance at the final time. The values φ and φ_6 that produce the optimal trajectory (trajectory that balances observability and fuel efficiency optimally for a particular mission) depend on the mission.

Table II Observability and Fuel Usage for Trajectories Minimizing the Trace of the Covariance at the Final Time

$\varphi \times 10^{-4}$	Integral of Thrust Acc. Squared $\times 10^{-4}$	Change in Integral of Thrust Acc. Squared $\times 10^{-7}$	Integral of Trace of Cov.	Change in Integral of Trace of Cov.	Trace of Cov. at Final Time $\times 10^{-3}$	Change in Final Trace of Cov. $\times 10^{-4}$
0	16.977	N/A	43.7312	N/A	1.8217	N/A
55	16.989	11.6	41.5930	-2.1382	1.5894	-2.323
110	17.013	35.6	40.1272	-3.6040	1.4411	-3.806
165	17.042	64.6	39.0203	-4.7109	1.3348	-4.869
200	17.062	84.2	38.4371	-5.2941	1.2807	-5.410
220	17.073	95.6	38.1359	-5.5953	1.2533	-5.6842

C. Periods without Measurements

The optimality conditions derived from using the calculus of variations for the case of continuous observations throughout the transfer are largely unchanged if there are periods when no observations are taken. The times when observations begin and end are not optimized and must be determined a priori. The time derivative of the covariance matrix instantaneously changes between (11) and (12), when observations begin and end. Consequently the Hamiltonian defined by (75) changes as well. Because the optimality conditions require the time derivative of the costates be given by (76), the time derivative of the costates changes as well. Both the augmented state and costate vectors are continuous at the times when observations start or stop. The other optimality conditions determined by the calculus of variations are unchanged.

$$\begin{aligned}
 H_a &= \lambda^T \tilde{\mathbf{X}} \\
 \dot{\lambda} &= -\frac{\partial H_a}{\partial \tilde{\mathbf{X}}}.
 \end{aligned}$$

(75)(76)

The same transfer as the first example, with all of the parameters unchanged except for the spectral density associated with the measurement, is used to illustrate the changes necessary when there are periods without observations. Continuous scalar range measurements are taken when the thousands digit of the elapsed time is an even number. Because the measurements are only taken half of the time, the spectral density associated with the measurement error is reduced by one-half, as specified in (77). This value of the spectral density causes the final value of the trace of the covariance on the mass optimal solution to be nearly the same in this example and the first example.

$$R = 0.05.$$

(77)

The results from optimizing the trajectory with periods that have no measurements are similar to the results obtained in the first example, when observations were made continuously throughout the trajectory. Table III provides a summary of the fuel cost and trace of the covariance at the final time for various values of φ . Notice that φ_6 is zero in this example. The trace of the covariance can be reduced by 30 percent for about a one-half percent increase in the integral of the thrust acceleration squared. The results indicate that the fuel cost of reducing the covariance at the final time is the same whether the measurements are continuous throughout the transfer or there are periods without measurements. The diminishing returns can be clearly seen in the Pareto front in Fig. 6. Figs. 7 and 8 show the trace of the covariance and costates for various values of φ . These figures, as well as the optimal trajectory and control profile, are remarkably similar to the figures for the same variables for the case of continuous measurements. The costates follow a curve with the same shape and phase behavior as the results from the first example.

Fig. 8. λ_1 versus time.

Table III Integral of Thrust Acceleration Squared and Trace of Covariance at Final Time with Periods of No Observations

φ	Integral of Thrust Squared $\times 10^{-4}$	Change in Integral of Thrust Acc. Squared $\times 10^{-10}$	Trace of Cov. at Final Time $\times 10^{-3}$	Change in Final Trace of Cov. $\times 10^{-6}$	$\frac{\Delta \text{Tr}[\mathbf{P}]}{\Delta J}$
0	16.97740	N/A	1.77833	N/A	N/A
.0001	16.97741	6	1.77285	-5.5	913.35
.0005	16.97753	134	1.75157	-26.8	166.25
.0010	16.97791	510	1.72632	-52.0	67.15
.0050	16.98720	9804	1.56459	-213.7	17.4
.0100	17.00772	30320	1.42504	-353.3	6.8
.0150	17.03281	55408	1.32372	-454.6	4.05
.0200	17.06007	82674	1.24538	-533.0	2.85
.0300	17.11718	139784	1.12974	-648.6	2
.0400	17.17500	197600	1.04657	-731.8	1.45
.0500	17.23222	254816	0.98271	-795.6	1.1

D. Discrete Scalar Range Measurements

The same transfer as the previous two examples is used to illustrate the discrete measurements case. The covariance associated with the noise in the measurement, given by (78), is different than the spectral density specified in (73) because the covariance term is not integrated over the time between each measurement. The value of the covariance associated with the noise in the measurement is selected so that the final value of the covariance for the mass optimal case is of the same order of magnitude as the mass optimal solution from the first example that uses continuous measurements. A discrete scalar range measurement from the center of the Earth to the spacecraft is taken every 400 seconds, beginning at $t = 400$ and ending at $t = 19,600$.

$$(78) \quad \tilde{\mathbf{R}} = .00025.$$

To illustrate the constraints specified by σ , (79), where s is defined in (80), gives the third unique constraint from (13). The expression represented by s is a scalar because the measurements can always be processed one at a time. The value of one element of the L matrix is given by (81). The other elements of σ and L can be determined from (13) and (39). When the observations are scalar range measurements, the determinant of B is given by (82), which can never be zero as long as there is noise in the measurement and as long as the observer never occupies the same location as the spacecraft. As a result one can solve (38) to determine the unique costates after each discrete measurement.

$$\begin{aligned}
 \sigma_3 &= P_{xx}(t_{k+}) - P_{xx}(t_{k-}) + \frac{S}{x_s^2 + y_s^2} \\
 &\times \left[P_{xx} \left(x_s^2 P_{xx} + x_s y_s P_{yx} \right) + P_{xy} \left(x_s y_s P_{xx} + y_s^2 P_{yx} \right) \right] |_{t_{k-}} \\
 S &= (\mathbf{H}(t_k) \mathbf{P}(t_{k-}) \mathbf{H}^T(t_k) + \mathbf{R})^{-1} \\
 L(6,3) &= \frac{S}{x_s^2 + y_s^2} [P_{xx} x_s y_s + P_{xy} y_s^2] |_{t_{k-}} \\
 |\mathbf{B}| &= \frac{\tilde{R}(x_s^2 + y_s^2)}{P_{xx} x_s^2 + P_{xy} x_s y_s + P_{yy} y_s^2 + \tilde{R}(x_s^2 + y_s^2)}.
 \end{aligned}
 \tag{79}(80)(81)(82)$$

The results for this discrete measurement example are similar to the results for continuous measurements. The integral of the thrust acceleration squared and the final value of the trace of the Cartesian covariance for the optimal trajectories for various values of φ are shown in Table IV. The trace of the covariance at the final time can be reduced by about 20 percent, with an increase of less than two-tenths of one percent in the integral of the thrust acceleration squared. Just as in the case of the continuous measurements, the fuel cost of further reducing the trace of the covariance at the final time increases significantly as φ is increased. The trace of the covariance and costates for various values of φ are shown in Figs. 9 and 10. Although the plot of the trace of the covariance is discontinuous, the shape of the graph and the time when the covariance on the mass optimal solution diverges from the covariance on the $\varphi = .5$ solution are nearly the same as the previous example with continuous observations. The results are similar for the control and the costates as well. Even though the covariance and costates are discontinuous at each measurement time, the controls still remain continuous at the measurement times. The plots for the other covariance elements and other costates are omitted because they are similar to the plots that are shown. The only elements that are significantly different are the plots for λ_8 , λ_9 , and λ_{10} , which are continuous.

Fig. 9. Trace of covariance versus time.

Fig. 10. λ_1 versus time.

Table IV Observability and Fuel Usage for Trajectories Minimizing the Trace of the Covariance at the Final Time with Discrete Observations

φ	Integral of Thrust Squared $\times 10^{-4}$	Change in Integral of Thrust Acc. Squared $\times 10^{-10}$	Trace of Cov. at Final Time $\times 10^{-3}$	Change in Final Trace of Cov. $\times 10^{-6}$	$\frac{\Delta \text{Tr}[\mathbf{P}]}{\Delta J}$
0	16.97740	N/A	1.82928	N/A	N/A
.0001	16.97741	12	1.82355	-6.1	510
.0005	16.97754	136	1.80220	-27.5	172.2
.0010	16.97791	512	1.77683	-52.8	67.45
.0050	16.98729	9888	1.61344	-216.2	17.45
.0100	17.00815	30752	1.47149	-358.2	6.8
.0150	17.03384	56436	1.36794	-461.7	4.05
.0200	17.06177	84374	1.28765	-542.0	2.85
.0300	17.12041	143010	1.16891	-660.8	2.05
.0400	17.17986	202458	1.08339	-746.3	1.45
.0500	17.23872	261324	1.01769	-812.0	1.1

E. Dynamic Errors Based on Thrust

The same planar transfer from LEO to near GEO is used to illustrate the effect of allowing the error in the dynamic equations to be a function of the controls. The observer is fixed at the center of the Earth taking continuous scalar range measurements of the spacecraft position throughout the trajectory. Because it more accurately reflects the operation of an engine, the second formulation, in which the controls are thrust acceleration magnitude and direction, is used to illustrate this example. The constant term in the spectral density matrix for the error in the dynamic equations w_1^2 , given by (72), is reduced to one-half its value in the previous examples. The constants ξ_1 and ξ_2 are set to 1.1×10^8 and 10^8 , respectively. These values were chosen so that errors in the engine thrust magnitude and direction and errors due to dynamics that were not modeled would be of approximately the same magnitude. ξ_1 and ξ_2 have different values in order to make the spectral density matrix $\mathbf{Q}$ nondiagonal.

Table V summarizes the effect that placing an increasing emphasis on the trace of the covariance at the final time has on the required thrust acceleration and the covariance. Again a relatively large decrease in the trace of the covariance at the final time can be obtained for a small increase in the thrust acceleration. As the covariance at the final time is reduced, it becomes more expensive to further reduce the final covariance. Following the trajectory that minimizes the trace of the covariance at the final time has nearly no affect on the trace of the covariance during the initial portion of the transfer, but it does significantly reduce the trace of the covariance at the final time.

Table V Observability and Fuel Usage for Trajectories Minimizing The Trace of the Covariance at the Final Time with Variable Errors in the Dynamics

$\varphi \times 10^{-3}$	Integral of Thrust Acc. Squared $\times 10^{-4}$	Change in Integral of Thrust Acc. Squared $\times 10^{-8}$	Integral of Trace of Cov.	Change in Integral of Trace of Cov. $\times 10^{-1}$	Trace of Cov. Final Time $\times 10^{-3}$	Change in Final Trace of Cov. $\times 10^{-4}$
0.000	16.9774	N/A	58.8494	N/A	4.1586	N/A
0.002	16.9802	28	57.9147	-9.347	4.0164	-1.422
0.004	16.9878	104	57.0755	-17.739	3.8897	-2.689
0.006	16.9990	216	56.3185	-25.309	3.7762	-3.824
0.008	17.0134	360	55.6321	-32.173	3.6740	-4.846
0.010	17.0300	526	55.0065	-38.429	3.5815	-5.771

Table VI shows the effect of placing an increasing emphasis on the integral of the trace of the covariance, instead of the trace of the covariance at the final time. In this case the integral of the trace of the covariance can be reduced by over eight percent, with less than a one-half percent increase in the integral of the thrust acceleration squared. Again the ratio of the percentage decrease in the integral of the trace of the covariance to the percentage increase in the integral of the thrust acceleration squared decreases significantly, as maximizing observability is emphasized. Figs. 11 and 12 illustrate the effect of modeling the error in the dynamic equations as a function of the thrust magnitude. As the integral of the trace of the covariance is weighted more heavily in the optimization process, the maximum value for the thrust acceleration magnitude decreases, thus introducing less uncertainty into the equations of motion.

Fig. 11. Thrust acceleration magnitude versus time.

Fig. 12. Maximum thrust acceleration magnitude versus time.

Table VI Observability and Fuel Usage for Trajectories Minimizing the Integral of the Trace of the Covariance with Variable Errors in the Dynamics

$\varphi_6 \times 10^{-7}$	Integral of Thrust Acc. Squared $\times 10^{-4}$	Change in Integral of Thrust Acc. Squared $\times 10^{-8}$	Integral of Trace of Cov.	Change in Integral of Trace of Cov. $\times 10^{-1}$	Trace of Cov. Final Time $\times 10^{-3}$	Change in Final Trace of Cov. $\times 10^{-4}$
0	16.9774	N/A	58.8494	N/A	4.159	N/A
3.83	16.9822	48.4	57.5619	-1.2875	3.981	-1.78
15.4	17.0406	632.2	54.4267	-4.4227	3.563	-5.96
17.9	17.0593	819	53.8654	-4.9840	3.491	-6.68

F. Earth Orbit to Lunar Orbit

A 36 hr planar transfer from an initial Earth orbit, with eccentricity equal to 0.05, semimajor axis equal to 42,000 km, and argument of perigee equal to zero, to a lunar orbit, with eccentricity equal to 0.1, semimajor axis equal to 3,476 km (two Moon radii), and argument of perigee equal to zero, is employed to demonstrate this application. Throughout the trajectory the gravitational effects from both the Earth and the Moon affect the spacecraft. The Moon is modeled as traveling in a circular orbit around the Earth at a distance of 384,400 km. At the beginning of the example transfer, the Moon has a specified mean anomaly, which is set to 190 deg. The effect of the initial mean anomaly on the optimal solution can be studied parametrically, or one could add this value to the list of free variables with only slight modifications to the optimality conditions.

During the transfer a series of scalar range measurements of the spacecraft state are taken. The observers are located at three locations on the surface of the Earth, which are separated by 120 deg. The locations of these observers are expressed as a function of time in (83). These observations are modeled as alternating periods of continuous measurements and no measurements, with each period lasting for one hour, until the final

two hours of the transfer, during which continuous observations are made. The measurements are taken by the observer who views the spacecraft with the highest elevation at the beginning of the measurement period. Consequently none of the measurements are occulted by the Earth. The cost function is again the sum of the integral of the thrust acceleration squared and the covariance associated with the semimajor axis of the spacecraft, with respect to the Moon at the end of the transfer, as specified by (84).

$$\begin{aligned} \mathbf{p}_1(t) &= 6378 \begin{bmatrix} \cos\left(\frac{2\pi t}{86400}\right) \\ \sin\left(\frac{2\pi t}{86400}\right) \end{bmatrix} \\ \mathbf{p}_2(t) &= 6378 \begin{bmatrix} \cos\left(\frac{2\pi t}{86400} + \frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi t}{86400} + \frac{2\pi}{3}\right) \end{bmatrix} \\ \mathbf{p}_3(t) &= 6378 \begin{bmatrix} \cos\left(\frac{2\pi t}{86400} + \frac{4\pi}{3}\right) \\ \sin\left(\frac{2\pi t}{86400} + \frac{4\pi}{3}\right) \end{bmatrix} \\ W &= J(t_f) + \varphi P_{aa}(t_f) = J(t_f) + \varphi \frac{\partial a}{\mathbf{X}} \mathbf{P}_x(t_f) \left(\frac{\partial a}{\mathbf{X}} \right). \end{aligned}$$

(83)(84)

The results from this optimization indicate that the covariance associated with the semimajor axis of the target lunar orbit can be reduced significantly with only a slight increase in thrust acceleration. Table VII documents the integral of the thrust acceleration squared and the covariance associated with the target orbit semimajor axis for several trajectories that increasingly weight the covariance in the cost function. The covariance associated with the semimajor axis of the target orbit can be reduced by over 75 percent, with an increase in the integral of the thrust acceleration squared of only slightly more than three-quarters of one percent. The Pareto front for this solution is shown in Fig. 13.

Fig. 13. Pareto optimal front.

Table VII Semimajor Axis Covariance and Thrust Acceleration for Near GEO to Lunar Orbit Transfers

$\varphi \times 10^{-4}$	Integral of Thrust Acc. Squared $\times 10^{-1}$	Change in Integral of Thrust Acc. Squared $\times 10^{-5}$	Semimajor Axis Cov. at Final Time $\times 10^{-3}$	Change in Semimajor Axis Cov. $\times 10^{-3}$
0	4.2646	N/A	32.82	N/A
3	4.2668	22	25.32	-7.50
6	4.2728	82	18.56	-14.26
9	4.2802	156	13.53	-19.29
12	4.2870	224	12.72	-20.10
15	4.2928	282	8.14	-24.68
18	4.2976	330	6.68	-26.14

The mass optimal transfer, with φ equal to zero, and the transfer, with φ equal to 1.8×10^{-4} , that optimize for fuel consumption and observability are shown in Figs. 14 and 15. Despite the fact that the covariance associated with the semimajor axis is reduced significantly, the transfer trajectory is not altered significantly.

Fig. 14. Earth orbit to lunar orbit transfer.

Fig. 15. Earth orbit to lunar orbit transfer (magnified).

Fig. 16 shows the magnitude of the thrust acceleration for the mass optimal transfer. Figs. 17 and 18 show the change in the thrust acceleration magnitude and the thrust

direction when φ changes from 0 to 1.8×10^{-4} . Notice that the thrust acceleration magnitude is altered by between one-tenth of one percent and 10 percent during different periods of the transfer.

Fig. 16. Thrust acceleration magnitude on mass optimal transfer versus time.

Fig. 17. Change in thrust acceleration magnitude between transfer with $\varphi=0.00018$ and mass optimal transfer versus time.

Fig. 18. Change in thrust direction between transfer with $\varphi=0.00018$ and mass optimal transfer versus time.

The covariance associated with the spacecraft semimajor axis, with respect to the Moon, along these two trajectories is shown in Fig. 19. This covariance is only plotted near the end of the trajectory because the spacecraft semimajor axis relative to the Moon does not have a significant meaning until the spacecraft approaches the Moon.

Fig. 19. Covariance associated with semimajor axis of the target lunar orbit versus time.

The costate associated with P_{yy} is plotted in Fig. 20. Notice how this costate is largely unchanged during the early part of the trajectory and then changes a great deal as the spacecraft approaches the Moon. These large changes near the end of the trajectory

make it difficult to determine the initial value of these covariance costates because a very small change in the initial value of the costate or numerical integration error during the long portion of the trajectory where the costate remains nearly constant can cause a large violation of the constraint on these costates at the final time. These difficulties can be significantly reduced by including the value of the costate at an intermediate time in the unknown parameter list and by constraining the value of the costate before and after this intermediate time to be the same. Fig. 21 shows the costate associated with P_{yy} near the end of the transfer for various values of φ . Notice that for this transfer from Earth orbit to lunar orbit, the covariance costates do not exhibit the same phase and amplitude behavior as they exhibit in previous transfers. The covariance costates do not simply increase in amplitude as the observability terms are weighted more heavily in the cost function.

Fig. 20. Covariance costate associated with P_{yy} versus time.

Fig. 21. Covariance costate associated with P_{yy} versus time (magnified).

SECTION V.

Conclusion

Adding either the trace of the covariance at discrete times or the integral of the trace of the covariance to the cost function allows one to determine trajectories with significantly improved observability with very little additional propellant required. These results indicate that the cost of further improving the observability increases as observability is increased.

The amount of fuel required to improve the observability along a trajectory by a given percentage depends on numerous factors, including the initial and final orbits as well as the transfer time. This research does not attempt to determine which trajectories require less propellant to improve observability. The models for the error in observations and errors in dynamic equations also affect the results. The effect of changing the ratio of terms in $\mathbf{R}$, $\mathbf{Q}$, and $\mathbf{P}(t_0)$ has not been investigated. Additionally these results assume scalar range measurements. Transitioning to different measurement types, multiple observers, or missions between different orbits may yield different results.

The differences between the optimal trajectories with, continuous measurements throughout the trajectory, alternating periods of continuous measurements and no measurements, and discrete measurements, are minimal for the example transfer from LEO to GEO. The optimal control histories and optimal costates are also similar for these three types of measurements. The results indicate that the exact times of the observations do not need to be known a priori in order to determine the fuel cost of decreasing the covariance by a given amount using this technique. The results do indicate that one must know the quality and quantity of the measurements that are taken. Additionally this study only examines trajectories in which the temporal spacing of the observations is equal. Further study is necessary to determine the effect of different measurement spacing.

The similarity between the results using discrete observations and the results using continuous observations indicates that the continuous model is sufficient to determine optimal trajectories, including observability. While one could theoretically use the method presented in this paper to solve an optimization problem with any number of discrete observations, one major difficulty limits the practical application of this method. In order to solve the TPBVP, one must compute the derivatives of the states and costates at the final time with respect to the costates at the initial time. As the number of observations increases, the number of jumps in the augmented state increases as well. These jumps in

the state elements and the corresponding jumps in the costate elements make it difficult to compute the derivatives with enough accuracy to determine optimal trajectories with thousands of discrete measurements. Therefore the similar results between discrete and continuous observations indicate the continuous model should be used in order to avoid the numerical difficulties of the discrete model, unless an extremely high level of accuracy is necessary.

For the case of dynamic errors based on the thrust, the fact that it is more difficult to reduce the integral of the trace of the covariance by changing the thrust profile should be expected. When the engine thrusts more in an effort to place the spacecraft on a trajectory that is more observable, this process introduces more error into the equations of motion (because the model for dynamic error assumes that the error is proportional to thrust magnitude), thus causing the uncertainty to increase.

The example transfer from Earth orbit to lunar orbit illustrates one of the important conclusions about designing orbits to increase knowledge of the spacecraft state at the end of the transfer. The trajectory where spacecraft knowledge is improved spends a longer time near the Moon, where the gradient of the gravitational forces from the Moon can help distinguish the trajectory. It is easier to distinguish between two trajectories, when the gradient of the equations of motion with respect to the state is larger.

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