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# Robust and Resilient Finite-Time Bounded Control of Discrete-Time Uncertain Nonlinear Systems

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## Abstract

Finite-time state-feedback stabilization is addressed for a class of discrete-time nonlinear systems with conic-type nonlinearities, bounded feedback control gain perturbations, and additive disturbances. First, conditions for the existence of a robust and resilient linear state-feedback controller for this class of systems are derived. Then, using linear matrix inequality techniques, a solution for the controller gain and the maximum allowable bound on the gain perturbation is obtained. The developed controller is robust for all unknown nonlinearities lying in a known hypersphere with an uncertain center and all admissible disturbances. Moreover, it is resilient against any bounded perturbations that may alter the controller's gain by at most a prescribed amount. The paper is concluded with a numerical example showcasing the applicability of the main result.

# Keywords

Discrete-time systems, LMIs, Nonlinear systems, State-feedback, Finite-time stability, Robust Control

## 1. Introduction

Finite-time stabilization via state-feedback of discrete-time nonlinear systems with additive disturbances and unknown nonlinearities lying within a hypersphere of uncertain center is presented. A system is said to be Finite-Time Stable (FTS) or more precisely Finite-Time Bounded (FTB) if, given a bound on the initial state of the system and the disturbance input, the state of the system does not exceed a given bound over a fixed time interval and for all admissible disturbances. Various developments and extensions in the field of FTS have been implemented, most of which have been applied to linear systems ([Amato and Ariola, 2005](#), [Amato et al., 2010a](#), [Amato et al., 2010b](#), [Amato et al., 2004](#), [Dorato et al., 1997](#), [Garcia et al., 2009](#), [Zhang and An, 2008](#), only to mention a few due to space limitations). However, to the best of our knowledge, the study of FTS or FTB of nonlinear systems is rarely addressed in the literature. Some of the work related to FTB of nonlinear systems is found in [Amato, Cosentino, and Merola \(2010\)](#), [Yang, Li, and Chen \(2009\)](#), and [Zhuang and Liu \(2010\)](#). In [ElBsat and Yaz \(2011\)](#), a robust and resilient FTB controller design for a class of discrete-time nonlinear systems with conic-type nonlinearities lying in a hypersphere with a known center, feedback gain perturbations, and additive disturbances is presented.

The work presented in this technical communique is an extension of that in [ElBsat and Yaz \(2011\)](#). Here, the center of the hypersphere, which contains the set of unknown nonlinear vector functions, is described by a linear system with uncertainty in its dynamics. Moreover, an analysis of the upper bound on the gain perturbation vector as a function of the vector's direction in a three-dimensional (3D) space is presented. The significance of the controller design developed here is that it requires the knowledge of a dynamical bound on the system's nonlinearity rather than its exact dynamics. Thus, the controller design developed is applicable to all nonlinear systems which are locally Lipschitz ([Khalil, 2002](#)). Furthermore, the controller is robust for all nonlinearities lying within the conic bound, all admissible disturbances, and all bounded perturbations affecting the center of the conic bound. It is also resilient against all bounded perturbations which may affect its gain and which may occur as a result of computational or implementation errors ([Takabashi, Dutra, Palhares, & Peres, 2000](#)).

Next, the system and controller models are introduced. In Section 3, the main results of this communique are presented followed by a simulation study illustrating the use of these results. Moreover, a study of the bound on the gain perturbation vector as a function of the vector's direction in a 3D space is introduced.

The following notation is used:  $x \in R^n$  is an  $n$ -dimensional vector,  $\|x\| = (x^T x)^{1/2}$  is the Euclidean norm,  $(\cdot)^T$  and  $(\cdot)^{-1}$  are the matrix transpose and inverse operators, respectively,  $A \in R^{m \times n}$  is an  $m \times n$  real matrix,  $A > 0$  ( $A < 0$ ) is a positive-definite (negative-definite) matrix,  $I$  is an identity matrix of appropriate dimensions, and  $N_0$  is the set of nonnegative integers.

## 2. Definition: finite-time boundedness

Generally, a system is said to be Finite-Time Bounded (FTB) if, given a bound on the initial state of the system and the disturbance input, the state of the system does not exceed a given bound over a fixed time interval and for all admissible additive disturbances. In this work, the definitions stated in the work of [Amato and Ariola \(2005\)](#) are adopted and are generalized to include nonlinear systems. Consider a system that is described by the following dynamics:

$$(1) \quad x_{k+1} = f(x_k, u_k, w_k)$$

where  $f$  is the vector function which is in general nonlinear.

System (1) is said to be FTB with respect to  $(\alpha_x, \alpha_w, \beta, R, N)$  where  $R > 0, \alpha_w \geq 0, 0 \leq \alpha_x \leq \beta$  and  $N \in N_0$  if

$$\begin{cases} x_0^T R x_0 \leq \alpha_x^2 \\ w_0^T w_0 \leq \alpha_w^2 \end{cases} \Rightarrow x_k^T R x_k \leq \beta^2 \forall k = 1, \dots, N.$$

### 3. System and control model

Consider the following discrete-time nonlinear system:

$$(2) \begin{cases} x_{k+1} = f(x_k, u_k, w_k) \\ w_{k+1} = \Phi w_k \end{cases}$$

where  $x_k \in W_n \subset R^n$  is the state vector,  $u_k \in W_m \subset R^m$  is the input vector,  $w_k \in W_r \subset R^r$  is the disturbance input, and  $\Phi \in R^{r \times r}$ , and  $W_m$  are open and connected sets. The disturbance is one of the known waveforms (Johnson, 1980), but it does not have to be of finite-energy type.  $f(x_k, u_k, w_k)$  is an unknown nonlinear vector function whose dynamics have the following conic sector description:

$$(3) \|f(x_k, u_k, w_k) - (Ax_k + Bu_k + Fw_k)\| \leq \|C_f x_k + D_f u_k + F_f w_k\|$$

for all time  $k \in N_0, x_k \in W_n, u_k \in W_m$ , and  $w_k \in W_r$  where  $A \in R^{n \times n}, B \in R^{n \times m}$ , and  $F \in R^{n \times r}$ , which are assumed to have the following expressions:

$$(4) \begin{cases} \tilde{A} = A + A_\Delta \\ \tilde{B} = B + B_\Delta \\ \tilde{F} = F + F_\Delta \end{cases} \text{ such that } \begin{cases} A_\Delta A_\Delta^T \leq \sigma_A^2 I \\ B_\Delta B_\Delta^T \leq \sigma_B^2 I \\ F_\Delta F_\Delta^T \leq \sigma_F^2 I. \end{cases}$$

The matrices  $A_\Delta, B_\Delta$ , and  $F_\Delta$  are unknown bounded perturbations with known scalar upper bounds  $\sigma_A, \sigma_B$ , and  $\sigma_F$ , respectively. Matrices  $A, B, F, C_f, D_f$ , and  $F_f$  are assumed to be known for the system in consideration. The inequality shown in (3) implies that the unknown nonlinearity lies in an  $n$ -dimensional hypersphere whose center is the linear system  $Ax_k + Bu_k + Fw_k$  with uncertain parameter matrices and whose radius is bounded by the right-hand side term of (3). Moreover, given system (2), a linear state-feedback controller with gain  $K \in R^{m \times n}$  is considered such that

$$(5) u_k = Kx_k$$

which leads to the following closed-loop system:

$$(6) \begin{cases} x_{k+1} = (\tilde{A} + \tilde{B}K)x_k + \tilde{F}w_k + \mathfrak{S}_k \\ w_{k+1} = \Phi w_k \end{cases}$$

where  $\mathfrak{S}_k = f(x_k, u_k, w_k) - (\tilde{A}x_k + \tilde{B}u_k + \tilde{F}w_k)$ .

## 4. Main results

The objective is to find a robust and resilient state-feedback controller that will render the closed-loop system FTB as long as the nonlinearity is within the hypersphere defined by (3). [Theorem 1](#) states the conditions for the existence of a robust linear state-feedback controller for the class of nonlinear systems described by (2).

### Theorem 1

Given sector condition (3), matrices  $A, B, F, C_f, D_f$ , and  $F_f$ , and the upper bounds  $\sigma_A, \sigma_B$ , and  $\sigma_F$ , system (6) is FTB with respect to  $(\alpha_x, \alpha_w, \beta, R, N)$ , if there exist positive-definite matrices  $Q_1 \in R^{n \times n}$  and  $Q_2 \in R^{r \times r}$ , a matrix  $Y \in R^{m \times n}$ , and positive scalars  $\gamma \geq 1, b_1, \delta, \alpha_1, \alpha_2$ , and  $\alpha_3$  such that the following conditions hold:

$$(7) M = M^T = [m_{ij}]_{i,j=1 \dots 8} > 0$$

$$(8) \begin{bmatrix} Q_1 - \delta R^{-1} & 0 \\ 0 & Q_2 - \delta I \end{bmatrix} > 0$$

$$(9) \delta R^{-1} \frac{\beta^2 \gamma^{-N}}{\alpha_x^2 + \alpha_w^2} - Q_1 > 0$$

Where

$$\begin{aligned} m_{11} &= \gamma Q_1, m_{13} = Q_1 A^T + Y^T B^T, m_{14} = Q_1 C_f^T + Y^T D_f^T, \\ m_{16} &= Q_1, m_{17} = Y^T, m_{22} = \gamma Q_2, m_{23} = Q_2 F^T, m_{24} = Q_2 F_f^T, \\ m_{25} &= Q_2 \Phi^T, m_{28} = Q_2, \\ m_{33} &= Q_1 - (b_1 + \alpha_1 \sigma_A^2 + \alpha_2 \sigma_B^2 + \alpha_3 \sigma_F^2) I, \\ m_{44} &= b_1 I, m_{55} = Q_2, m_{66} = \alpha_1 I, m_{77} = \alpha_2 I, m_{88} = \alpha_3 I, \end{aligned}$$

and the unspecified submatrices are equal to zero matrices with appropriate dimensions. The controller gain is given by  $K = Y Q_1^{-1}$ .

### Sketch of Proof

Assume that  $x_0^T R x_0 \leq \alpha_x^2, w_0^T w_0 \leq \alpha_w^2$ , and that  $x_k^T R x_k \leq \beta^2 \forall k = 1, \dots, N$ . Consider the energy function,

$$(10) V_k = x_k^T P_1 x_k + w_k^T P_2 w_k \text{ such that } V_{k+1} < \gamma V_k$$

where  $P_1 > 0, P_2 > 0$ , and  $\gamma \geq 1$ .

Moreover, consider the inequality shown in (3) which can be rewritten as follows:

$$(11) \mathfrak{S}_k^T \mathfrak{S}_k \leq (A_f x_k + F_f w_k)^T (A_f x_k + F_f w_k)$$

where  $A_f = C_f + D_f K$ .

From (10), replacing  $x_{k+1}$  and  $w_{k+1}$  with the equations of system (2), and applying Schur's complement the following matrix inequality is obtained.

$$(12) \begin{bmatrix} h_{11} & h_{12} \\ h_{12}^T & h_{22} \end{bmatrix} > \begin{bmatrix} 0 & -\mathfrak{S}_k^T P_1 \\ -P_1 \mathfrak{S}_k & 0 \end{bmatrix}$$

Where

$$h_{11} = \gamma(x_k^T P_1 x_k + w_k^T P_2 w_k) - w_k^T \Phi^T P_2 \Phi w_k, h_{22} = P_1, \text{ and } h_{12} = (A_c x_k + \tilde{F} w_k)^T P_1.$$

For any  $b_1 > 0$ , it is true that

$$(13) \begin{bmatrix} b_1^{-1} \tilde{\mathcal{S}}_k^T \tilde{\mathcal{S}}_k & 0 \\ 0 & b_1 P_1^2 \end{bmatrix} \geq \begin{bmatrix} 0 & -\tilde{\mathcal{S}}_k^T P_1 \\ -P_1 \tilde{\mathcal{S}}_k & 0 \end{bmatrix}.$$

Using (13), the following is a sufficient condition for (12):

$$(14) \begin{bmatrix} h_{11} & h_{12} \\ h_{12}^T & h_{22} \end{bmatrix} > \begin{bmatrix} b_1^{-1} \tilde{\mathcal{S}}_k^T \tilde{\mathcal{S}}_k & 0 \\ 0 & b_1 P_1^2 \end{bmatrix}.$$

Moreover, based on (11), (14) will still be satisfied, if the following inequality holds.

$$(15) \begin{bmatrix} h_{11} - b_1^{-1} (A_f x_k + F_f w_k)^T (A_f x_k + F_f w_k) & h_{12} \\ h_{12}^T & h_{22} - b_1 P_1^2 \end{bmatrix} > 0.$$

Applying Schur's complement to (15), substituting the expressions of  $h_{11}$ ,  $h_{12}$ , and  $h_{22}$ , then rearranging the obtained expression in a quadratic format in  $[x_k^T \ w_k^T]$  yields a positive-definite matrix, which can be rewritten as follows:

$$(16) \begin{bmatrix} \gamma P_1 - b_1^{-1} A_f^T A_f & -b_1^{-1} A_f^T F_f \\ -b_1^{-1} F_f^T A_f & \gamma P_2 - \Phi^T P_2 \Phi - b_1^{-1} F_f^T F_f \end{bmatrix} - \begin{bmatrix} A_c^T P_1 \\ \tilde{F}^T P_1 \end{bmatrix} (P_1 - b_1 P_1^2)^{-1} \begin{bmatrix} P_1 A_c & P_1 \tilde{F} \end{bmatrix} > 0.$$

It is worth noting that the condition  $P_1 - b_1 P_1^2 > 0$  due to Schur's complement will be implicitly satisfied and thus it would be redundant to consider it as one of the conditions for the existence of the robust controller developed.

By applying Schur's complement to (16), and after some algebraic manipulations and substituting for  $A_f$  and  $A_c$ , we obtain

$$(17) \begin{bmatrix} \gamma Q_1 & 0 & Q_1 \tilde{A}^T + Y^T \tilde{B}^T & Q_1 C_f^T + Y^T D_f^T & 0 \\ * & \gamma Q_2 & Q_2 \tilde{F}^T & Q_2 F_f^T & Q_2 \Phi^T \\ * & * & Q_1 - b_1 I & 0 & 0 \\ * & * & * & b_1 I & 0 \\ * & * & * & * & Q_2 \end{bmatrix} > 0$$

where \* denotes the elements that need to be added in order to have a symmetric matrix,  $Q_1 = P_1^{-1}$ ,  $Q_2 = P_2^{-1}$ , and  $Y = K Q_1$ .

Substituting (4) in (17), rearranging the obtained inequality and after showing that for any positive scalars  $\alpha_1, \alpha_2$ , and  $\alpha_3$ :

$$(18) \begin{bmatrix} -\alpha_1^{-1}Q_1^2 - \alpha_2^{-1}Y^TY & 0 & -Q_1A_\Delta^T - Y^TB_\Delta^T & 0 & 0 \\ * & \alpha_3^{-1}Q_2^2 & Q_2F_\Delta^T & 0 & 0 \\ * & * & \Psi & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix} \leq 0$$

where  $\Psi = \alpha_1A_\Delta A_\Delta^T + \alpha_2B_\Delta B_\Delta^T + \alpha_3F_\Delta F_\Delta^T$ , we arrive at condition (7).

Now, we proceed to show the derivation of conditions (8), (9). Applying successive substitutions of (10) for  $k = 1, 2, \dots, N$ , knowing that  $\gamma \geq 1$ , replacing  $V_k$  and  $V_0$  with their corresponding expressions, and since  $x_k^T P_1 x_k < x_k^T P_1 x_k + w_k^T P_2 w_k$ , we obtain the following:

$$(19) x_k^T P_1 x_k < \gamma^N (x_0^T P_1 x_0 + w_0^T P_2 w_0).$$

In (19), introduce the term  $R^{1/2}R^{-1/2}$  to the left- and right-hand side of  $P_1$ , express the right-hand side of the inequality in a quadratic format, and apply Rayleigh's inequality, which states that given  $Q > 0$ , then  $\lambda_{\min}(Q)x_k^T x_k < x_k^T Q x_k < \lambda_{\max}(Q)x_k^T x_k$  is true, to obtain the following:

$$(20) \lambda_{\min}(R^{-1/2}P_1R^{-1/2})x_k^T R x_k < \gamma^N \lambda_{\max} \left( \begin{bmatrix} R^{-1/2}P_1R^{-1/2} & 0 \\ 0 & P_2 \end{bmatrix} \right) (\alpha_x^2 + \alpha_w^2).$$

In order for  $x_k^T R x_k < \beta^2$  to be satisfied, then

$$(21) \lambda_{\max} \left( \begin{bmatrix} R^{-1/2}P_1R^{-1/2} & 0 \\ 0 & P_2 \end{bmatrix} \right) < \frac{\beta^2 \gamma^{-N}}{(\alpha_x^2 + \alpha_w^2)} \lambda_{\min}(R^{-1/2}P_1R^{-1/2})$$

must hold. Let  $\delta^{-1} > 0$  such that

$$(22) \lambda_{\max} \left( \begin{bmatrix} R^{-1/2}P_1R^{-1/2} & 0 \\ 0 & P_2 \end{bmatrix} \right) < \delta^{-1}$$

and

$$(23) \delta^{-1} < \frac{\beta^2 \gamma^{-N}}{(\alpha_x^2 + \alpha_w^2)} \lambda_{\min}(R^{-1/2}P_1R^{-1/2}).$$

Then, conditions (8), (9) can be obtained from (22), (23) respectively through some algebraic manipulations and the proof of Theorem 1 is concluded.

Next, the controller gain is assumed to have additive bounded perturbations and conditions for the existence of a finite-time controller that is not only robust but also *resilient* are derived. Consider system (6) with a controller gain  $K$  instead of  $K$ :

$$(24) \begin{cases} \tilde{x}_{k+1} = (A + BK)\tilde{x}_k + Fw_k + \mathfrak{S}_k \\ w_{k+1} = \Phi w_k \end{cases}$$

where  $\mathfrak{S}_k$  satisfies inequality (11) and

$$(25) \tilde{K} = K_r + K_\Delta$$

where  $K_r$  is the controller gain and  $K_\Delta$  is an unknown additive bounded gain perturbation with a scalar upper bound  $\sigma_K$  such that

$$(26) K_\Delta^T K_\Delta \leq \sigma_K^2 I.$$

## Theorem 2

Given sector condition (3), matrices  $A, B, F, C_f, D_f$ , and  $F_f$ , and the upper bounds  $\sigma_A, \sigma_B$ , and  $\sigma_F$ , system (24) is FTB with respect to  $(\alpha_x, \alpha_w, \beta, R, N)$ , if there exist positive-definite matrices  $Q_1 \in R^{n \times n}$  and  $Q_2 \in R^{r \times r}$ , a matrix  $Y_r \in R^{m \times n}$ , and positive scalars  $\gamma \geq 1, b_1, b_2, b_3, \alpha_1, \alpha_2, \alpha_3$  and  $\delta$  such that conditions (8), (9), and the following hold:

$$(27) Z = Z^T = [z_{ij}]_{i,j=1 \dots 9} > 0$$

where

$$\begin{aligned} z_{11} &= \gamma Q_1, z_{13} = Q_1 A^T + Y_r^T B^T, z_{14} = Q_1 C_f^T + Y_r^T D_f^T, \\ z_{16} &= Q_1, z_{17} = Y_r^T, z_{19} = Q_1, z_{22} = \gamma Q_2, z_{23} = Q_2 F^T, \\ z_{24} &= Q_2 F_f^T, z_{25} = Q_2 \Phi^T, z_{28} = Q_2, \\ z_{33} &= Q_1 - (b_1 + \alpha_1 \sigma_A^2 + \alpha_2 \sigma_B^2 + \alpha_3 \sigma_F^2) I - b_2 B B^T, \\ z_{34} &= -b_2 B D_f^T, z_{37} = -b_2 B, z_{44} = b_1 I - b_2 D_f D_f^T, \\ z_{47} &= -b_2 D_f, z_{55} = Q_2, z_{66} = \alpha_1 I, z_{77} = (\alpha_2 - b_2) I, \\ z_{88} &= \alpha_3 I, z_{99} = b_3 I \end{aligned}$$

and the unspecified submatrices are equal to zero matrices of appropriate dimensions. The controller gain is given by:

$$(28) K_r = Y_r Q_1^{-1}.$$

The bound on the maximum allowable controller gain perturbation is given by:

$$(29) \sigma_K = \sqrt{b_2 b_3^{-1}}.$$

Consider Theorem 1 and replace  $Y$  by  $\tilde{Y}$  where  $\tilde{Y} = \tilde{K} Q_1 = Y_r + Y_\Delta$ ,  $Y_r = K_r Q_1$ , and  $Y_\Delta = K_\Delta Q_1$ . Then, (7) can be rewritten as the equivalent condition

$$(30) L > \begin{bmatrix} 0 & 0 & -Y_{\Delta}^T B^T & -Y_{\Delta}^T D_f^T & 0 & 0 & -Y_{\Delta}^T & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & 0 \end{bmatrix}$$

where  $L$  is the same matrix as  $M$  with  $Y$  evaluated at  $Y_r$ ,  $L = M|_{Y=Y_r}$ .

For an arbitrary  $b_2 > 0$ , it is true that

$$(31) \begin{bmatrix} b_2^{-1/2} Y_{\Delta}^T \\ 0 \\ b_2^{1/2} B \\ b_2^{1/2} D_f \\ 0 \\ 0 \\ b_2^{1/2} I \\ 0 \end{bmatrix} \begin{bmatrix} b_2^{-1/2} Y_{\Delta}^T \\ 0 \\ b_2^{1/2} B \\ b_2^{1/2} D_f \\ 0 \\ 0 \\ b_2^{1/2} I \\ 0 \end{bmatrix}^T \geq 0.$$

Using (31), the following condition is sufficient for (30) to hold.

$$(32) L > \begin{bmatrix} b_2^{-1} Y_{\Delta}^T Y_{\Delta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & b_2 B B^T & b_2 B D_f^T & 0 & 0 & b_2 B & 0 \\ * & * & * & b_2 D_f D_f^T & 0 & 0 & b_2 D_f & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & b_2 I & 0 \\ * & * & * & * & * & * & * & 0 \end{bmatrix}.$$

Now, substituting the expression of  $Y_{\Delta}$  in (32), using (26), applying some algebraic manipulations, and letting  $b_3 = b_2 \sigma_K^{-2}$ , it can be shown that if (27) holds, (32) still holds. The derivation of conditions (8), (9) is the same as that shown in the proof of [Theorem 1](#). This concludes the proof of [Theorem 2](#).

Given  $(\alpha_x, \alpha_w, \beta, R, N)$ , system (24), matrices  $A, B, F, C_f, D_f$ , and  $F_f$ , the upper bounds  $\sigma_A, \sigma_B$ , and  $\sigma_F$ , and for a fixed value of  $\gamma$ , conditions (27), (8), (9) constitute a set of LMIs with unknown variables  $Q_1, Q_2, b_1, b_2, b_3, \alpha_1, \alpha_2, \alpha_3$ , and  $Y_r$ . Thus, a controller gain and a bound on the gain perturbation for which the LMIs are feasible can be solved for and obtained from (28), (29) respectively. In the next section, a numerical example is provided to illustrate the applicability of the developed controller design.

## 5. Simulation studies

### 5.1. Application to Chua's circuit

Consider the following closed-loop system based on the discretized state-space model corresponding to Chua's circuit ([Chua, Wu, Hung, & Zhong, 1993](#)).

$$(33) \begin{cases} \tilde{x}_{k+1} = \tilde{A}x_k + \tilde{B}u_k + \tilde{F}w_k + \tilde{\mathfrak{S}}_k \\ w_{k+1} = \Phi w_k \end{cases}$$

where  $\tilde{A}$ ,  $\tilde{B}$ , and  $\tilde{F}$  are given by [\(4\)](#),

$$A = \begin{bmatrix} 1 - T\alpha_c(1+b) & T\alpha_c & 0 \\ T & 1-T & T \\ 0 & -T\beta_c & 1-T\mu \end{bmatrix}, B = T[2 \quad 5 \quad 4]^T, x_k = [x_k^1 \quad x_k^2 \quad x_k^3]^T, F = T[1 \quad 1 \quad 1]^T, \Phi = 0.99,$$

and

$$\tilde{\mathfrak{S}}_k = 0.5T\alpha_c(a-b)(|x_k^1 + 1| - |x_k^1 - 1|)[1 \quad 0 \quad 0]^T$$

$x_k^i$  is the  $i$ th state variable at time instant

$k$ ,  $\alpha_c = 9.1$ ,  $\beta_c = 16.5811$ ,  $\mu = 0.138083$ ,  $a = -1.3659$ ,  $b = -0.7408$  ([Azemi & Yaz, 2001](#)), and  $T = 0.05s$  is the sampling period. Since  $|x_k^1 + 1| - |x_k^1 - 1| \leq |2x_k^1|$ , then  $\tilde{\mathfrak{S}}_k^T \tilde{\mathfrak{S}}_k \leq (T\alpha_c(a-b)x_k^1)^2$ , from which the values of  $C_f$ ,  $D_f$ , and  $F_f$  in [\(3\)](#) can be easily derived. Moreover, the upper bound values on  $A_\Delta$ ,  $B_\Delta$ , and  $F_\Delta$  are considered to be  $\sigma_A = 0.02$ ,  $\sigma_B = 0.02$ , and  $\sigma_F = 0.03$ , respectively. Therefore, consider the following:

$$A_\Delta = 0.008I, B_\Delta = [0 \quad 0.019 \quad 0]^T, F_\Delta = [0 \quad 0.029 \quad 0]^T.$$

Given  $(\alpha_x, \alpha_w, \beta, R, N)$ , the feasibility of the LMIs while varying  $\gamma^{-1}$  over the range  $(0,1]$  is examined. A solution exists as long as there exists at least one value of  $\gamma$  for which the LMIs are feasible. Otherwise, the design parameters must be modified. In this work, the simulations are conducted in MATLAB® (R2010a), and the LMIs obtained are solved using the Robust Control Toolbox™ V3.4.1 in the MATLAB® version indicated. It is worth noting that as the order of the system increases, it is inherent that the dimensions of the relevant LMIs will increase. However, with the ongoing advances in computational capabilities and speed of computers, an increase in the dimensions of the LMIs to be solved does not present any computational extensiveness of the approach.

For the system considered and the set of parameters  $\alpha_x = 1.5$ ,  $\alpha_w = 1$ ,  $\beta = 11$ ,  $R = I$ ,  $N = 20$ , a solution for the controller gain is found for  $\gamma = 1.002$  where

$$K_r = [-3.3782 \quad -3.4785 \quad -0.1466] \text{ and } \sigma_K = 0.021.$$

System [\(33\)](#) is simulated with the perturbed controller applied during the first  $N = 20$  time steps then removed. [Fig. 1](#) shows the norm of the state of the system with respect to time in both the closed-loop and open-loop cases. The norm of the state remains within the prescribed bound  $\beta = 11$  for every time step over the interval during which the controller is applied, despite the perturbations in its gain. Thus, the controller

developed is performing as expected during the finite-time interval, and the system returns to its controller-free dynamics afterwards.

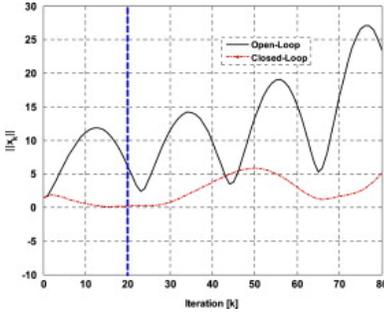


Fig. 1. Evolution of  $\|x_k\|$  over time for the open-loop and closed-loop cases. The vertical dashed line indicates where the controller is removed.

## 5.2. Controller gain perturbation magnitude analysis

In this section, the maximum allowable perturbation in the gain of the controller developed as a function of the position of the perturbation vector in a 3D space is examined. Condition (26) on the controller gain perturbation vector implies that  $\|K_\Delta\|$  must be less than or equal to  $\sigma_K$  in the case of a 3D system with a single input.

Therefore, in a 3D space, the solution for  $\sigma_K$  obtained earlier would be the minimum norm of the perturbation vector for every direction of  $K_\Delta$ . However, this also means that, in certain directions,  $\|K_\Delta\|$  may have a maximum value, which implies a possibility of a higher upper bound on the allowable perturbations in the controller gain.

In order to determine  $\|K_\Delta\|$  as a function of its direction,  $K_\Delta$  is expressed in spherical coordinates as shown below.

$$(34) K_\Delta = \|K_\Delta\| \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix}$$

where  $K_\Delta = \|K_\Delta\|$ ,  $-90^\circ \leq \theta \leq +90^\circ$ , and  $-180^\circ \leq \phi \leq +180^\circ$ .

Moreover, conditions (7), (8), (9) are feasible for the solution of the unknown variables,  $Q_1, Q_2, Y = Y_r, b_1, \delta, \alpha_1, \alpha_2$ , and  $\alpha_3$  obtained from (27), (8), (9) with the value of  $K_r$  assigned to  $K$  and, consequently, the value of  $Y_r$  assigned to  $Y$ . With all the variables in (7) known, the controller gain is perturbed by adding  $K_\Delta$  to it.

For a fixed direction of  $K_\Delta$  (i.e. one set of values of  $\theta$  and  $\phi$ ),  $K_\Delta$  is incrementally increased until (7) is no longer feasible. Then, the values of  $\theta$  and  $\phi$  are varied, and the previous steps are repeated until the ranges of  $\theta$  and  $\phi$  are covered.

The result obtained is shown in Fig. 2. The minimum value of  $K_\Delta$  is 0.022, which corresponds to a 4.55% difference from the value obtained for  $\sigma_K$  using the inequalities developed in Theorem 2.

Furthermore, Fig. 2 shows that, for example in the direction of  $\phi = 75^\circ$  and  $\theta = 87^\circ$ , the controller gain can be perturbed up to 1.305. Even though this result may reflect conservativeness in the results given in Theorem 2, it still shows that the controller design obtained is resilient against perturbations, whose upper bound is at least given by  $\sigma_K$ .

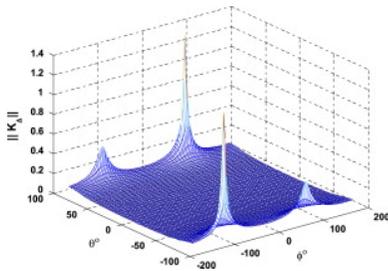


Fig. 2. Norm of the controller gain perturbation vector  $K_A$  as a function of its position in 3D space.

## 6. Conclusion

A robust and resilient FTB controller design is developed for a class of nonlinear systems with conic-type nonlinearities of uncertain center, known waveform type disturbances, and additive gain perturbations. A solution for the controller gain and the bound on the maximum allowable gain perturbation is obtained using LMI techniques. It is worth noting that the conditions arrived at in this paper reduce to the conditions existing in the literature on the finite-time bounded control of discrete-time linear systems. This fact can be shown by setting the right-hand side term of the conic sector condition and the bounds on the perturbations to zero. Thus, the class of nonlinear systems considered here and the associated results serve as a generalization of previous results. This is due to the fact that this class of systems, in addition to representing several nonlinearities that arise in the control literature, represents simple discrete-time linear systems considered by others in previous works.

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