2019

Recent Developments in Distribution Theory: A Brief Survey and Some New Generalized Classes of distributions

Zubair Ahmad

Gholamhossein G. Hamedani

Nadeem Shafique Butt

Recent Developments in Distribution Theory: A Brief Survey and Some New Generalized Classes of distributions

Zubair Ahmad
Department of Statistics, Yazd University,
Yazd, Iran
z.ferry21@gmail.com

G. G. Hamedani
Department of Mathematics, Statistics and Computer Science, Marquette University,
WI 53201-1881, Milwaukee, USA
gholamhoss.hamedani@marquette.edu

Nadeem Shafique Butt
Department of Family and Community Medicine,
King Abdulaziz University
nshafique@kau.edu.sa

Abstract

The generalization of the classical distributions is an old practice and has been considered as precious as many other practical problems in statistics. These generalizations started with the introduction of the additional location, scale or shape parameters. In the last couple of years, this branch of statistics has received a great deal of attention and quite a few new generalized classes of distributions have been introduced. We present a brief survey of this branch and introduce several new families as well.

Keyword: Generalized classes of distributions; Exponentiated family; Marshall Olkin family; Transmuted family; Kumaraswamy family; Alpha power transformation; Zubair-G family; Construction of new families.

1. Introduction

The recent development in distribution theory stresses on problem solving faced by the researchers and proposes a variety of models so that lifetime data sets can be better assessed and investigated in different applied areas. In other words, there is a need to introduce useful models for the better exploration of the real phenomenon of nature. Nowadays, the trends and practices in proposing new probability models totally differ in comparison to the models suggested before 1997. One main objective for proposing, extending or generalizing (models or their classes) is to explain how the lifetime phenomenon arises in fields like physics, computer science, insurance, public health, medical, engineering, biology, industry, communications, life-testing and many others. The well-known and fundamental distributions such as exponential, Rayleigh, Weibull and gamma are very limited in their characteristics and are unable to show wide flexibility. For example, the exponential distribution is capable of modeling with constant hazard function, whereas, the Rayleigh distribution has increasing hazard function only. However, the Weibull is much flexible and capable of modeling with increasing, decreasing or constant hazard function.

Unfortunately, the Weibull model is not capable of modeling with non-monotonic (such as unimodal, modified unimodal or bathtub shaped) failure rate function. The gamma distribution does not have a closed form of cumulative distribution function (cdf) which causes difficulties in describing its mathematical properties. For complex phenomenon in human mortality studies, reliability studies, lifetime testing, engineering modeling, electronic sciences and biological surveys, the failure rate behavior can be bathtub, upside-down bathtub and other shaped but not usually monotone increasing or decreasing. Thus, in order to cope with both monotonic and non-monotonic failure rate shapes, researchers have proposed several generalized classes of distributions which are very flexible to study needful properties of the model and its fitness. In the last two decades, several generalization approaches were adopted and practiced, which have received increased attention.

The objectives of the present study are three-fold: Firstly, we present an up-to-date account of the extended classes of distributions for the readers of modern distribution theory. Secondly, this survey will motivate the researchers to fill up the gap and to furnish their work in remaining applied areas. Thirdly, we propose some new classes of distributions which might be helpful as a tutorial to the beginners of the generalized modeling art.

The rest of the article is organized as follows. In Section 2, some extended classes of distributions are reviewed. In section 3, we present some new families. Section 4 presents certain characterizations of the distributions listed in Section 3. Finally, concluding remarks are provided in Section 5.

2. Review of the existing family of distributions

In this section, we present up-to-date review of the extended families of distributions.

2.1. The exponentiated family of distributions

Mudholkar and Srivastava (1993) proposed another method of introducing an extra parameter to a two-parameter Weibull distribution. The cumulative distribution function of the Mudholkar and Srivastava (1993)'s proposed exponentiated family has the following form

$$G(x;\alpha,\xi)=F(x;\xi)^{a}, \qquad \alpha,\xi>0,\ \ x\in\mathbb{R}, \tag{1}$$

where $\alpha>0$ is an extra shape parameter. Due to the presence of an extra shape parameter, the proposed exponentiated distributions are more flexible than the traditional models. Using (1), a number of modifications of the existing distributions have been proposed in the literature. A brief list of these modifications is presented in Table 1:

Table 1: Contributed work on exponentiated distributions

S. No.	Year	Distribution	Author(s)
1	2001	Exponentiated Exponential	Gupta and Kundu (2001)
2	2005	Exponentiated beta	Nadarajah (2005)
3	2005	Exponentiated Pareto	Nadarajah (2005)
4	2006	exponentiated lognormal	Shirke and Kakde (2006)
5	2006	exponentiated Fréchet	Nadarajah and Kotz (2006)
6	2006	eexponentiated Gumbel	Nadarajah (2006)
7	2007	exponentiated Gamma	Nadarajah and Gupta (2007)
8	2011	exponentiated generalized gamma	Cordeiro et al. (2011)
9	2013	exponentiated Lomax Poisson	Ramos et al. (2013)
10	2013	exponentiated modified Weibull extension	Sarhan and Apaloo (2013)
11	2013	exponentiated generalized class	Cordeiro et al. (2013)
12	2016	exponentiated Weibull-Pareto	Afify et al. (2016)
13	2013	exponentiated Kumaraswamy	Lemonte et al. (2013)
14	2014	Exponentiated Kumaraswamy-Dagum	Huang and Oluyede (2014)
15	2014	Exponentiated Half-Logistic family	Cordeiro et al. (2014)
16	2015	Exponentiated Power Lindley	Ashour and Eltehiwy (2015)
17	2015	Exponentiated power Lindley	Ashour and Eltehiwy (2015)
18	2015	exponentiated generalized modified Weibull	Aryal and Elbatal (2015)
19	2015	Exponentiated Burr XII Poisson	da Silva et al. (2015)
20	2015	Exponentiated Generalized Gumbel	Andrade et al. (2015)
21	2015	exponentiated transmuted generalized Rayleigh	Nofal et al. (2015)
22	2015	exponentiated flexible Weibull extension	El-Gohary et al. (2015)
23	2016	Exponentiated Gumbel Type-2	Okorie et al.(2016)
24	2016	Exponentiated Gompertz Generated Family	Cordeiro et al. (2016)
25	2017	Exponentiated Generalized Weibull Gompertz	El-Bassiouny et al. (2017)
26	2017	Exponentiated power Lindley Poisson	Pararai et al. (2017)
27	2017	Exponentiated inverse flexible Weibull extension	Morshedy and El-Bassiouny (2017)
28	2017	Exponentiated Lomax Geometric	Hassan and Abd-Allah (2017)
29	2018	Exponentiated Inverse Power Lindley	Jan et al. (2018)
30	2018	Exponentiated Weibull-Lomax	Hassan and Abd-Allah (2018)

2.2. The Marshall-Olkin family of distributions

Marshall and Olkin (1997) pioneered a simple method of adding a single parameter to a family of distributions and several authors used their method to extend well-known distributions in the last few years. If $\bar{F}(x;\xi)$ and $F(x;\xi)$ denote the survival function (sf) and cumulative distribution function of a parent distribution depending on the vector parameter ξ, then the sf of Marshall and Olkin (MO) family is defined by

$$\bar{G}(x;\sigma,\xi)=\frac{\sigma\bar{F}(x;\xi)}{1-\bar{\sigma}\bar{F}(x;\xi)}, \qquad \xi,\sigma>0,\ x\in\mathbb{R}, \tag{2}$$

where, $\bar{\sigma}=1-\sigma$. Clearly, for $\sigma=1$, we obtain the baseline distribution, i.e., $\bar{F}(x;\xi)=\bar{G}(x;\xi)$. Using (2), the extended versions of the existing distributions have been proposed. Based on the MO family, a detail review of the existing distributions is provided in Table 2:

Table 2: Contributed work on Marshall-Olkin distributions

S. No.	Year	Distribution	Author(s)
1	2003	Marshall Olkin Pareto	Alice and Jose (2003)
2	2005	Marshall Olkin extended Pareto	Ghitany (2005)
3	2005	Marshall Olkin semi Weibull	Alice and Jose (2005)
4	2005	Marshall Olkin Logistic	Alice and Jose (2005)
5	2005	Marshall Olkin extended Weibull	Ghitany et al. (2005)
6	2007	Marshall-Olkin gamma	Ristic et al. (2007)
7	2007	Marshall-Olkin extended Lomax	Ghitany (2007)
8	2009	Marshall-Olkin beta	Jose et al. (2009)
9	2011	Marshall-Olkin extended exponential	Rao et al. (2011)
10	2010	Marshall Olkin q-Weibull	Jose et al. (2010)
11	2011	Marshall-Olkin extended uniform	Jose and Krishnu (2011)
12	2013	Marshall-Olkin Extended Log-Logistic	Gui (2013)
13	2013	Marshall-Olkin Extended Zipf	Casany and Casellas (2013)
14	2013	Marshall-Olkin power log-normal	Gui (2013)
16	2013	Marshall-Olkin extended Weibull	Cordeiro and Lemonte (2013)
17	2014	Marshall-Olkin extended Weibull family	Santos-Neto et al. (2014)
18	2014	Marshall Olkin extended Burr type XII	Al-Saiari et al. (2014)
19	2014	Marshall-Olkin discrete uniform	Sandhya and Prasanth (2014)
20	2015	Marshall-Olkin generalized exponential	Ristic and Kundu (2015)
21	2015	Marshall–Olkin exponential Weibull	Pogány et al. (2015)
22	2016	Marshall-Olkin Extended Burr Type III	Kumar (2016)
23	2016	Marshall-Olkin Flexible Weibull Extension	Mustafa et al. (2016)
24	2016	Marshall–Olkin gamma–Weibull	Saboor and Pogány (2016)
25	2016	Marshall-Olkin Additive Weibull	Afify et al. (2016)
26	2017	Marshall-Olkin Extended Generalized Gompertz	Benkhelifa (2016)
27	2017	Marshall-Olkin Log-Logistic Extended Weibull	Lepetu et al. (2017)
28	2017	Marshall-Olkin generalized Erlang-truncated exponential	Okorie et al. (2017)
29	2017	Marshall-Olkin Burr X family	Jamal et al. (2017)
30	2018	Marshall-Olkin Extended Inverse Power Lindley	Hibatullah (2018)
31	2018	Marshall-Olkin Extended Inverse Weibull	Pakungwati et al. (2018)
32	2018	Marshall-Olkin Half Logistic	Yeğen and Özel (2018)
33	2018	Marshall-Olkin generalized-G family	Yousof et al. (2018)

2.3. Transmuted family of distributions

Shaw and Buckley (2009) pioneered another prominent method of adding a parameter into a family of distributions and several authors used their method to extend well-known distributions in the last couple of years. If $F(x;\xi)$ denotes the cdf of a parent distribution depending on the vector parameter ξ, then the cdf of the transmuted family is given by

$$G(x;\lambda,\xi)=(1+\lambda)F(x;\xi)-\lambda F(x;\xi)^2, \qquad \xi>0,\ |\lambda|\le 1,\ x\in\mathbb{R}. \tag{3}$$

From (3), for $\lambda=0$, we obtain the baseline distribution, i.e., $F(x;\xi)=G(x;\xi)$. Using (3), the extended versions of the existing distributions have been proposed, for detail we refer to Tahir and Cordeiro (2016).

2.4. Cubic Transmuted family of distributions

Granzotto et al. (2017) proposed a new method of generating distributions called Cubic Transmutation method. Let X_1, X_2 and X_3 be independent and identically random variables with distribution $F(x;\xi)$. Then, the ranking cubic transmutation map is given by

$$G(x;\lambda,\xi)=\lambda_1 F(x;\xi)+(\lambda_2-\lambda_1)F(x;\xi)^2+(1-\lambda_2)F(x;\xi)^3, \qquad \xi>0,\ x\in\mathbb{R}, \tag{4}$$

with $\lambda_1\in[0,1]$ and $\lambda_2\in[-1,1]$.

Recently, Aslam et al. (2018) proposed Cubic transmuted-G family by using the T-*X* idea of Alzaatreh (2013).

2.5. A General Transmuted family of distributions

Recently, Rahman et al. (2018) proposed a general transmuted family of distributions, is defined by

$$G(x;\lambda_i,\xi)=F(x;\xi)+(1-F(x;\xi))\sum_{i=1}^{k}\lambda_i F(x;\xi)^i \qquad \xi>0,\ x\in\mathbb{R}, \tag{5}$$

with $\lambda_i\in[-1,1]$ for $i = 1;2; \cdots ; k$ and $-k\le\sum_{i=1}^{k}\lambda_i\le 1.$ The general transmuted family reduces to the base distribution for $\lambda_i=0$ for $i = 1;2; \cdots ; k$.

2.6. Kumaraswamy-G family of distributions

Kumaraswamy (1980) (for short Ku) proposed a two-parameter distribution on (0,1), called Kumaraswamy distribution, is defined by

$$G(x;\alpha,\beta,\xi)=1-(1-x^{\alpha})^{\beta}, \qquad \xi>0,\ x\in(0,1), \tag{6}$$

where $\alpha>0$ and $\beta>0$ are shape parameters. The density function corresponding to (6) is

$$g(x;\alpha,\beta,\xi)=\alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}, \qquad x\in(0,1). \tag{7}$$

The Ku density has the same basic shape properties as to the beta distribution: $\alpha>1$ and $\beta>1$ (unimodal); $\alpha<1$ and $\beta<1$ (bathtub); $\alpha>1$ and $\beta\le 1$ (increasing); $\alpha\le 1$ and $\beta>1$ (decreasing) and $\alpha=\beta=1$ (constant). Using (7), for an arbitrary baseline distribution function $F(x;\xi)$, Cordeiro and Castro proposed the cdf of the Kumaraswamy-G (Ku-G) family

$$G(x;\alpha,\beta,\xi)=1-\left(1-F(x;\xi)^{\alpha}\right)^{\beta}, \qquad \alpha,\beta,\xi>0, \quad x\in\mathbb{R}, \tag{8}$$

Using (8), a number of modifications of the existing distributions have been proposed in the literature. A brief list of these modifications is presented in Table 3:

Table 3: Contributed work on Ku-G distributions.

S. No.	Year	Distribution	Author(s)
1	2010	Kumaraswamy Weibull	Cordeiro et al. (2010)
2	2011	Kumaraswamy Generalized Gamma	Pascoa et al. (2011)
3	2012	Kumaraswamy-Log-Logistic	Santana et al. (2012)
4	2012	Kumaraswamy Pareto	Pereira et al. (2012)
5	2012	Kumaraswamy Gumbel	Cordeiro et al. (2012)
6	2012	Kumaraswamy Birnbaum-Saunders	Saulo et al. (2012)
7	2013	Kumaraswamy Generalized Lomax	Shams (2013)
8	2013	Kumaraswamy-Generalized Exponentiated Pareto	Shams (2013)
9	2013	Kumaraswamy generalized linear failure rate	Elbatal (2013)
10	2013	Kumaraswamy Pareto	Elbatal (2013)
11	2013	Kumaraswamy Burr XII	Paranaba et al. (2013)
12	2013	Kumaraswamy Generalized Pareto	Nadaraja and Eljabri (2013)
13	2014	Kumaraswamy Inverse Rayleigh	Roges et al. (2014)
14	2014	Kumaraswamy-geometric distribution	Akinsete et al. (2014)
15	2014	Kumaraswamy modified Weibull	Cordeiro et al. (2014)
16	2014	Kumaraswamy Lindley	Merovci and Sharma (2014)
17	2014	Kumaraswamy Inverse Weibull	Shahbaz et al. (2014)
18	2014	Kumaraswamy generalized Rayleigh	Gomes et al. (2014)
19	2014	Kumaraswamy exponentiated Lomax	Elbatal and Kareem (2014)
20	2015	Kumaraswamy Modified Inverse Weibull	Pararai et al. (2015)
21	2015	Kumaraswamy Lindley-Poisson	Alizadeh et al. (2015)
22	2015	Kumaraswamy odd log-logistic	Alizadeh et al. (2015)
23	2015	Kumaraswamy Modified Inverse Weibull	Aryal and Elbatal (2015)
24	2016	Kumaraswamy Gompertz Makeham	Chukwu and Ogunde (2016)
25	2016	Kumaraswamy Laplace	Nassar (2016)
26	2016	Kumaraswamy Exponentiated Inverse Rayleigh	Haq (2016)
27	2016	Kumaraswamy transmuted-G	Afify et al. (2016)
28	2016	Kumaraswamy Flexible Weibull Extension	El-Damcese et al. (2016)
29	2016	Kumaraswamy exponential Weibull	Cordeiro et al. (2016)
30	2016	Kumaraswamy generalized power Weibull	Selim and Badr (2016)
31	2016	Kumaraswamy Weibull-G family	Hassan and Elgarhy (2016)
32	2016	Kumaraswamy-Burr Type III	Behairy et al. (2016)
33	2017	Kumaraswamy transmuted Pareto	Chhetri et al. (2017)
34	2017	Inverted Kumaraswamy	AL-Fattah et al. (2017)
35	2017	Kumaraswamy-Burr III	Kumar et al. (2017)
36	2017	Kumaraswamy Inverse Exponential	Oguntunde et al. (2017)

S. No.	Year	Distribution	Author(s)
37	2017	Kumaraswamy transmuted exponentiated modified Weibull	Al-Babtain et al. (2017)
38	2017	Kumaraswamy Half-Logistic	Usman et al. (2017)
39	2018	Kumaraswamy Exponentiated U-Quadratic	Muhammad et al. (2018)
40	2018	Kumaraswamy exponentiated Chen	Khan et al. (2018)
41	2018	Kumaraswamy odd Burr-G family	Nasir et al. (2018)
42	2018	Kumaraswamy Marshall-Olkin Log-Logistic	Cakmakyapan et al. (2018)

2.7. T-*X* Family approach

Eugene et al. (2002) introduced the beta generated method that uses the beta distribution with parameters a and b as the generator to develop the beta generated distributions. The distribution of a beta-generated random variable *X* is defined as

$$G(x;a,b,\xi)=\int_0^{F(x;\xi)} r(t)\,dt, \qquad a,b,\xi>0, \tag{9}$$

where *r(t)* is the pdf of a beta random variable and $F(x;\xi)$ is the cdf of any random variable *X*. Alzaatreh et al. (2013) proposed another method of generating families of continuous distributions called T-*X* family by replacing the beta pdf with a pdf, *b*(*t*), of a continuous random variable and applying a function $W\{F(x;\xi)\}$ that satisfies some certain conditions. Using the T-*X* idea, several new classes of distributions have been introduced in the literature. Table 4 provides some $W[F(x;\xi)]$ functions for some members of the T-*X* family.

Table 4. Some members of the T-*X* family

$W\{F(x;\xi)\}$	Range of *T*	Members of T-*X* family
$F(x;\xi)$	[0, 1]	Beta-G (Eugene et al., 2002), Mc-G (Alexander et al., 2012)
$-\log[F(x;\xi)]$	(0, ∞)	Gamma-G Type-2 (Risti´c and Balakrishnan, 2012)
$-\log[1-F(x;\xi)]$	(0, ∞)	Gamma-G Type-1 (Zografos and Balakrishnan, 2009)
$\frac{F(x;\xi)}{1-F(x;\xi)}$	(0, ∞)	Gamma-G Type-3 (Torabi and Montazeri, 2012)
$-\log[1-F^{\alpha}(x;\xi)]$	(0, ∞)	Exponentiated T-X (Alzaghal et al., 2013)
$\log\left\{\frac{F(x;\xi)}{1-F(x;\xi)}\right\}$	(-∞ ∞)	Logistic-G (Torabi and Montazeri, 2014)
$\log[-\log\{1-F(x;\xi)\}]$	(-∞ ∞)	The Logistic-X Family (Tahir et al., 2015)
$\frac{-\log\{1-F(x;\xi)\}}{1-F(x;\xi)}$	(0, ∞)	New Weibull-X Family (Ahmad et al., 2018)

2.8. Alpha Power Transformation

Mahdavi and Kundu (2017) proposed a new method for introducing statistical distributions via the cdf given by

$$G(x;\alpha,\xi)=\frac{\alpha^{F(x;\xi)}-1}{\alpha-1}, \qquad \alpha,\xi>0,\ \alpha\neq 1,\ x\in\mathbb{R}. \qquad (10)$$

Using (10), some new extensions of the parent distributions have been introduced. A list of distributions based on alpha power transformation is provided in Table 5.

Table 5: Contributed work on alpha power transformation.

S. No.	Year	Distribution	Author(s)
1	2017	Alpha power exponential Weibull	Rahman and El-Bassiouny (2017)
2	2018	Alpha power inverted exponential	Unal et al. (2018)
3	2018	Alpha power transformed Lindley	Dey et al. (2018)
4	2018	Alpha power inverse Weibull	Ramadan and Walaa (2018)
5	2019	Alpha power transformed inverse Lindley	Dey et al. (2019)
6	2019	Alpha power transformed Frechet	Nasiru et al. (2019)
7	2019	Alpha power transformed power Lindley	Hassan et al. (2019)

2.9. The Zubair-G family

Recently, Ahmad (2018) proposed another method for generating new distributions via the cdf given by

$$G(x;\alpha,\xi)=\frac{e^{\alpha F(x;\xi)^2}-1}{e^{\alpha}-1}, \qquad \alpha,\xi>0,\ \ x\in\mathbb{R}. \qquad (11)$$

Using (11), some new modified versions of the parent distributions have been proposed. A list of distributions based on the Zubair-G method is provided in Table 6.

Table 6: Contributed work on the Zubair-G family.

S. No.	Year	Distribution	Author(s)
1	2019	$\alpha-$Zubair-G family	Kyurkchiev et al. (2019)
2	2019	Zubair-G distribution with baseline Lomax	Pavlov et al. (2019)
3	2019	Zubair-G distribution with baseline Ghosh–Bourguignon's extended Burr XII	Rahneva et al. (2019)

3. New Proposed Families

As we discussed in Section 2, the distribution theory has received serious consideration in the literature. We carry further this branch of statistics and propose some new methods for generating new distributions. We can define a general form of cdf via the expression

$$G(x;\xi)=\frac{e^{R(x;\xi)}-1}{e-1},\qquad \xi>0,\ \ x\in\mathbb{R},\tag{12}$$

where, $R(x;\xi)$ is a baseline cdf. We can also take $R(x;\xi)$ as any function of cdf, which obey the properties of cdf, or we may combine two or more distribution functions to propose a new class of distributions. For the sake of simplicity we omit the dependency on the vector parameter and we simply write $R(x;\xi)=R(x)$, $G(x;\xi)=G(x)$ and $F(x;\xi)=F(x)$. From (12), we can also define a new function as

$$G(x)=\frac{e^{\alpha R(x)}-1}{e^{\alpha}-1},\qquad \alpha>0,\ \ x\in\mathbb{R}.\tag{13}$$

Taking $R(x)=F(x)^2$ in (13), we arrive at the Zubair-G distribution.

3.1. The extended Zubair-G family

In this sub-section, we define a new family of distributions, called the extended Zubair-G (EZ-G) family via taking $R(x)=\alpha F(x)^2+\beta F(x)$ in (12). The cdf of the EZ-G family is given by

$$G(x)=\frac{e^{\alpha F(x)^2+\beta F(x)}-1}{e^{\alpha+\beta}-1},\qquad \alpha,\beta>0,\ \ x\in\mathbb{R},\tag{14}$$

where $\alpha>0$ and $\beta>0$ are the additional parameters. The density corresponding to (14) is

$$g(x)=\frac{f(x)\left(2\alpha F(x)+\beta\right)e^{\alpha F(x)^2+\beta F(x)}}{e^{\alpha+\beta}-1},\qquad x\in\mathbb{R}.\tag{15}$$

Using (15), we can generate the extended version of the existing distributions. We discuss some special sub-models of the EZ-G class by considering $F(x;\xi)$ as the cdf of the baseline model. In Table 7, we define $R(x)$ for the sub-models of the EZ-G class of distributions.

Table 7: Special sub-models of the EZ-G family

S. No.	Baseline model	$R(x)$	Proposed model	Status
1	Weibull	$\alpha\left(1-e^{-\eta x^{\gamma}}\right)^2+\beta\left(1-e^{-\eta x^{\gamma}}\right)$	EZ-Weibull	New
2	Lomax	$\alpha\left(1-(1+bx)^{-a-1}\right)^2+\beta\left(1-(1+bx)^{-a-1}\right)$	EZ-Weibull	New
3	uniform	$\alpha\left(x/\eta\right)^2+\beta\left(x/\eta\right)$	EZ-uniform	New
4	Exponential	$\alpha\left(1-e^{-\eta x}\right)^2+\beta\left(1-e^{-\eta x}\right)$	EZ- exponential	New
5	Rayleigh	$\alpha\left(1-e^{-\eta x^2}\right)^2+\beta\left(1-e^{-\eta x^2}\right)$	EZ- Rayleigh	New

6	Linear failure rate	$\alpha\left(1-e^{-\eta x^2-\gamma x}\right)^2+\beta\left(1-e^{-\eta x^2-\gamma x}\right)$	EZ- Linear failure rate	New
7	Pareto	$\alpha\left(1-\left(\eta/x\right)^a\right)^2+\beta\left(1-\left(\eta/x\right)^a\right)$	EZ-Pareto	New
8	Burr	$\alpha\left(1-\left(1+x^b\right)^{-a}\right)^2+\beta\left(1-\left(1+x^b\right)^{-a}\right)$	EZ-Burr	New
9	Topp–Leone	$\alpha\left(x^b\left(2-x^b\right)\right)^2+\beta\left(x^b\left(2-x^b\right)\right)$	EZ-Topp-Leone	New

3.2. The Cosine-*X* family of distributions

Taking $R(x)=1-\cos\left(\frac{\pi}{2}F(x)\right)$ in (9), we define the cosine-*X* family as

$$G(x)=\frac{e^{1-\cos\left(\frac{\pi}{2}F(x)\right)}-1}{e-1},\qquad x\in\mathbb{R}, \tag{16}$$

The pdf corresponding to (16), is given by

$$g(x)=\frac{\pi}{(e-1)2}f(x)\sin\left(\frac{\pi}{2}F(x)\right)e^{1-\cos\left(\frac{\pi}{2}F(x)\right)},\qquad x\in\mathbb{R}. \tag{17}$$

3.3. The Cosine exponentiated-*X* family of distributions

A random variable *X* is said to follow the Cosine exponentiated-*X* distribution if its cdf is given by

$$G(x)=\frac{e^{1-\cos\left(\frac{\pi}{2}F(x)^a\right)}-1}{e-1},\qquad a>0,\ x\in\mathbb{R}, \tag{18}$$

with pdf

$$g(x)=\frac{a\pi}{(e-1)2}f(x)F(x)^{a-1}\sin\left(\frac{\pi}{2}F(x)^a\right)e^{1-\cos\left(\frac{\pi}{2}F(x)^a\right)},\qquad x\in\mathbb{R}. \tag{19}$$

3.4. The extended Cosine-*X* family of distributions

A random variable *X* is said to follow the extended Cosine-*X* (for short 'EC-*X*') if its cdf is given by

$$G(x)=\frac{e^{\alpha\left(1-\cos\left(\frac{\pi}{2}F(x)\right)\right)^2}-1}{e^{\alpha}-1},\qquad \alpha>0,\ x\in\mathbb{R}, \tag{20}$$

with density function

$$g(x)=\frac{\pi}{e^{\alpha}-1}f(x)\sin\left(\frac{\pi}{2}F(x)\right)\left(1-\cos\left(\frac{\pi}{2}F(x)\right)\right)e^{\alpha\left(1-\cos\left(\frac{\pi}{2}F(x)\right)\right)^2},\qquad \alpha>0,\ x\in\mathbb{R}. \tag{21}$$

3.5. The extended Cosine exponentiated-*X* family of distributions

A random variable *X* is said to follow the extended cosine exponentiated-*X* (for short 'ECE-*X*') distribution, if its cdf is given by

$$G(x)=\frac{e^{\alpha\left(1-\cos\left(\frac{\pi}{2}F(x)^a\right)\right)^2}-1}{e^{\alpha}-1},\qquad \alpha,a>0,\ x\in\mathbb{R},\tag{22}$$

with density function

$$g(x)=\frac{a\alpha\pi}{e^{\alpha}-1}f(x)F(x)^{a-1}\sin\left(\frac{\pi}{2}F(x)^a\right)\left(1-\cos\left(\frac{\pi}{2}F(x)^a\right)\right)e^{\alpha\left(1-\cos\left(\frac{\pi}{2}F(x)^a\right)\right)^2},\qquad x\in\mathbb{R}.\tag{23}$$

3.6. Another extended Cosine-*X* family of distributions

Taking $R(x)=\alpha\left(1-\cos\left(\frac{\pi}{2}F(x)\right)\right)^2+\beta\left(1-\cos\left(\frac{\pi}{2}F(x)\right)\right)$ in (14), we define the cdf of the another extended cosine-*X* (for short 'AEC-*X*') family as

$$G(x)=\frac{e^{\alpha\left(1-\cos\left(\frac{\pi}{2}F(x)\right)\right)^2+\beta\left(1-\cos\left(\frac{\pi}{2}F(x)\right)\right)}-1}{e^{\alpha+\beta}-1},\qquad \alpha,\beta>0,\ x\in\mathbb{R}.\tag{24}$$

The pdf of the AEC-*X* family can easily be obtained by simply differentiating (24).

3.7. Another extended Cosine exponentiated-*X* family

Taking $R(x)=\alpha\left(1-\cos\left(\frac{\pi}{2}F(x)^a\right)\right)^2+\beta\left(1-\cos\left(\frac{\pi}{2}F(x)^a\right)\right)$ in (14), we introduce another extended cosine exponentiated-*X* (for short 'AECE-*X*') via the cdf

$$G(x)=\frac{e^{\alpha\left(1-\cos\left(\frac{\pi}{2}F(x)^a\right)\right)^2+\beta\left(1-\cos\left(\frac{\pi}{2}F(x)^a\right)\right)}-1}{e^{\alpha+\beta}-1},\qquad \alpha,\beta,a>0,\ x\in\mathbb{R}.\tag{25}$$

By differentiating (25), we get the density function of the AECE-*X* family.

3.8. The extended transmuted-G family

Let $T(x)$ be the cdf of the transmuted distribution family. Then we define the extended transmuted-G family (for short 'ET-G') by taking $R(x)=\alpha T(x)^2+\beta T(x)$ in (14), as follows

$$G(x)=\frac{e^{\alpha T(x)^2+\beta T(x)}-1}{e^{\alpha+\beta}-1},\qquad \alpha,\beta,\xi>0,\ x\in\mathbb{R},\tag{26}$$

3.9. The extended Kumaraswamy-G family

Let $K(x)$ be the cdf of the Kumaraswamy distributions. Then, we define the extended Kumaraswamy family (for short 'EKu-G') by taking $R(x)=\alpha K(x)^2+\beta K(x)$ in (14), as follows

$$G(x)=\frac{e^{\alpha K(x)^2+\beta K(x)}-1}{e^{\alpha+\beta}-1},\qquad \alpha,\beta,\xi>0,\ x\in\mathbb{R}.\tag{27}$$

3.10. The alpha power transformed Cosine-*X* family

We define an extended form of the alpha power transformed family by

$$G(x)=\frac{\alpha^{R(x)}-1}{\alpha-1},\qquad \alpha>0,\ \alpha\neq1,\ x\in\mathbb{R},\tag{28}$$

where, $R(x)$ may be any function of cdf satisfying the conditions stated in section 2. Here, we define a new family, called the alpha power transformed cosine-*X* (for short 'APTC-*X*') family by taking $R(x)=1-\cos\left(\frac{\pi}{2}F(x)\right)$ in (28).

$$G(x)=\frac{\alpha^{1-\cos\left(\frac{\pi}{2}F(x)\right)}-1}{\alpha-1}, \qquad \alpha>0,\ \alpha\neq 1,\ x\in\mathbb{R}. \tag{29}$$

The pdf of the APTC-*X* can easily be obtained by simply differentiating (29).

3.11. The alpha power transformed Cosine exponentiated-*X* family

A random variable *X* is said to have the alpha power transformed cosine exponentiated-*X* (for short 'APTCE-*X*') family, if its cdf is given by

$$G(x)=\frac{\alpha^{1-\cos\left(\frac{\pi}{2}F(x)^a\right)}-1}{\alpha-1}, \qquad \alpha,a>0,\ \alpha\neq 1,\ x\in\mathbb{R}, \tag{30}$$

with density function

$$g(x)=\frac{a\pi}{2(\alpha-1)}f(x)F(x)^{a-1}\sin\left(\frac{\pi}{2}F(x)^a\right)\alpha^{1-\cos\left(\frac{\pi}{2}F(x)^a\right)}, \qquad x\in\mathbb{R}. \tag{31}$$

3.12. The extended alpha power transformed-*X* family

Taking $R(x)=\alpha_1F(x)^2+\beta F(x)$ in (28), we introduce the extended alpha power transformed-*X* (for short 'EAPT-*X*') via the cdf

$$G(x)=\frac{\alpha^{\alpha_1F(x)^2+\beta F(x)}-1}{\alpha-1}, \qquad \alpha,\alpha_1,\beta>0,\ \alpha\neq 1,\ x\in\mathbb{R}. \tag{33}$$

By differentiating (33), we get the density function of the EAPT-*X* family.

4. Characterization Results

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Thus, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in different directions. This section deals with various characterizations of 12 proposed distributions listed in Section 3. These characterizations are based on a simple relationship between two truncated moments. It should be mentioned that one important advantage of our characterization is that the cdf need not have a closed, and moreover, it depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation. In the subsection 4.1 we provide the characterizations of the Extended Zubair-G (EZ-G) family of distributions. Similar characterizations can be stated for the other 11 distributions.

4.1. Characterizations based on two truncated moments

This subsection deals with the characterizations of the EZ-G distribution based on the ratio of two truncated moments. Our first characterization employs a theorem of Glänzel (1987); see Theorem 1 of Appendix A.

Proposition 4.1. Let $X:\Omega\rightarrow\mathbb{R}$ be a continuous random variable and let $q_1(x)\equiv 1$ and $q_2(x)=e^{\alpha F(x)^2+\beta F(x)}$ for $x\in\mathbb{R}$. Then, the random variable X has pdf (12) if and only if the function η defined in Theorem 1 is of the form

$$\eta(x)=\frac{1}{2}\left(e^{\alpha+\beta}+e^{\alpha F(x)^2+\beta F(x)}\right), \qquad x\in\mathbb{R}.$$

Proof. Suppose the random variable X has pdf (12), then

$$(1-F(x))E(q_1(X)\mid X\geq x)=\frac{1}{e^{\alpha+\beta}-1}\left(e^{\alpha+\beta}-e^{\alpha F(x)^2+\beta F(x)}\right), \qquad x\in\mathbb{R},$$

and

$$(1-F(x))E(q_2(X)\mid X\geq x)=\frac{1}{2\left(e^{\alpha+\beta}-1\right)}\left(e^{2(\alpha+\beta)}-e^{2\left(\alpha F(x)^2+\beta F(x)\right)}\right), \qquad x\in\mathbb{R}.$$

Further,

$$\eta(x)q_1(x)-q_2(x)=\frac{1}{2}\left(e^{\alpha+\beta}-e^{\alpha F(x)^2+\beta F(x)}\right)>0, \qquad \text{for} \quad x\in\mathbb{R}.$$

Conversely, if η is of the above form, then

$$s'(x)=\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\frac{f(x)\left(2\alpha F(x)+\beta\right)e^{\alpha F(x)^2+\beta F(x)}}{e^{\alpha+\beta}-e^{\alpha F(x)^2+\beta F(x)}}, \qquad x\in\mathbb{R},$$

and hence

$$s(x)=-\log\left(e^{\alpha+\beta}-e^{\alpha F(x)^2+\beta F(x)}\right), \qquad x\in\mathbb{R}.$$

Now, in view of Theorem 1, X has density (12).

Corollary 4.1. Let $X:\Omega\rightarrow\mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 4.1. The random variable X has pdf (12) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the following differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\frac{f(x)\left(2\alpha F(x)+\beta\right)e^{\alpha F(x)^2+\beta F(x)}}{e^{\alpha+\beta}-e^{\alpha F(x)^2+\beta F(x)}}, \qquad x\in\mathbb{R}.$$

Corollary 4.2. The general solution of the differential equation in Corollary 4.1 is

$$\eta(x)=\left(e^{\alpha+\beta}-e^{\alpha F(x)^2+\beta F(x)}\right)\left[-\int\frac{f(x)\left(2\alpha F(x)+\beta\right)e^{\alpha F(x)^2+\beta F(x)}}{\left(1-F(x;\xi)\right)^{\alpha+1}}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 4.1 with D=1/2. However, it should also be noted that there are other triplets $\left(q_1(x),q_2(x),\eta(x)\right)$ satisfying the conditions of Theorem 1.

5. Concluding Remarks

The need of compounding and generalizing distributions were first felt in the financial and actuarial science and later in many other fields which researchers adopted this approach for lifetime and reliability modeling. In this way, the possible available compound and generalized G-classes are surveyed and using these basic principles nearly 12 new classes are proposed. The goal of providing a variety of new class classes is to test the flexibility of the proposed models to cope with the data available in complex situations. The parameters inducted in this way might be helpful in describing the phenomenon generated from real-lifetime data sets. We expect that these distributions will be an addition to the art of constructing useful probability models. One can imagine its motivation and usefulness in the fields which are not touched so far. Lastly, we offer more choices to the learners and practitioners of modeling to compare different models and to illustrate usefulness of old and new classes of distributions.

References

1. Abdel Rahman, N. S. and El-Bassiouny, A.H. (2017). On the alpha-power exponential Weibull distribution, The 2nd National of Mathematics and Its Applications (NCMA17), 1-18.
2. Abdul-Moniem, I. B. (2012). Recurrence relations for moments of lower generalized order statistics from exponentiated Lomax distribution and its characterization. Journal of Mathematical and Computational Science, 2(4), 999-1011.
3. Afify, A. Z., Cordeiro, G. M., Yousof, H. M., Alzaatreh, A., & Nofal, Z. M. (2016). The Kumaraswamy transmuted-G family of distributions: properties and applications. Journal of Data Science, 14(2), 245-270.
4. Afify, A. Z., Cordeiro, G. M., Yousof, H. M., Saboor, A., & Ortega, E. M. (2016). The Marshall-Olkin additive Weibull distribution with variable shapes for the hazard rate. Hacettepe Journal of Mathematics and Statistics, Forthcoming.
5. Afify, A. Z., Yousof, H. M., Hamedani, G. G., & Aryal, G. (2016). The exponentiated Weibull-Pareto distribution with application. J. Stat. Theory Appl, 15, 328-346.
6. Ahmad, Z. (2018). The Zubair-G Family of Distributions: Properties and Applications. Annals of Data Science, 1-14.
7. Ahmad, Z., Elgarhy, M., & Hamedani, G. G. (2018). A new Weibull-X family of distributions: properties, characterizations and applications. Journal of Statistical Distributions and Applications, 5(1), 5.
8. Akinsete, A., Famoye, F., & Lee, C. (2014). The Kumaraswamy-geometric distribution. Journal of statistical distributions and applications, 1(1), 17.
9. Al-Babtain, A., Fattah, A. A., Ahmed, A. H. N., & Merovci, F. (2017). The Kumaraswamy-transmuted exponentiated modified Weibull distribution. Communications in Statistics-Simulation and Computation, 46(5), 3812-3832.

10. Alexander, C., Cordeiro, G. M., Ortega, E. M., & Sarabia, J. M. (2012). Generalized beta-generated distributions. Computational Statistics & Data Analysis, 56(6), 1880-1897.
11. AL-Fattah, A. M., EL-Helbawy, A. A., & AL-Dayian, G. R. (2017). INVERTED KUMARASWAMY DISTRIBUTION: PROPERTIES AND ESTIMATION. Pakistan Journal of Statistics, 33(1).
12. Alice, T., & Jose, K. K. (2003). Marshall-Olkin Pareto processes. Far East Journal of Theoretical Statistics, 9(2), 117-132.
13. Alice, T., & Jose, K. K. (2005). Marshall-Olkin logistic processes. STARS Int J, 6, 1-11.
14. Alizadeh, M., Emadi, M., Doostparast, M., Cordeiro, G. M., Ortega, E. M., & Pescim, R. R. (2015). A new family of distributions: the Kumaraswamy odd log-logistic, properties and applications. Hacettepa Journal of Mathematics and Statistics, forthcomig.
15. Alizadeh, M., Tahir, M. H., Cordeiro, G. M., Mansoor, M., Zubair, M., & Hamedani, G. G. (2015). The Kumaraswamy Marshal-Olkin family of distributions. Journal of the Egyptian Mathematical Society, 23(3), 546-557.
16. Al-Saiari, A. Y., Baharith, L. A., & Mousa, S. A. (2014). Marshall-Olkin extended Burr type XII distribution. International Journal of Statistics and probability, 3(1), 78.
17. Alzaatreh, A., Lee, C., & Famoye, F. (2013). A new method for generating families of continuous distributions. Metron, 71(1), 63-79.
18. Alzaghal, A., Famoye, F., & Lee, C. (2013). Exponentiated T-X family of distributions with some applications. International Journal of Statistics and Probability, 2(3), 31.
19. Andrade, T., Rodrigues, H., Bourguignon, M., & Cordeiro, G. (2015). The exponentiated generalized Gumbel distribution. Revista Colombiana de Estadística, 38(1), 123-143.
20. Aryal, G., & Elbatal, I. (2015). Kumaraswamy modified inverse Weibull distribution: Theory and application. Applied Mathematics & Information Sciences, 9(2), 651.
21. Aryal, G., & Elbatal, I. (2015). On the exponentiated generalized modified Weibull distribution. Communications for Statistical Applications and Methods, 22(4), 333-348.
22. Ashour, S. K., & Eltehiwy, M. A. (2015). Exponentiated power Lindley distribution. Journal of advanced research, 6(6), 895-905.
23. Ashour, S. K., & Eltehiwy, M. A. (2015). Exponentiated power Lindley distribution. Journal of advanced research, 6(6), 895-905.
24. Aslam, M., Hussain, Z., & Asghar, Z. (2018). Cubic transmuted-g family of distributions and its properties. Stochastics and Quality Control, 33(2), 103-112.

25. Batal, I. E. L., & Kareem, A. (2014). Statistical properties of Kumaraswamy exponentiated Lomax distribution. Journal of modern Mathematics and statistics, 8(1), 1-7.
26. Behairy, S. M., AL-Dayian, G. R., & EL-Helbawy, A. A. (2016). The Kumaraswamy-Burr type III distribution: properties and estimation. British Journal of Mathematics, 14, 1-21.
27. Benkhelifa, L. (2016). The Marshall-Olkin extended generalized Gompertz distribution. arXiv preprint arXiv:1603.08242.
28. Cakmakyapan, S., Ozel, G., El Gebaly, Y. M. H., & Hamedani, G. G. (2018). The Kumaraswamy Marshall-Olkin Log-Logistic Distribution with Application. Journal of Statistical Theory and Applications, 17(1), 59-76.
29. Chhetri, S. B., Akinsete, A. A., Aryal, G., & Long, H. (2017). The Kumaraswamy transmuted Pareto distribution. Journal of Statistical Distributions and Applications, 4(1), 11.
30. Chukwu, A. U., & Ogunde, A. A. (2016). On Kumaraswamy Gompertz Makeham distribution. American Journal of Mathematics and Statistics, 6(3), 122-127.
31. Cordeiro, G. M., & Lemonte, A. J. (2013). On the Marshall–Olkin extended Weibull distribution. Statistical papers, 54(2), 333-353.
32. Cordeiro, G. M., Alizadeh, M., & Ortega, E. M. (2014). The exponentiated half-logistic family of distributions: Properties and applications. Journal of Probability and Statistics, 2014.
33. Cordeiro, G. M., Alizadeh, M., Nascimento, A. D., & Rasekhi, M. (2016). The Exponentiated Gompertz Generated Family of Distributions: Properties and Applications. Chilean Journal of Statistics (ChJS), 7(2).
34. Cordeiro, G. M., Nadarajah, S., & Ortega, E. M. (2012). The Kumaraswamy Gumbel distribution. Statistical Methods & Applications, 21(2), 139-168.
35. Cordeiro, G. M., Ortega, E. M., & da Cunha, D. C. (2013). The exponentiated generalized class of distributions. Journal of Data Science, 11(1), 1-27.
36. Cordeiro, G. M., Ortega, E. M., & Nadarajah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data. Journal of the Franklin Institute, 347(8), 1399-1429.
37. Cordeiro, G. M., Ortega, E. M., & Silva, G. O. (2011). The exponentiated generalized gamma distribution with application to lifetime data. Journal of statistical computation and simulation, 81(7), 827-842.
38. Cordeiro, G. M., Ortega, E. M., & Silva, G. O. (2014). The Kumaraswamy modified Weibull distribution: theory and applications. Journal of Statistical Computation and Simulation, 84(7), 1387-1411.
39. Cordeiro, G. M., Saboor, A., Khan, M. N., Ozel, G., & Pascoa, M. A. (2016). The Kumaraswamy Exponential–Weibull Distribution: Theory and

Applications. Hacettepe journal of mathematics and statistics, 45(4), 1203-1229.

40. da Silva, R. V., Gomes-Silva, F., Ramos, M. W. A., & Cordeiro, G. M. (2015). The exponentiated Burr XII Poisson distribution with application to lifetime data. International Journal of Statistics and Probability, 4(4), 112.
41. De Pascoa, M. A., Ortega, E. M., & Cordeiro, G. M. (2011). The Kumaraswamy generalized gamma distribution with application in survival analysis. Statistical methodology, 8(5), 411-433.
42. De Santana, T. V. F., Ortega, E. M., Cordeiro, G. M., & Silva, G. O. (2012). The Kumaraswamy-log-logistic distribution. Journal of Statistical Theory and Applications, 11(3), 265-291.
43. Dey, S., Ghosh, I., & Kumar, D. (2018). Alpha-Power Transformed Lindley Distribution: Properties and Associated Inference with Application to Earthquake Data. Annals of Data Science, 1-28.
44. Dey, S., Nassar, M., & Kumar, D. (2019). Alpha power transformed inverse Lindley distribution: A distribution with an upside-down bathtub-shaped hazard function. Journal of Computational and Applied Mathematics, 348, 130-145.
45. El-Bassiouny, A. H., El-Damcese, M., Mustafa, A., & Eliwa, M. S. (2017). Exponentiated generalized Weibull-Gompertz distribution with application in survival analysis. J. Stat. Appl. Probab, 6, 7-16.
46. Elbatal, I. (2013). Kumaraswamy generalized linear failure rate distribution. Indian J Comput Appl Math, 1(1), 61-78.
47. Elbatal, I. (2013). The Kumaraswamy exponentiated Pareto distribution. Economic Quality Control, 28(1), 1-8.
48. El-Damcese, M. A., Mustafa, A., El-Desouky, B. S., & Mustafa, M. E. (2016). The Kumaraswamy flexible Weibull extension. rn, 55, 7.
49. El-Gohary, A., El-Bassiouny, A. H., & El-Morshedy, M. (2015). Exponentiated flexible Weibull extension distribution. rn, 55, 7.
50. Eugene, N., Lee, C., & Famoye, F. (2002). Beta-normal distribution and its applications. Communications in Statistics-Theory and methods, 31(4), 497-512.
51. Ghitany, M. E. (2005). Marshall-Olkin extended Pareto distribution and its application. International Journal of Applied Mathematics, 18(1), 17.
52. Ghitany, M. E., Al-Awadhi, F. A., & Alkhalfan, L. A. (2007). Marshall–Olkin extended Lomax distribution and its application to censored data. Communications in Statistics—Theory and Methods, 36(10), 1855-1866.
53. Ghitany, M. E., Al-Hussaini, E. K., & Al-Jarallah, R. A. (2005). Marshall–Olkin extended Weibull distribution and its application to censored data. Journal of Applied Statistics, 32(10), 1025-1034.

54. Glänzel, W. (1987). A characterization theorem based on truncated moments and its application to some distribution families. In Mathematical statistics and probability theory (pp. 75-84). Springer, Dordrecht.
55. Glänzel, W., & Hamedani, G. G. (2001). Characterizations of univariate continuous distributions. Studia Scientiarum Mathematicarum Hungarica, 37(1-2), 83-118.
56. Gomes, A. E., da-Silva, C. Q., Cordeiro, G. M., & Ortega, E. M. (2014). A new lifetime model: the Kumaraswamy generalized Rayleigh distribution. Journal of statistical computation and simulation, 84(2), 290-309.
57. Granzotto, D. C. T., Louzada, F., & Balakrishnan, N. (2017). Cubic rank transmuted distributions: inferential issues and applications. Journal of Statistical Computation and Simulation, 87(14), 2760-2778.
58. Gui, W. (2013). A Marshall-Olkin power Log-Normal distribution and its applications to survival data. International Journal of Statistics and Probability, 2(1), 63.
59. Gui, W. (2013). Marshall-Olkin extended log-logistic distribution and its application in minification processes. Appl Math Sci, 7(80), 3947-3961.
60. Gupta, R. D., & Kundu, D. (2001). Exponentiated exponential family: an alternative to gamma and Weibull distributions. Biometrical Journal: Journal of Mathematical Methods in Biosciences, 43(1), 117-130.
61. Haq, M. A. (2016). Kumaraswamy Exponentiated Inverse Rayleigh Distribution. Math. Theo. Model, 6(3), 93-104.
62. Hassan, A. S., & Abd-Allah, M. (2018). Exponentiated Weibull-Lomax Distribution: Properties and Estimation. Journal of Data Science, 277, 298.
63. Hassan, A. S., & Abdelghafar, M. A. (2017). Exponentiated Lomax geometric distribution: Properties and applications. Pakistan Journal of Statistics and Operation Research, 13(3), 545-566.
64. Hassan, A. S., & Elgarhy, M. (2016). Kumaraswamy Weibull-generated family of distributions with applications. Advances and Applications in Statistics, 48(3), 205.
65. Hassan, A. S., Elgarhy, M., Mohamd, R. E., & Alrajhi, S. (2019). On the Alpha Power Transformed Power Lindley Distribution. Journal of Probability and Statistics, 2019.
66. Hibatullah, R., Widyaningsih, Y., & Abdullah, S. (2018, October). Marshall-Olkin extended inverse power Lindley distribution with applications. In AIP Conference Proceedings (Vol. 2021, No. 1, p. 060025). AIP Publishing.
67. Huang, S., & Oluyede, B. O. (2014). Exponentiated Kumaraswamy-Dagum distribution with applications to income and lifetime data. Journal of Statistical Distributions and Applications, 1(1), 8.

68. Jamal, F., Tahir, M. H., Alizadeh, M., & Nasir, M. A. (2017). On Marshall-Olkin Burr X family of distribution. Tbilisi Mathematical Journal, 10(4), 175-199.
69. Jan, R., Jan, T. R., & Ahmad, P. B. (2018). Exponentiated Inverse Power Lindley Distribution and its Applications. arXiv preprint arXiv:1808.07410.
70. Jose, K. K., & Krishna, E. (2011, October). Marshall-Olkin extended uniform distribution. In ProbStat Forum (Vol. 4, No. October, pp. 78-88).
71. Jose, K. K., Joseph, A., & Ristić, M. M. (2009). A Marshall-Olkin beta distribution and its applications. Journal of Probability and Statistical Science, 7(2), 173-186.
72. Jose, K. K., Naik, S. R., & Ristić, M. M. (2010). Marshall–Olkin q-Weibull distribution and max–min processes. Statistical papers, 51(4), 837-851.
73. Khan, M. S., King, R., & Hudson, I. L. (2018). Kumaraswamy exponentiated Chen distribution for modelling lifetime data. Appl. Math, 12(3), 617-623.
74. Kumar, D. (2016). Ratio and inverse moments of Marshall-Olkin extended Burr type XII distribution based on lower generalized order statistics. Journal of Data Science, 14(1), 53-66.
75. Kumar, D., Kumar, M., Saran, J., & Jain, N. (2017). The Kumaraswamy-Burr III distribution based on upper record values. American Journal of Mathematical and Management Sciences, 36(3), 205-228.
76. Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes. Journal of Hydrology, 46(1-2), 79-88.
77. Kyurkchiev, N., Iliev, A., & Rahnev, A. (2019). Comments on a Zubair–G family of cumulative lifetime distributions. Some extensions. Communications in applied analysis, 23(1), 1-20.
78. Lemonte, A. J., Barreto-Souza, W., & Cordeiro, G. M. (2013). The exponentiated Kumaraswamy distribution and its log-transform. Brazilian Journal of Probability and Statistics, 27(1), 31-53.
79. Lepetu, L., Oluyede, B. O., Makubate, B., Foya, S., & Mdlongwa, P. (2017). Marshall-Olkin Log-Logistic Extended Weibull Distribution: Theory, Properties and Applications. Journal of Data Science, 15(4).
80. Mahdavi, A., & Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution. Communications in Statistics-Theory and Methods, 46(13), 6543-6557.
81. Marshall, A. W., & Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Biometrika, 84(3), 641-652.
82. Merovci, F. A. T. O. N., & Sharma, V. K. (2014). The Kumaraswamy-Lindley Distribution: Properties and Applications.

83. Mudholkar, G. S., & Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. IEEE transactions on reliability, 42(2), 299-302.
84. Muhammad, M., Muhammad, I., & Yaya, A. M. (2018). The Kumaraswamy Exponentiated U-Quadratic Distribution: Properties and Application. Asian Journal of Probability and Statistics, 1(3), 1-17.
85. Mustafa, A., El-Desouky, B. S., & AL-Garash, S. (2016). The Marshall-Olkin Flexible Weibull Extension Distribution. arXiv preprint arXiv:1609.08997.
86. Nadarajah, S. (2005). Exponentiated beta distributions. Computers & Mathematics with Applications, 49(7-8), 1029-1035.
87. Nadarajah, S. (2005). Exponentiated Pareto distributions. Statistics, 39(3), 255-260.
88. Nadarajah, S. (2006). The exponentiated Gumbel distribution with climate application. Environmetrics: The official journal of the International Environmetrics Society, 17(1), 13-23.
89. Nadarajah, S., & Eljabri, S. (2013). The Kumaraswamy GP distribution. Journal of Data Science, 11(4), 739-766.
90. Nadarajah, S., & Gupta, A. K. (2007). The exponentiated gamma distribution with application to drought data. Calcutta Statistical Association Bulletin, 59(1-2), 29-54.
91. Nadarajah, S., & Kotz, S. (2006). The exponentiated type distributions. Acta Applicandae Mathematica, 92(2), 97-111.
92. Nasir, A., Bakouch, H. S., & Jamal, F. (2018). Kumaraswamy odd Burr G family of distributions with applications to reliability data. Studia Scientiarum Mathematicarum Hungarica, 55(1), 94-114.
93. Nasiru, S., Mwita, P. N. and Ngesa, O. (2019). Alpha Power Transformed Frechet Distribution. Applied Mathematics & Information Sciences, 13(1):129-141.
94. Nassar, M. M. (2016). The Kumaraswamy Laplace Distribution. Pakistan Journal of Statistics and Operation Research, 12(4), 609-624.
95. Nofal, Z. M., & Abd El Hadi, N. E. (2015). Exponentiated Transmuted Generalized Raleigh Distribution: A New Four Parameter Rayleigh Distribution. Pakistan journal of statistics and operation research, 11(1), 115-134.
96. Oguntunde, P. E., Adejumo, A., & Owoloko, E. A. (2017). Application of Kumaraswamy Inverse Exponential Distribution to Real Lifetime Data. International Journal of Applied Mathematics and Statistics.
97. Okorie, I. E., Akpanta, A. C., & Ohakwe, J. (2016). The exponentiated Gumbel type-2 distribution: properties and application. International Journal of Mathematics and Mathematical Sciences, 2016.

98. Okorie, I. E., Akpanta, A. C., & Ohakwe, J. (2017). Marshall-Olkin generalized Erlang-truncated exponential distribution: Properties and applications. Cogent Mathematics, 4(1), 1285093.
99. Pakungwati, R. M., Widyaningsih, Y., & Lestari, D. (2018, November). Marshall-Olkin Extended Inverse Weibull Distribution and Its Application. In Journal of Physics: Conference Series (Vol. 1108, No. 1, p. 012114). IOP Publishing.
100. Paranaíba, P. F., Ortega, E. M., Cordeiro, G. M., & Pascoa, M. A. D. (2013). The Kumaraswamy Burr XII distribution: theory and practice. Journal of Statistical Computation and Simulation, 83(11), 2117-2143.
101. Pararai, M., Oluyedede, B. O., & Warahena-Liyanage, G. (2015). Kumaraswamy Lindley-Poisson distribution: theory and applications. Asian Journal of Mathematics and Applications, 2015.
102. Pavlov, N., Kyurkchiev, N., Iliev, A., Rahnev, A. and Rahven, A. (2019). A Note on the Zubair-G Family with baseline Lomax Cumulative Distribution Function. Some Applications. communications in applied analysis, 120(3):471-486.
103. Pereira, M. B., Silva, R. B., Zea, L. M., & Cordeiro, G. M. (2012). The Kumaraswamy Pareto distribution. arXiv preprint arXiv:1204.1389.
104. Pérez-Casany, M., & Casellas, A. (2013). Marshall-Olkin Extended Zipf Distribution. arXiv preprint arXiv:1304.4540.
105. Pogány, T. K., Saboor, A., & Provost, S. (2015). The Marshall-Olkin exponential Weibull distribution. Hacettepe Journal of Mathematics and Statistics, 44(6), 1579.
106. Rahman, M. M., Al-Zahrani, B., & Shahbaz, M. Q. (2018). A General Transmuted Family of Distributions. Pakistan Journal of Statistics and Operation Research, 14(2).
107. Rahneva, O., Terzieva, T. and Golev, A. (2019). Investigations on the Zubair–family with baseline Ghosh–Bourguignon's extended Burr XII cumulative sigmoid. some applications. Neural, Parallel, and Scientific Computations, 27(1), 11-22.
108. Ramadan and Walaa (2018). On the Alpha-Power Inverse Weibull Distribution. International Journal of Computer Applications, 181(11),
109. Ramos, M. W. A., Marinho, P. R. D., da Silva, R. V., & Cordeiro, G. M. (2013). The exponentiated Lomax Poisson distribution with an application to lifetime data. Advances and Applications in Statistics, 34(2), 107-135.
110. Rao, G. S., Ghitany, M. E., & Kantam, R. R. L. (2011). An economic reliability test plan for Marshall-Olkin extended exponential distribution. Applied Mathematical Sciences, 5(3), 103-112.

111. Ristić, M. M., & Balakrishnan, N. (2012). The gamma-exponentiated exponential distribution. Journal of Statistical Computation and Simulation, 82(8), 1191-1206.
112. Ristić, M. M., & Kundu, D. (2015). Marshall-Olkin generalized exponential distribution. Metron, 73(3), 317-333.
113. Ristic, M. M., Jose, K. K., & Ancy, J. (2007). A Marshall–Olkin gamma distribution and minification process. Stress Anxiety Res Soc, 11, 107-117.
114. Roges, D. L., de Gusmao, F., & Diniz, C. (2014). The Kumaraswamy inverse Rayleigh distribution. Unpublished manuscript.
115. Saboor, A., & Pogány, T. K. (2016). Marshall–Olkin gamma–Weibull distribution with applications. Communications in Statistics-Theory and Methods, 45(5), 1550-1563.
116. Sandhya, E., & Prasanth, C. B. (2014). Marshall-Olkin discrete uniform distribution. Journal of probability, 2014.
117. Santos-Neto, M., Bourguignon, M., Zea, L. M., Nascimento, A. D., & Cordeiro, G. M. (2014). The Marshall-Olkin extended Weibull family of distributions. Journal of Statistical Distributions and Applications, 1(1), 9.
118. Sarhan, A. M., & Apaloo, J. (2013). Exponentiated modified Weibull extension distribution. Reliability Engineering & System Safety, 112, 137-144.
119. Saulo, H., Leão, J., & Bourguignon, M. (2012). The Kumaraswamy Birnbaum–Saunders Distribution. Journal of Statistical Theory and Practice, 6(4), 745-759.
120. Selim, M. A., & Badr, A. M. (2016). The Kumaraswamy generalized power Weibull distribution. Math. Theo. Model, 6, 110-124.
121. Shahbaz, M. Q., Shahbaz, S., & Butt, N. S. (2012). The Kumaraswamy–inverse Weibull distribution. Pakistan journal of statistics and operation research, 8(3), 479-489.
122. Shams, T. M. (2013). The Kumaraswamy-generalized exponentiated Pareto distribution. European Journal of Applied Sciences, 5(3), 92-99.
123. Shams, T. M. (2013). The Kumaraswamy-generalized Lomax distribution. Middle-East Journal of Scientific Research, 17(5), 641-646.
124. Shaw, W. T., & Buckley, I. R. (2009). The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. arXiv preprint arXiv:0901.0434.
125. Shirke, D. T., & Kakade, C. S. (2006). On exponentiated lognormal distribution. International Journal of Agricultural and Statistical Sciences, 2(2), 319-326.
126. Tahir, M. H., & Cordeiro, G. M. (2016). Compounding of distributions: a survey and new generalized classes. Journal of Statistical Distributions and Applications, 3(1), 13.

127. Tahir, M. H., Cordeiro, G. M., Alzaatreh, A., Mansoor, M., & Zubair, M. (2016). The Logistic-X family of distributions and its applications. Communications in Statistics-Theory and Methods, 45(24), 7326-7349.
128. Thomas, A., & Jose, K. K. (2005). Marshall-Olkin semi-Weibull minification processes. Recent Adv Stat Theory Appl, 1, 6-17.
129. Torabi, H., & Hedesh, N. M. (2012). The gamma-uniform distribution and its applications. Kybernetika, 48(1), 16-30.
130. Torabi, H., & Montazeri, N. H. (2014). The logistic-uniform distribution and its applications. Communications in Statistics-Simulation and Computation, 43(10), 2551-2569.
131. Unal, C., Cakmakyapan, S., & Ozel, G. (2018). Alpha Power Inverted Exponential Distribution: Properties and Application. Gazi University Journal of Science, 31(3).
132. Usman, R. M., Haq, M. A., & Talib, J. (2017). Kumaraswamy half-logistic distribution: properties and applications. J Stat Appl Probab, 6, 597-609.
133. Yeğen, D., & Özel, G. (2018). Marshall-Olkin Half Logistic Distribution with Theory and Applications. Alphanumeric Journal, 6(2), 407-416.
134. Yousof, H. M., Afify, A. Z., Nadarajah, S., Hamedani, G., & Aryal, G. R. (2018). The Marshall-Olkin generalized-G family of distributions with Applications. Statistica, 78(3), 273-295.
135. Zografos, K., & Balakrishnan, N. (2009). On families of beta-and generalized gamma-generated distributions and associated inference. Statistical Methodology, 6(4), 344-362.

Appendix A.

Theorem 1. Let $(\Omega, \mathcal{F}, P)$ be a given probability space and let $H = [d; e]$ be an interval for some $d<e$ ($d=-\infty;\ e=\infty$ might as well be allowed). Let $X: \Omega \to H$ be a continuous random variable with the distribution function F and let $q_1(x)$ and $q_2(x)$ be two real functions defined on H such that

$$E\left(q_2(X) \mid X \geq x\right) = E\left(q_1(X) \mid X \geq x\right)\eta(x), \qquad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1(H),\ \eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions $q_1,\ q_2$ and η particularly

$$F(x) = \int_a^x C\left|\frac{\eta^{/}(u)}{\eta(u)q_1(u) - q_2(u)}\right| \exp\left(-s(u)\right) du,$$

where the function $s(u)$ is a solution of the differential equation $s' = \frac{\eta' q_1}{\eta q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

Note that the result, however, holds also when the interval H is not closed, since the condition is on the interior of H.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see, Glänzel (1990)), in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions q_{1n}, q_{2n} and η_n $(n \in \mathbb{N})$ satisfy the conditions of Theorem 1 and let $q_{1n} \to q_1$, $q_{2n} \to q_2$ for some continuously differentiable real functions q_1 and q_2. Let, finally, X be a random variable with distribution $F(x)$. Under the condition that q_{1n} and q_{2n} are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if η_n converges to η, where

$$\eta(x) = \frac{E\left(q_2(X) \mid X \ge x\right)}{E\left(q_1(X) \mid X \ge x\right)}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions q_1, q_2 and η respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Levy-Smirnov distribution if $\alpha \to \infty$ as was pointed out in Glänzel and Hamedani (2001).

A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions q_1, q_2 and, specially, η should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose η as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.