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## A NEW GENERALIZED-UPPER RECORD VALUES- $G$ FAMILY OF LIFETIME DISTRIBUTIONS

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**ABSTRACT.** A new family of lifetime distributions is introduced via distribution of the upper record values, the well-known concept in survival analysis and reliability engineering. Some important properties of the proposed model including quantile function, hazard function, order statistics are obtained in a general setting. A special case of this new family is proposed by considering the exponential and Weibull distribution as the parent distributions. In addition estimating unknown parameters of specialized distribution is examined from the perspective of the traditional statistics. A simulation study is presented to investigate the bias and mean square error of the maximum likelihood estimators. Moreover, one example of real data set is studied; point and interval estimations of all parameters are obtained by maximum likelihood and bootstrap (parametric and non-parametric) procedures. Finally, the superiority of the proposed model in terms of the parent exponential distribution over other known distributions is shown via the example of real observations.

### 1. INTRODUCTION

The statistical distribution theory has been widely explored by researchers in recent years. Given the fact that the data from our surrounding environment follow various statistical models, it is necessary to extract and develop appropriate high-quality models. In addition, sometimes it is necessary to provide applications from

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existing models. For more details, see the Samuel et al. (2018) and Ababneh et al. (2018).

Recently, Alzaatreh et al. (2013) have introduced a new model of lifetime distributions, which the researchers refer to its special case as *generalized-G* distribution. It is based on the combination of one arbitrary *CDF*  $F$  of a continuous random variable  $X$  with the baseline *CDF*  $G$ . The integration form of new *CDF*  $H$  is stated as

$$H(x) = \frac{1}{F(1)} \int_{-\infty}^{G(x)} f(t) dt, \quad x \in R, \quad (1)$$

where  $f$  is the corresponding density function of  $F$  and  $F(1) = P(X \leq 1)$ . This interesting method attracted the attention of some researchers. Generating new model based on this method resulted in creating very flexible statistical modeling.

The upper and lower record values, in a sequence of independent and identically distributed (iid) random variables  $X_1, X_2, \dots$ , have applications in different areas of applied probability and reliability engineering. Let  $X_i$ 's have a common absolutely continuous distribution  $G$  with survival function  $\bar{G}$ . Define a sequence of record times  $U(n), n = 1, 2, \dots$ , as follows:

$$U(n+1) = \min \{j : j > U(n), X_j > X\}, \quad n \geq 1,$$

with  $U(1) = 1$ . Then, the sequence of upper record values  $\{R^n, n \geq 1\}$  is defined by  $R^n = X_{U(n)}, n \geq 1$ , where  $R^1 = X_1$ . The survival function of  $R^n$  is given by

$$\bar{G}_n^U(t) = \bar{G}(t) \sum_{x=0}^{n-1} \frac{[-\log \bar{G}(t)]^x}{x!}, \quad t \geq 0, \quad n = 1, 2, \dots$$

The corresponding *CDF* of the random variable  $R^n$  is

$$G_n^U(t) = 1 - \bar{G}(t) \sum_{x=0}^{n-1} \frac{[-\log \bar{G}(t)]^x}{x!}, \quad t \geq 0, \quad n = 1, 2, \dots$$

Here, we introduce a new family of lifetime distributions by compounding *CDF* of upper record values  $G_n^U(t)$  of a parent distribution  $G$  and an arbitrary *CDF*  $F$  with *PDF*  $f$ .

This new model will be denote by  $G-UR-G$  (or  $GURG$ ) distribution. One of our main motivation to introduce this new category of distributions is to provide more flexibility for fitting real datasets in comparing with other well-known classic statistical distributions.

We first derive the fundamental and statistical properties of  $GURG$  in a general setting and then we propose a special case of this model by considering Weibull distribution instead of the parent distribution  $G$  and exponential distribution instead of the parent distribution  $F$  for fixed value  $n = 2$ . It is referred to as  $GURWE$  distribution. We provide a comprehensive discussion about the statistical and reliability properties of the new  $GURWE$  model. Furthermore, we consider Maximum

likelihood and bootstrap estimation procedures to estimate the unknown parameters of the new model for complete data set. In addition, the asymptotic confidence intervals and parametric and non-parametric bootstrap confidence intervals are calculated.

2. NEW GENERAL MODEL AND ITS PROPERTIES

In this section, we provide the structure of our new model and some of its main properties in a general setting. Motivated by the idea of Alzaatreh et al. (2013), a new class of statistical distributions is proposed. The new model is constructed by implementing Alzaatreh idea to the upper record value distribution  $G_n^U(t)$ . Let the non-negative random variable  $X$  have *CDF* and *PDF*  $F$  and  $f$ , respectively. In view of (1), the *CDF* of new general class of lifetime distributions is defined as:

$$\begin{aligned} H(x, n) &= \frac{1}{F(1)} \int_0^{G_n^U(x)} f(t) dt \\ &= \frac{F(G_n^U(x))}{F(1)}, \quad x \geq 0, \quad n = 1, 2, \dots \end{aligned} \tag{2}$$

The (*PDF*) is

$$h(x, n) = \frac{g_n^U(x)}{F(1)} f(G_n^U(x)), \quad x > 0, \quad n = 1, 2, \dots \tag{3}$$

where  $g_n^U(x)$  is the *PDF* of the  $n$ - upper record value distribution and

$$g_n^U(x) = g(x) \frac{[-\log \bar{G}(x)]^{n-1}}{(n-1)!}, \quad x > 0, \quad n = 1, 2, \dots \tag{4}$$

Using (2) and (4), the survival  $\bar{H}(x, n)$  and the hazard rate  $r(x, n)$  functions for *GURG* distribution are given, respectively, by:

$$\bar{H}(x, n) = 1 - \frac{F(G_n^U(x))}{F(1)}$$

and

$$r(x, n) = \frac{g_n^U(x) f(G_n^U(x))}{F(1) - F(G_n^U(x))}, \quad x > 0, \quad n = 1, 2, \dots$$

The  $p$ th quantile  $x_p$  of the *GURG* distribution can be obtained from

$$x_p = G_n^{U^{-1}}(F^{-1}(F(1)p)),$$

where  $G_n^{U^{-1}}$  is the inverse function of *CDF*  $G_n^U$ .

3. SPECIAL CASE BASED ON THE PARENT WEIBULL AND EXPONENTIAL DISTRIBUTIONS

Let  $G(x) = 1 - e^{-\alpha x^\beta}$ ,  $F(x) = 1 - e^{-\lambda x}$  and  $n = 2$ . From (2) we have:

$$\begin{aligned} H(x) &= H(x, n = 2) \\ &= \frac{1}{F(1)} \int_{-\infty}^{G(x)+\bar{G}(x) \log \bar{G}(x)} f(t) dt \\ &= \frac{1}{1 - e^{-\lambda}} [1 - e^{-\lambda(G(x)+\bar{G}(x) \log \bar{G}(x))}] \\ &= \frac{1}{1 - e^{-\lambda}} e^{-\lambda(1 - e^{-\alpha x^\beta} (1 + \alpha x^\beta))}, \quad x \geq 0. \end{aligned} \tag{5}$$

The corresponding *PDF* is :

$$h(x) = \frac{\alpha^2 \beta \lambda}{1 - e^{-\lambda}} x^{2\beta - 1} e^{-\alpha x^\beta} e^{-\lambda(1 - e^{-\alpha x^\beta} (1 + \alpha x^\beta))},$$

where  $x > 0$ ,  $\alpha, \lambda, \beta > 0$ .

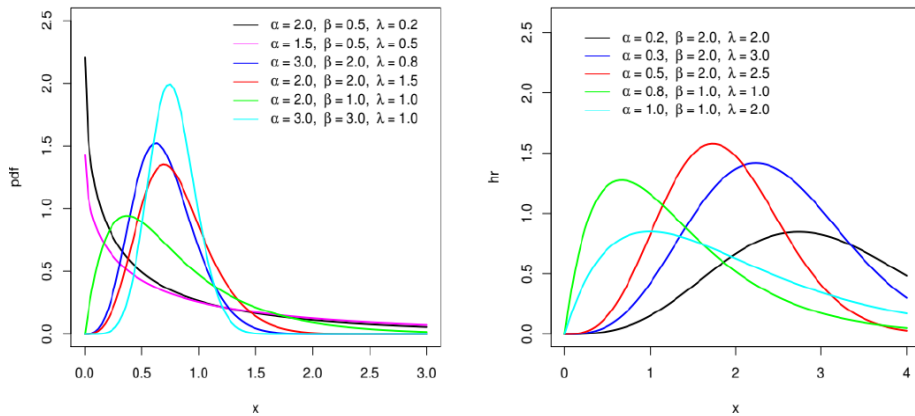


FIGURE 1. Plots of the  $GUREW(\alpha, \beta, \lambda)$  density (left) and failure rate function (right) for selected values of  $\alpha, \beta, \lambda$ .

The survival and hazard rate functions are

$$\bar{H}(x) = 1 - \frac{1}{1 - e^{-\lambda}} \left( 1 - e^{-\lambda(1 - e^{-\alpha x^\beta} (1 + \alpha x^\beta))} \right),$$

and

$$r(x) = \frac{h(x)}{\bar{H}(x)} = \frac{\frac{\alpha^2 \beta \lambda}{1 - e^{-\lambda}} x^{2\beta - 1} e^{-\alpha x^\beta} e^{-\lambda(1 - e^{-\alpha x^\beta} (1 + \alpha x^\beta))}}{1 - \frac{1}{1 - e^{-\lambda}} \left( 1 - e^{-\lambda(1 - e^{-\alpha x^\beta} (1 + \alpha x^\beta))} \right)},$$

respectively.

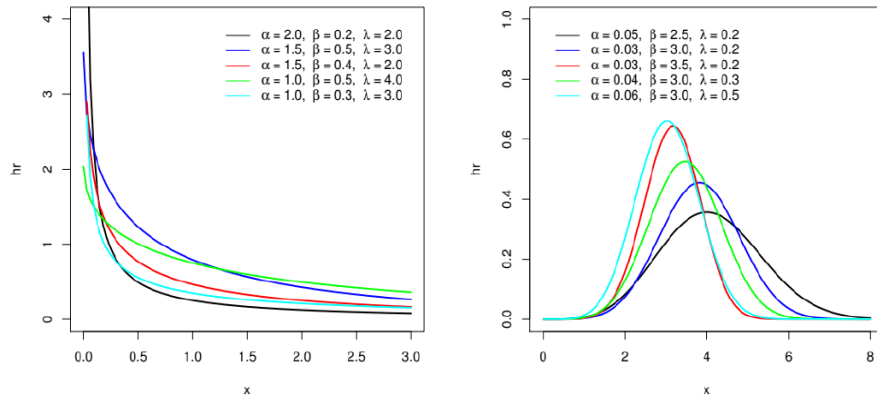


FIGURE 2. Plots of failure rate function for selected values of the parameters.

**3.1. Some properties of the GUREW distribution.** In this section, we obtain some properties of the GUREW distribution, such as quantiles, moments, moment generating function and order statistics distribution. The characterizations of GUREW distribution are presented in subsection 3.5.

**3.2. Quantiles.** For the GUREW distribution, the  $p$ th quantile  $x_p$  is the solution of  $H(x_p) = p$ , hence

$$x_p = \left( -\frac{1}{\alpha} - \frac{1}{\alpha} W_{-1} \left( -e^{-1} \left( 1 + \frac{1}{\lambda} \log(1 - (1 - e^{-\lambda})p) \right) \right) \right)^{1/\beta}, \quad 0 \leq p \leq 1,$$

which is the base of generating GUREW random variates, where  $W_{-1}$  denotes the negative branch of the Lambert function.

**3.3. Moments and Moment generating function.** In this subsection, moments and related measures including coefficients of variation, skewness and kurtosis are presented. Tables of values for the first six moments, standard deviation ( $SD$ ), coefficient of variation ( $CV$ ), coefficient of skewness ( $CS$ ) and coefficient of kurtosis ( $CK$ ) are also presented. The  $r$ th moment of the GUREW distribution, denoted by  $\mu'_r$ , is

$$\mu'_r = E(X^r) = \sum_{k=0}^{\infty} \sum_{t=0}^k \sum_{j=0}^t \frac{(-1)^k (-1)^t \binom{k}{t} \binom{t}{j} \alpha^{j+1} \lambda^{k+1}}{(1 - e^{-\lambda})(t + 1)} E_{X_W} [X^{r+(j+1)\beta}],$$

where  $X_W \sim Weibull(\alpha(t + 1), \beta)$  and  $E_{X_W}[X^{r+(j+1)\beta}] = \frac{\Gamma(1 + \frac{r+j\beta+i\beta}{\beta})}{(\alpha(t+1))^{r+(j+1)\beta}}$ .  
 The variance,  $CV$ ,  $CS$ , and  $CK$  are given by

$$\sigma^2 = \mu'^2 - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu'^2 - \mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1}, \tag{6}$$

$$CS = \frac{E[(X - \mu)^3]}{[E(X - \mu)^2]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}}, \tag{7}$$

and

$$CK = \frac{E[(X - \mu)^4]}{[E(X - \mu)^2]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2}, \tag{8}$$

respectively. Table 1 lists the first six moments of the *GUREW* distribution for selected values of the parameters, when  $\alpha = 3$ . Table 2 lists the first six moments of the *GUREW* distribution for selected values of the parameters, when  $\beta = 0.5$ . These values can be determined numerically using R.

The moment generating function of the *GUREW* distribution is given by

$$E(e^{tX}) = \sum_{k=0}^{\infty} \sum_{t=0}^k \sum_{j=0}^t \frac{(-1)^t (-1)^k \lambda^{k+1} \binom{k}{t} \binom{t}{j} \Gamma(j+2)}{k!(t+1)^j (1 - e^{-\lambda})} E_{X_G}[e^{tX^{1/\beta}}],$$

where  $X_G \sim Gamma(j + 2, \alpha(t + 1))$ .

TABLE 1. Moments of the *GUREW* distribution for selected parameter values when  $\alpha = 3$ .

$\mu'_r$	$\beta = 0.5, \lambda = 0.5$	$\beta = 0.5, \lambda = 1.5$	$\beta = 1.5, \lambda = 0.5$	$\beta = 1.5, \lambda = 1.5$
$\mu'_1$	0.5671076	0.3992706	0.6760747	0.5884041
$\mu'_2$	1.179537	0.709571	0.5688418	0.4399216
$\mu'_3$	5.38217	3.07405	0.5671076	0.3992693
$\mu'_4$	42.7557	24.04752	0.6488483	0.4256478
$\mu'_5$	521.4693	291.9992	0.832901	0.519231
$\mu'_6$	9033.24	5051.822	1.179537	0.709571
SD	0.9262429	0.7417237	0.3343124	0.3061082
CV	1.6332755	1.8576967	0.4944904	0.5202346
CS	4.706703	5.762384	0.8405167	1.051078
CK	44.17227	65.22099	3.881779	4.575555

TABLE 2. Moments of the *GUREW* distribution for selected parameter values when  $\beta = 0.5$ .

$\mu'_r$	$\alpha = 0.5, \lambda = 0.5$	$\alpha = 1, \lambda = 1$	$\alpha = 1.5, \lambda = 1.5$	$\alpha = 2, \lambda = 2$
$\mu'_1$	20.33652	4.299195	1.597077	0.7470063
$\mu'_2$	1478.852	74.72371	11.35314	2.721693
$\mu'_3$	218312	2993.463	196.7392	25.74113
$\mu'_4$	48809999	212208.3	6156.162	449.2581
$\mu'_5$	13959548045	23124985	299005.4	12245.5
$\mu'_6$	4.637651e+12	3540458803	20691289	476369.5
SD	32.638596	7.499375	2.966898	1.470943
CV	1.604925	1.744367	1.857705	1.969118
CS	4.168103	5.189165	5.762391	6.433476
CK	30.14692	53.11153	65.221	81.28218

**3.4. Order statistics.** Order statistics play an important role in probability and statistics. In this subsection, we present the distribution of the  $i$ th order statistic from the *GUREW* distribution. The *PDF* of the  $i$ th order statistic from the *GUREW PDF*,  $f_{GUREW}(x)$ , is given by

$$\begin{aligned}
 f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} f_{GUREW}(x) [F_{GUREW}(x)]^{i-1} [1 - F_{GUREW}(x)]^{n-i} \\
 &= \frac{n!}{(i-1)!(n-i)!} f_{GUREW}(x) \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m [F_{GUREW}(x)]^{m+i-1}.
 \end{aligned}$$

Using the binomial expansion

$$[1 - F_{GUREW}(x)]^{n-i} = \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m [F_{GUREW}(x)]^m,$$

we have

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m [F_{GUREW}(x)]^{m+i-1} f_{GUREW}(x)$$

**3.5. Characterization Results.** This section is devoted to the characterizations of the *GUREW* distribution in different directions: (i) based on the ratio of two truncated moments; (ii) in terms of the reverse hazard function and (iii) based on the conditional expectation of certain function of the random variable. Note that (i) can be employed also when the cdf does not have a closed form. We would also like to mention that due to the nature of *GUREW* distribution, our characterizations may be the only possible ones. We present our characterizations (i) – (iii) in three subsections.



3.5.1. *Characterizations based on two truncated moments.* This subsection deals with the characterizations of *GUREW* distribution based on the ratio of two truncated moments. Our first characterization employs a theorem due to Glänzel (1987), see Theorem 1 of Appendix A. The result, however, holds also when the interval  $H$  is not closed, since the condition of the Theorem is on the interior of  $H$ .

**Proposition 3.5.1.** Let  $X : \Omega \rightarrow (0, \infty)$  be a continuous random variable and let  $q_1(x) = x^{-\beta} e^{\lambda(1 - e^{-\alpha x^\beta}(1 + \alpha x^\beta))}$  and  $q_2(x) = q_1(x) e^{-\alpha x^\beta}$  for  $x > 0$ . The random variable  $X$  has PDF (6) if and only if the function  $\xi$  defined in Theorem 1 is of the form

$$\xi(x) = \frac{1}{2} e^{-\alpha x^\beta}, \quad x > 0.$$

Proof. Suppose the random variable  $X$  has PDF (6), then

$$(1 - F(x)) E[q_1(X) | X \geq x] = \frac{\alpha \lambda}{1 - e^{-\lambda}} e^{-\alpha x^\beta}, \quad x > 0,$$

and

$$(1 - F(x)) E[q_2(X) | X \geq x] = \frac{\alpha \lambda}{2(1 - e^{-\lambda})} e^{-2\alpha x^\beta}, \quad x > 0.$$

Further,

$$\xi(x) q_1(x) - q_2(x) = -\frac{q_1(x)}{2} e^{-\alpha x^\beta} < 0, \quad \text{for } x > 0.$$

Conversely, if  $\xi$  is of the above form, then

$$s'(x) = \frac{\xi'(x) q_1(x)}{\xi(x) q_1(x) - q_2(x)} = \alpha \beta x^{\beta-1}, \quad x > 0,$$

and consequently

$$s(x) = \alpha x^\beta, \quad x > 0.$$

Now, according to Theorem 1,  $X$  has density (6).

**Corollary 3.5.1.** Let  $X : \Omega \rightarrow (0, \infty)$  be a continuous random variable and let  $q_1(x)$  be as in Proposition A.1. The random variable  $X$  has PDF (6) if and only if there exist functions  $q_2$  and  $\xi$  defined in Theorem 1 satisfying the following differential equation

$$\frac{\xi'(x) q_1(x)}{\xi(x) q_1(x) - q_2(x)} = \alpha \beta x^{\beta-1}, \quad x > 0.$$

**Corollary 3.5.2.** The general solution of the differential equation in Corollary 3.5.1 is

$$\xi(x) = e^{\alpha x^\beta} \left[ - \int \alpha \beta x^{\beta-1} e^{-\alpha x^\beta} (q_1(x))^{-1} q_2(x) dx + D \right],$$

where  $D$  is a constant. We like to point out that one set of functions satisfying the above differential equation is given in Proposition 3.5.1 with  $D = 0$ . Clearly, there are other triplets  $(q_1, q_2, \xi)$  which satisfy conditions of Theorem 1.

3.5.2. *Characterization in terms of reverse hazard function.* The reverse hazard function,  $r_F$ , of a twice differentiable distribution function,  $F$ , is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \quad x \in \text{support of } F.$$

In this subsection we present a characterization of GUREW distribution in terms of the reverse hazard function.

**Proposition 3.5.2.** Let  $X : \Omega \rightarrow (0, \infty)$  be a continuous random variable. The random variable  $X$  has PDF (6) if and only if its reverse hazard function  $r_F(x)$  satisfies the following differential equation

$$r'_F(x) + \alpha\beta x^{\beta-1} r_F(x) = \alpha^2 \beta \lambda (2\beta - 1) x^{2(\beta-1)} e^{-\alpha x^\beta}, \quad x > 0.$$

Proof. If  $X$  has PDF (6), then clearly the above differential equation holds. Now, if this equation holds, then

$$\frac{d}{dx} \left\{ e^{\alpha x^\beta} r_F(x) \right\} = \alpha^2 \beta \lambda \frac{d}{dx} \left\{ x^{2\beta-1} \right\}, \quad x > 0.,$$

from which we obtain the reverse hazard function corresponding to the PDF (6).

3.5.3. *Characterization based on the conditional expectation of certain function of the random variable.* In this subsection we employ a single function  $\psi$  of  $X$  and characterize the distribution of  $X$  in terms of the truncated moment of  $\psi(X)$ . The following proposition has already appeared in Hamedani's previous work (2013), so we will just state it here which can be used to characterize GUREW distribution.

**Proposition 3.5.3.** Let  $X : \Omega \rightarrow (e, f)$  be a continuous random variable with cdf  $F$ . Let  $\psi(x)$  be a differentiable function on  $(e, f)$  with  $\lim_{x \rightarrow f^-} \psi(x) = 1$ . Then for  $\delta \neq 1$ ,

$$E[\psi(X) \mid X \leq x] = \delta \psi(x), \quad x \in (e, f)$$

implies that

$$\psi(x) = (F(x))^{\frac{1}{\delta}-1}, \quad x \in (e, f).$$

**Remark 3.5.1.** For  $(e, f) = (0, \infty)$ ,  $\psi(x) = \frac{1}{(1-e^{-\lambda})^{1/\lambda}} e^{-\left(1-e^{-\alpha x^\beta}\right)(1+\alpha x^\beta)}$  and  $\delta = \frac{\lambda}{\lambda+1}$ , Proposition 3.5.3 provides a characterization of GUREW.

#### 4. INFERENCE PROCEDURE

In this section, we consider estimation of the unknown parameters of the  $GUREW(\alpha, \beta, \lambda)$  distribution via maximum likelihood method and bootstrap estimation.

**4.1. Maximum likelihood estimation.** Let  $x_1, \dots, x_n$  be a random sample from the *GUREW* distribution and  $\Delta = (\alpha, \beta, \lambda)$  be the vector of parameters. The log-likelihood function is given by

$$L = L(\Delta) = 2n \log \alpha + n \log \beta + n \log \frac{\lambda}{1 - e^{-\lambda}} + (2\beta - 1) \sum_{i=1}^n \log x_i - \alpha \sum_{i=1}^n x_i^\beta - \lambda \sum_{i=1}^n \left(1 - e^{-\alpha x_i^\beta (1 + \alpha x_i^\beta)}\right). \quad (9)$$

The elements of the score vector are given by

$$\frac{dL}{d\alpha} = \frac{2n}{\alpha} - \sum_{i=1}^n x_i^\beta - \lambda \alpha \sum_{i=1}^n x_i^{2\beta} e^{-\alpha x_i^\beta} = 0,$$

$$\begin{aligned} \frac{dL}{d\beta} &= \frac{n}{\beta} + 2 \sum_{i=1}^n \log x_i - \alpha \sum_{i=1}^n x_i^\beta \log x_i \\ &\quad - \alpha^2 \lambda \sum_{i=1}^n x_i^{2\beta} \log x_i e^{-\alpha x_i^\beta} = 0, \end{aligned}$$

and

$$\frac{dL}{d\lambda} = \frac{n}{\lambda} - \frac{ne^{-\lambda}}{1 - e^{-\lambda}} - \sum_{i=1}^n \left(1 - e^{-\alpha x_i^\beta (1 + \alpha x_i^\beta)}\right) = 0.$$

respectively.

The maximum likelihood estimate,  $\hat{\Delta}$  of  $\Delta = (\alpha, \beta, \lambda)$  is obtained by solving the nonlinear equations  $\frac{dL}{d\alpha} = 0$ ,  $\frac{dL}{d\beta} = 0$ ,  $\frac{dL}{d\lambda} = 0$  simultaneously. These equations do not have closed forms so, the values of the parameters  $\alpha, \lambda$  and  $\beta$  must be found using iterative methods. Therefore, the maximum likelihood estimate,  $\hat{\Delta}$  of  $\Delta = (\alpha, \beta, \lambda)$  can be determined using an iterative method such as the Newton-Raphson procedure.

**4.2. Bootstrap estimation.** The parameters of the fitted distribution can be estimated by parametric (resampling from the fitted distribution) or non-parametric (resampling with replacement from the original data set) bootstraps resampling (see Efron and Tibshirani, 1994). These two parametric and nonparametric bootstrap procedures are described as below.

**Parametric bootstrap procedure:**

- (1) Estimate  $\theta$  (vector of unknown parameters), say  $\hat{\theta}$ , by using the *MLE* procedure based on a random sample.
- (2) Generate a bootstrap sample  $\{X_1^*, \dots, X_m^*\}$  using  $\hat{\theta}$  and obtain the bootstrap estimate of  $\theta$ , say  $\hat{\theta}^*$ , from the bootstrap sample based on the *MLE* procedure.

- (3) Repeat Step 2  $NBOOT$  times.
- (4) Order  $\hat{\theta}^*_1, \dots, \hat{\theta}^*_{NBOOT}$  as  $\hat{\theta}^*_{(1)}, \dots, \hat{\theta}^*_{(NBOOT)}$ . Then obtain  $\gamma$ -quantiles and  $100(1 - \alpha)\%$  confidence intervals for the parameters.

In the case of  $GUREW$  distribution, the parametric bootstrap estimators (PBs) of  $\alpha, \beta$  and  $\lambda$ , are  $\hat{\alpha}_{PB}, \hat{\beta}_{PB}$  and  $\hat{\lambda}_{PB}$ , respectively.

**Nonparametric bootstrap procedure**

- (1) Generate a bootstrap sample  $\{X^*_1, \dots, X^*_m\}$ , with replacement from the original data set.
- (2) Obtain the bootstrap estimate of  $\theta$  with MLE procedure, say  $\hat{\theta}^*$ , by using the bootstrap sample.
- (3) Repeat Step 2  $NBOOT$  times.
- (4) Order  $\hat{\theta}^*_1, \dots, \hat{\theta}^*_{NBOOT}$  as  $\hat{\theta}^*_{(1)}, \dots, \hat{\theta}^*_{(NBOOT)}$ . Then obtain  $\gamma$ -quantiles and  $100(1 - \alpha)\%$  confidence intervals for the parameters.

In the case of  $GUREW$  distribution, the nonparametric bootstrap estimators (NPBs) of  $\alpha, \beta$  and  $\lambda$ , are  $\hat{\alpha}_{NPB}, \hat{\beta}_{NPB}$  and  $\hat{\lambda}_{NPB}$ , respectively.

5. ALGORITHM AND A SIMULATION STUDY

In this section, we give an algorithm for generating the random data  $x_1, \dots, x_n$  from the GUREW distribution and hence a simulation study is done to evaluate the performance of the MLEs.

5.1. **Algorithm.** Here, we obtain an algorithm for generating the random data  $x_1, \dots, x_n$  from the GUREW distribution as follows.

The algorithm is based on generating random data from the inverse CDF of the GUREW distribution.

- Generate  $U_i \sim Uniform(0, 1); i = 1, \dots, n$ ,
- set

$$X_i = \left( -\frac{1}{\alpha} - \frac{1}{\alpha} W_{-1} \left( -e^{-1} \left( 1 + \frac{1}{\lambda} \log(1 - (1 - e^{-\lambda})U_i) \right) \right) \right)^{1/\beta},$$

where  $W_{-1}$  denote the negative branch of the Lambert function.

5.2. **Monte Carlo simulation study.** Here, we assess the performance of the MLE's of the parameters with respect to the sample size  $n$  for the  $GUREW$  distribution. The assessment of the performance is based on a simulation study via Monte Carlo method. Let  $\hat{\alpha}, \hat{\beta}$  and  $\hat{\lambda}$  be the MLEs of the parameters  $\alpha, \beta$  and  $\lambda$ , respectively. We calculate the mean square error (MSE) and bias of the MLE's of the parameters  $\alpha, \beta$  and  $\lambda$  based on the simulation results of 2000 independent replications. Results are summarized in Table 3 for different values of  $\alpha, \beta$  and  $\lambda$ . From Table 3 the results verify that MSE of the MLE's of the parameters decrease

TABLE 3. MSEs and Average biases(values in parentheses) of the simulated estimates.

		$\alpha = 2$	$\beta = 0.5$	$\lambda = 0.5$
$n$	30	0.5782 (0.1286)	3.7220 (0.0262)	0.0082 (0.0070)
	50	0.5484 (0.1170)	3.7073 (-0.0058)	0.0057 (-0.0058)
	100	0.4796 (0.1137)	3.4564 (-0.0365)	0.0040 (-0.0117)
	200	0.3955 (0.0774)	3.1149 (0.0061)	0.0030 (-0.0134)
		$\alpha = 2$	$\beta = 1$	$\lambda = 1.5$
$n$	30	0.8754 (0.3646)	4.8570 (0.0593)	0.2866 (-0.5077)
	50	0.8185 (0.3408)	4.3000 (0.0907)	0.2953 (-0.5256)
	100	0.6055 (0.2053)	4.0039 (0.3363)	0.2927 (-0.5301)
	200	0.4345 (0.1305)	3.1229 (0.4160)	0.2856 (-0.5276)
		$\alpha = 1.5$	$\beta = 1.5$	$\lambda = 1.5$
$n$	30	0.6282 (0.3410)	4.6770 (-0.5706)	0.0650 (-0.0135)
	50	0.5480 (0.2777)	4.6123 (-0.3817)	0.0438 (-0.0450)
	100	0.4122 (0.1702)	4.1145 (-0.1196)	0.0284 (-0.0428)
	200	0.3085 (0.1111)	3.1749 (-0.0368)	0.0188 (-0.0417)
		$\alpha = 0.5$	$\beta = 2$	$\lambda = 2$
$n$	30	0.1439 (0.1702)	8.8891 (-0.6927)	0.1047 (-0.0097)
	50	0.1472 (0.1552)	5.5487 (-0.5219)	0.0773 (-0.0453)
	100	0.1072 (0.1136)	4.2505 (-0.2866)	0.0450 (-0.0544)
	200	0.0815 (0.0845)	3.2902 (-0.1761)	0.0296 (-0.0529)
		$\alpha = 1$	$\beta = 1$	$\lambda = 1$
$n$	30	0.2922 (0.1631)	4.0435 (-0.2354)	0.0297 (-0.0041)
	50	0.2635 (0.1397)	4.1884 (-0.1596)	0.0213 (-0.0189)
	100	0.2186 (0.0984)	3.6548 (0.0001)	0.0140 (-0.0253)
	200	0.1774 (0.0895)	3.1011 (-0.0176)	0.0100 (-0.0303)
		$\alpha = 0.5$	$\beta = 1.5$	$\lambda = 1.5$
$n$	30	0.1469 (0.1499)	4.5239 (-0.5508)	0.0641 (-0.0076)
	50	0.1378 (0.1378)	4.3793 (-0.3924)	0.0444 (-0.0400)
	100	0.1094 (0.0994)	3.8707 (-0.1563)	0.0289 (-0.0439)
	200	0.0675 (0.0696)	2.8043 (-0.1028)	0.0172 (-0.0396)

with respect to sample size  $n$  for all the parameters. So, the MLEs of  $\alpha$ ,  $\beta$  and  $\lambda$  are consistent estimators.

## 6. PRACTICAL DATA APPLICATION

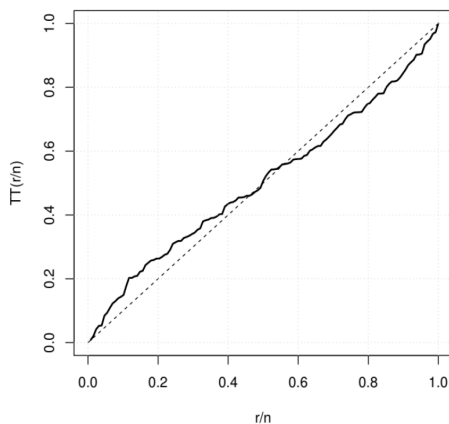
In this section, we present an application of the *GUREW* distribution to a practical data set to illustrate its flexibility among a set of competitive models. In order to achieve this goal, we consider a real data set corresponding to the remission times (in months) of a random sample of 128 bladder cancer patients. These data were previously studied by Lee and Wang (2003). This data set consists of the following observations:

0.08 0.20 0.40 0.50 0.51 0.81 0.90 1.05 1.19 1.26 1.35 1.40 1.46 1.76 2.02 2.02 2.07 2.09  
 2.23 2.26 2.46 2.54 2.62 2.64 2.69 2.69 2.75 2.83 2.87 3.02 3.25 3.31 3.36 3.36 3.48 3.52 3.57  
 3.64 3.70 3.82 3.88 4.18 4.23 4.26 4.33 4.34 4.40 4.50 4.51 4.87 4.98 5.06 5.09 5.17 5.32 5.32  
 5.34 5.41 5.41 5.49 5.62 5.71 5.85 6.25 6.54 6.76 6.93 6.94 6.97 7.09 7.26 7.28 7.32 7.39  
 7.59 7.62 7.63 7.66 7.87 7.93 8.26 8.37 8.53 8.65 8.66 9.02 9.22 9.47 9.74 10.06 10.34 10.66  
 10.75 11.25 11.64 11.79 11.98 12.02 12.03 12.07 12.63 13.11 13.29 13.80 14.24 14.76 14.77  
 14.83 15.96 16.62 17.12 17.14 17.36 18.10 19.13 20.28 21.73 22.69 23.63 25.74 25.82 26.31  
 32.15 34.26 36.66 43.01 46.12 79.05

**Graphical measure:** The total time test (*TTT*) plot due to Aarset (1987) is an important graphical approach to verify whether the data can be applied to a specific distribution or not. According to Aarset (1987), the empirical version of the *TTT* plot is given by plotting  $T(r/n) = [\sum_{i=1}^r y_{i:n} + (n-r)y_{r:n}] / \sum_{i=1}^n y_{i:n}$  against  $r/n$ , where  $r = 1, \dots, n$  and  $y_{i:n} (i = 1, \dots, n)$  are the order statistics of the sample. Aarset (1987) showed that the hazard function is constant if the *TTT* plot is graphically presented as a straight diagonal, the hazard function is increasing (or decreasing) if the *TTT* plot is concave (or convex). The hazard function is U-shaped if the *TTT* plot is convex and then concave, if not, the hazard function is unimodal. The *TTT* plots for data set is presented in Fig 3. These plots indicate that the empirical hazard rate functions of the data set is upside-down bathtub shapes. Therefore, the *GUREW* distribution is appropriate to fit this data set.

**6.1. Bootstrap inference for *GUREW* parameters.** In this section, we obtain point and 95% confidence interval (CI) estimation of the *GUREW* parameters by parametric and non-parametric bootstrap methods. We provide results of bootstrap estimation in Table 4 for the complete data set. It is interesting to observe the joint distribution of the bootstrapped values in a scatter plot in order to understand the potential structural correlation between the parameters. The corresponding plots of the bootstrap estimation are shown in Fig 4.

**6.2. MLE inference and comparison with other models.** Now, we fit the *GUREW* distribution to a data set and compare it with Lidley, Generalized Lindley (*GL*), Gamma Lindley (*GaL*), Power Lindley (*PL*), Exponential Lindley (*EL*), gamma, generalized exponential, exponential and Weibull distributions. Table 5 shows the *MLEs* of the parameters, log-likelihood, Akaike information criterion (*AIC*), Cramrvon Mises(*W\**), AndersonDarling (*A\**) and *p-value(P)* statistics

FIGURE 3. Scaled-*TTT* plot of the data set.TABLE 4. Bootstrap point and interval estimation of the parameters  $\alpha$ ,  $\beta$  and  $\lambda$ .

	parametric bootstrap		non-parametric bootstrap	
	point estimation	CI	point estimation	CI
$\alpha$	0.183	(0.055,0.336)	0.172	(0.052,0.313)
$\beta$	0.770	(0.602,0.913)	0.765	(0.578,0.906)
$\lambda$	3.703	(0.775,46.763)	3.898	(1.100,66.902)

for the data set. The *GUREW* distribution provides the best fit for the data set as it shows the lowest AIC,  $A^*$  and  $W^*$  than other considered models. The relative histograms, fitted *GUREW*, Lindley, *GL*, *GaL*, *EXP*, *PL*, *EL*, gamma, generalized exponential and Weibull *PDFs* for data are plotted in Fig 5. The plots of the empirical and fitted survival functions,  $P - P$  plots and  $Q - Q$  plots for the *GUREW* and other fitted distributions are displayed in Fig 5 and Fig 6 respectively. These plots also support the results in Table 5. We compare the *GUREW* model with a set of competitive models, namely:

(i) Lindley distribution (Lindley, 1958). The one-parameter Lindley density function is given by

$$f(x; \beta) = \frac{\beta^2}{1 + \beta} (1 + x) e^{-\beta x}; \quad x > 0,$$

where  $\beta > 0$ .

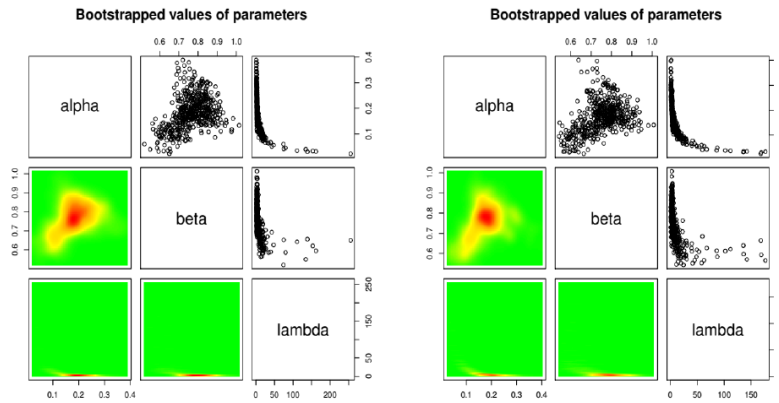


FIGURE 4. Parametric (left) and non-parametric (right) bootstrapped values of parameters of the *GUREW* distribution for the real data.

(ii) Generalized Lindley distribution (*GL*) (Zakerzadeh and Dolati, 2009) . The three-parameter *GL* density function is given by

$$f(x; \theta, \alpha, \beta) = \frac{\theta^{\alpha+1}}{(\theta + \beta)\Gamma(\alpha + 1)} x^{\alpha-1} (\alpha + \beta x) e^{-\theta x}; \quad x > 0,$$

where  $\theta > 0, \alpha > 0$  and  $\beta > 0$ .

(iii) Exponentiated Lindley distribution (*EL*) (Nadarajah et al., 2011). The two-parameter *EL* density function is given by

$$f(x; \theta, \alpha) = \frac{\alpha\theta^2}{(1 + \theta)} (1 + x)e^{-\theta x} \left[ 1 - \left( 1 + \frac{\theta x}{1 + \theta} \right) e^{-\theta x} \right]^{\alpha-1}; \quad x > 0,$$

where  $\theta > 0$  and  $\alpha > 0$ .

(iv) Power Lindley distribution (*PL*) (Ghitany et al., 2013). The two-parameter *PL* density function is given by

$$f(x; \theta, \alpha) = \frac{\alpha\theta^2}{\theta + 1} (1 + x^\alpha)x^{\alpha-1} e^{-\theta x^\alpha}; \quad x > 0$$

where  $\alpha > 0$  and  $\theta > 0$ .

(v) Gamma Lindley distribution (*GaL*) (Zeghdoudi and Nedjar, 2015). The two-parameter *GaL* density function is given by

$$f(x; \theta, \alpha) = \frac{\theta^2}{\alpha(1 + \theta)} [(\alpha + \alpha\theta - \theta)x + 1]e^{-\theta x}; \quad x > 0,$$

where  $\theta > 0$  and  $\alpha > 0$ .

(vi) The two-parameter Weibull distribution is given by

$$f(x; \alpha, \beta) = \frac{\alpha}{\beta} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha}; \quad x > 0$$



where  $\alpha > 0$  and  $\beta > 0$ .

(vii) The two-parameter Gamma distribution is given by

$$f(x; \alpha, \theta) = \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-(x/\theta)}; \quad x > 0$$

where  $\alpha > 0$  and  $\theta > 0$  and  $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ .

(viii) The one parameter Exponential distribution is given by

$$f(x; \lambda) = \lambda e^{-\lambda x}$$

where  $\lambda > 0$ .

(ix) The two-parameter generalized exponential (*GE*) distribution is given by

$$f(x; \alpha, \lambda) = \alpha \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha-1}; \quad x > 0$$

where  $\alpha > 0$  and  $\lambda > 0$ .

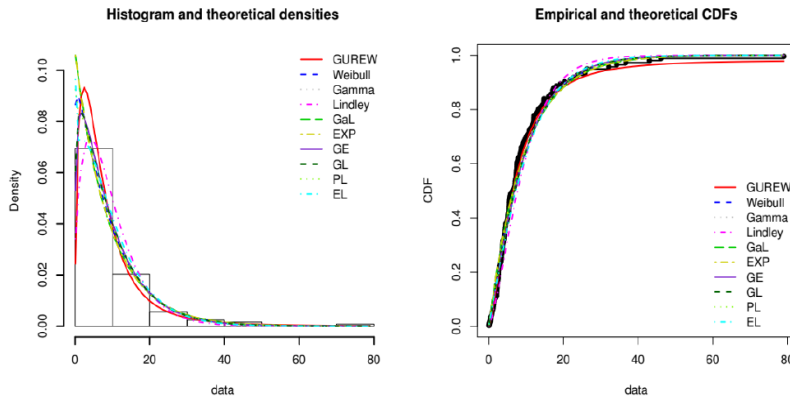


FIGURE 5. Estimated densities and Empirical and Estimated cdf for the data set.

### 7. CONCLUSION

In this article, a new model for the lifetime distributions is introduced and its main properties are discussed. A special submodel of this family is taken up by considering exponential distributions in place of the parent distribution  $F$  and Weibull distribution in place of the parent distribution  $G$ . We show that the proposed distribution has variability of hazard rate shapes such as increasing, decreasing and upside-down bathtub shapes. From a practical point of view, we show that the proposed distribution is more flexible than some commonly known statistical distributions for a given data set.

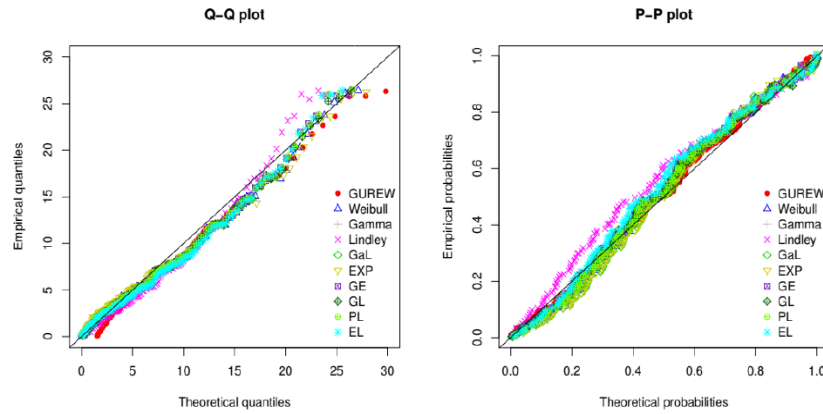


FIGURE 6. Q-Q and P-P plots for the data set.

TABLE 5. Parameter estimates (standard errors), log-likelihood values and goodness of fit measures

Model	MLEs of parameters (s.e)	Log-likelihood	AIC	BIC	A*	W*	K.S	P
<i>GUREW</i>	$\hat{\alpha} = 0.17(0.06)$ $\hat{\beta} = 0.77(0.08)$ $\hat{\lambda} = 3.94(3.02)$	-409.78	825.56	834.12	0.13	0.01	0.03	0.99
<i>Lindley</i>	$\hat{\beta} = 0.19(0.01)$	-419.52	841.05	843.91	2.78	0.51	0.11	0.06
<i>GL</i>	$\hat{\theta} = 1.25e - 01(1.72e - 02)$ $\hat{\alpha} = 1.71e - 01(1.30e - 01)$ $\hat{\beta} = 3.03e - 05(8.38e + 03)$	-413.36	832.73	841.29	0.77	0.13	0.07	0.49
<i>PL</i>	$\hat{\theta} = 0.29(0.03)$ $\hat{\alpha} = 0.83(0.04)$	-413.35	830.70	836.41	0.78	0.12	0.06	0.59
<i>EL</i>	$\hat{\theta} = 0.16(0.01)$ $\hat{\alpha} = 0.73(0.09)$	-416.28	836.57	842.27	1.32	0.24	0.09	0.21
<i>GaL</i>	$\hat{\theta} = 0.10(0.02)$ $\hat{\alpha} = 0.09(0.03)$	-414.34	832.68	838.38	1.17	0.17	0.08	0.31
<i>GE</i>	$\hat{\alpha} = 1.21(0.14)$ $\hat{\lambda} = 0.69(0.09)$	-413.07	830.15	835.85	0.71	0.12	0.07	0.51
<i>EXP</i>	$\hat{\lambda} = 0.10(0.009)$	-414.34	830.68	833.53	1.17	0.17	0.08	0.31
<i>Weibull</i>	$\hat{\alpha} = 1.04(0.06)$ $\hat{\beta} = 9.56(0.85)$	-414.08	832.17	837.87	0.95	0.15	0.06	0.55
<i>Gamma</i>	$\hat{\alpha} = 1.17(0.13)$ $\hat{\theta} = 0.12(0.01)$	-413.36	830.73	836.43	0.77	0.13	0.07	0.49

### Appendix A

**Theorem 1.** Let  $(\Omega, \mathcal{F}, \mathbf{P})$  be a given probability space and let  $H = [a, b]$  be an interval for some  $d < b$  ( $a = -\infty, b = \infty$  might as well be allowed). Let  $X : \Omega \rightarrow H$  be a continuous random variable with the distribution function  $F$  and

let  $q_1$  and  $q_2$  be two real functions defined on  $H$  such that

$$\mathbf{E}[q_2(X) \mid X \geq x] = \mathbf{E}[q_1(X) \mid X \geq x] \xi(x), \quad x \in H,$$

is defined with some real function  $\eta$ . Assume that  $q_1, q_2 \in C^1(H)$ ,  $\xi \in C^2(H)$  and  $F$  is twice continuously differentiable and strictly monotone function on the set  $H$ . Finally, assume that the equation  $\xi q_1 = q_2$  has no real solution in the interior of  $H$ . Then  $F$  is uniquely determined by the functions  $q_1, q_2$  and  $\xi$ , particularly

$$F(x) = \int_a^x C \left| \frac{\xi'(u)}{\xi(u) q_1(u) - q_2(u)} \right| \exp(-s(u)) \, du,$$

where the function  $s$  is a solution of the differential equation  $s' = \frac{\xi' q_1}{\xi q_1 - q_2}$  and  $C$  is the normalization constant, such that  $\int_H dF = 1$ .

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