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NEW MODIFIED SINGH-MADDALA DISTRIBUTION: DEVELOPMENT, PROPERTIES, CHARACTERIZATIONS AND APPLICATIONS

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ABSTRACT

In this paper, a new five-parameter extended Burr XII model called new modified Singh-Maddala (NMSM) is developed from cumulative hazard function of the modified log extended integrated beta hazard (MLEIBH) model. The NMSM density function is left-skewed, right-skewed and symmetrical. The Lambert W function is used to study descriptive measures based on quantile, moments, and moments of order statistics, incomplete moments, inequality measures and residual life function. Different reliability and uncertainty measures are also theoretically established. The NMSM distribution is characterized via different techniques and its parameters are estimated using maximum likelihood method. The simulation studies are performed on the basis of graphical results to illustrate the performance of maximum likelihood estimates (MLEs) of the parameters. The significance and flexibility of NMSM distribution is tested through different measures by application to two real data sets.

Key Words: Moments; Lambert W function; Reliability; Characterizations; Maximum Likelihood Estimation.

1. Introduction

Burr (1942) suggested 12 distributions as Burr family to fit cumulative frequency functions on frequency data. Burr distributions XII, III and X are frequently used. Burr-XII (BXII) distribution has wide applications in modeling insurance data in finance and business and modeling failure time data in reliability, survival analysis and acceptance sampling plans.

During recent decades, many continuous univariate distributions have been developed, however, various data sets from reliability, engineering, environmental, financial, biomedical sciences, among other areas, do not follow these distributions. Therefore, modified, extended and generalized distributions and their applications to problems in these areas is a clear need of day.

The modified, extended and generalized distributions are obtained by the introduction of some transformation or addition of one or more parameters to the baseline distribution. These new developed distributions provide better fit to the data than the competing models.

The probability density function (pdf) of BXII distribution has unimodal or decreasing shaped as well as monotone hazard rate function (hrf). However, these properties are inadequate, since the empirical approaches to real data are often non-monotone hrf shapes such as inverted bathtub hazard rate, bathtub, and various shaped specifically in the lifetime applications. Thus, various modified, extended and generalized forms of BXII distribution with extra shape and scale parameters are available in the literature such as multivariate BXII (Takahasi; 1965), BXII and related (Tadikamalla;1980), doubly truncated Lomax (Saran and Pushkarna;1999), doubly truncated BXII (Begum and Parvin; 2002), extended BXII (Usta; 2013), extended three-parameter BXII (Shao et al.; 2004), six-parameter generalized BXII (Olapade; 2008), beta BXII (Paranaíba et al.; 2011), extended BXII (Usta; 2013), Kumaraswamy BXII (Paranaíba et al.; 2013), generalized log-Burr family (Akhtar and Khan; 2014), BXII geometric (Korkmaz and Erişoğlu, 2014), McDonald BXII (Gomes et al.;2015), three-Parameter BXii (Okasha and Matter; 2015), BXII power series (Silva and Cordeiro; 2015), three-parameter BXII Distribution (Thupeng; 2016), BXII-Poisson (Muhammad; 2016), extensions of the BXII (Cadena; 2017), new extended BXII (Ghosh and Bourguignon; 2017), gamma BXII (Guerra et al.; 2017), BXII (Kumar; 2017), BXII modified Weibull (Mdlongwa et al.; 2017), BXII (Kayal etal.;2017), five-parameter BXII (Mead and Afify; 2017), new BXII distribution (Yari and Tondpour; 2017), four parameter BXII (Afify et al.; 2018), BXII system of densities (Cordeiro et al.;2018), Odd Lindley BXII (Abouelmagd et al., 2018 and Korkmaz et al., 2018), BXII (Gunasekera;2018), Modified log BXII (Bhatti et al.;2018), BXII (Chiang et al.;2018), BXII (Chen and Singh;2018) and BXII (Keighley et al.;2018).

The main goal of this paper is to obtain a more flexible distribution for lifetime applications called NMSM distribution. The basic motivations for the NMSM distribution are: (i) to generate distributions with left skewed, right-skewed and symmetrical shaped as well as high kurtosis; (ii) to have increasing, decreasing, bathtub and inverted bathtub hazard rate function (iii) to serve as the best alternative model to other current models to explore and model the real data in economics, life testing, reliability, survival analysis manufacturing and other areas of research and (iv) to provide better fits than other models.

Our interest is to study NMSM distribution along with its properties, applications and examine the usefulness of this distribution for modeling phenomena compared to the competing models.

This paper is divided into the following sections. In section 2, NMSM distribution is developed from the cumulative hazard rate of function of MLEIBH model. NMSM distribution is also developed via different transformations and compounding of generalized modified Weibull (GMW) and gamma distribution. In section 3, NMSM distribution is studied in terms of basic structural properties, descriptive measures based on quintiles, some plots and sub-models. In section 4, moments, moments of order statistics, incomplete moments, inequality measures, residual and reverse residual life function and some other properties are theoretically derived. In section 5, stress-strength reliability, multicomponent stress-strength reliability of model and uncertainty measures are studied. In section 6, NMSM distribution is characterized via (i) Conditional expectation; (ii) truncated moment; (iii) Hazard function and (iv) Mills ratio. In section 7, parameters of NMSM distribution are estimated using maximum likelihood method. In Section 8, the simulation studies are performed on the basis of graphical results to illustrate the performance of maximum likelihood estimates (MLEs) of the NMSM distribution. Section 9 deals with the study of goodness of fit of NMSM distribution via different methods. Conclusion and remarks are given in section 10.

2. DEVELOPMENT OF NMSM DISTRIBUTION

The cumulative hazard rate for the integrated beta hazard (IBH) model (Lai et al.; 1998 and Lai et al.; 2016) is

$$H(x) = x^{\beta}(1-ax)^{b}, \;\; \beta > 0,\, a > 0,\, b < 0,\; 0 \le x \le \frac{1}{a}. \tag{1}$$

The cumulative hazard rate for the MLEIBH model is

$$H(x) = \frac{\alpha}{\gamma}\ln\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}(1-ax)^{b}\right]. \tag{2}$$

Setting $H(x) = -\ln(1-F(x))$, we reach at

$$-\ln(1-F(x)) = \frac{\alpha}{\gamma}\ln\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}(1-ax)^{b}\right],$$

$$F(x) = 1-\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}(1-ax)^{b}\right]^{-\frac{\alpha}{\gamma}} \;\; x \ge 0.$$

For $a = \frac{1}{m},\, b = -m\,\lambda, m \to \infty,\; (1-ax)^{b} \to e^{\lambda x},\; \lim_{m\to\infty}\left(1-\frac{x}{m}\right)^{-m\,\lambda} = e^{\lambda x},$

We finally obtain the cumulative distribution function (cdf) of NMSM distribution as

$$F(x) = 1-\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{-\frac{\alpha}{\gamma}} \;\; x \ge 0, \tag{3}$$

where α, γ, β are positive shape parameters and $\theta > 0, \lambda > 0$ are scale parameters. The corresponding probability density function is

$$f(x) = \alpha\frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{-\frac{\alpha}{\gamma}-1}, x > 0. \tag{4}$$

Clearly, the ordinary Singh Maddala distribution is obtained for $\gamma = 1 \, and \, \lambda = 0$. Hence, we generalize Singh Maddala distribution with extra shape parameters γ and extra factor $e^{\lambda x}$ and called it as NMSM distribution. In addition, we have the ordinary burr XII distribution for $\gamma = 1, \theta = 1 \, and \, \lambda = 0$. Other sub-models have been marked in Section 3.

2.1 The Lambert W Function

The lambert W function was first introduced by Johann Heinrich Lambert (1758) in Lambert's Transcendental equation ($x = x^m + q$) and Euler (1783) studied special case of we^w. The Lambert W function also called product logarithm or Omega function is a set of functions written as the solution of the equation

$$W(z)\exp(W(z)) = z,$$

where z is a complex number. For $z \geq -\frac{1}{e}$, the $W(z)$ is a real function. The real branch in which $z \in (-\infty, -1]$ is called the negative branch and denoted by W_{-1}. The real branch in which $z \in [-1, \infty)$ is called the principal branch and denoted by W_0.

The Lambert W relation is useful to (i) enumerate trees (combinatorics); (ii) find maxima of Planck, Bose-Einstein and Fermi-Dirac distributions; (iii) solve delay differential equations $y'(t) = a\,y(t-1)$ and obtain solution for Michaelis-Menten kinetics (time-course kinetics analysis).

2.2 Transformations and Compounding

In this sub-section, NMSM distribution is derived through (a) certain exponential random variable; (b) ratio of exponential and gamma random variables and (c) GMW and gamma distributions.

Lemma (i) If Y has exponential random variable with scale parameter 1, then

$$X = \frac{\beta}{\lambda} W_0 \left\{ \frac{\lambda\theta}{\beta} \left\{ \gamma^{-1} \left[\exp(\gamma Y/\alpha) \right] - \gamma^{-1} \right\}^{\frac{1}{\beta}} \right\} \sim NMSM(\alpha, \beta, \gamma, \theta, \lambda).$$

(ii) If Z_1 has exponential random variable with scale parameter 1 and Z_2 has gamma random variable with shape parameter $\frac{\alpha}{\gamma}$ and scale parameter 1, then

$$X = \frac{\beta}{\lambda} W_0 \left[\frac{\lambda\theta}{\beta} \left(\frac{Z_1}{\gamma Z_2} \right)^{\frac{1}{\beta}} \right] \sim NMSM(\alpha, \beta, \gamma, \theta, \lambda).$$

(iii) Let X/τ has generalized modified Weibull random variable with $\beta, \gamma, \theta, \lambda$ parameters and τ has gamma random variable with α, γ parameters, then integrating the effect of τ with the help of $f(x; \alpha, \beta, \gamma, \theta, \lambda) = \int_0^\infty gmw(x; \alpha, \beta, \theta/\tau) g(\tau; \alpha, \gamma) d\tau$, so

$X \sim NMSM\left(\alpha,\beta,\gamma,\theta,\lambda\right).$

3. STRUCTURAL PROPERTIES OF NMSM DISTRIBUTION

The survival, hazard, cumulative hazard and reverse hazard functions of a random variable X with NMSM distribution are given, respectively, by

$$S(x)=\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{-\frac{\alpha}{\gamma}}, \tag{5}$$

$$h(x)=\alpha\frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{-1}, \tag{6}$$

$$r(x)=\frac{d}{dx}\ln\left\{1-\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{-\frac{\alpha}{\gamma}}\right\}, \tag{7}$$

and

$$H(x)=\frac{\alpha}{\gamma}\ln\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]. \tag{8}$$

The Mills ratio and elasticity $e(x)=\frac{d\ln F(x)}{d\ln x}=xr(x)$ for NMSM distribution are, respectively, given by

$$m(x)=\alpha^{-1}x(\beta+\lambda x)^{-1}\left(\frac{\theta}{x}\right)^{\beta}e^{-\lambda x}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]. \tag{9}$$

$$e(x)=\frac{d}{d\ln x}\ln\left\{1-\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{-\frac{\alpha}{\gamma}}\right\}. \tag{10}$$

The elasticity of NMSM distribution shows the behavior of the accumulation of probability in the domain of the random variable.

The quantile function of NMSM distribution is $x_q=\frac{\lambda}{\beta}W_0\left\{\frac{\lambda\theta}{\beta}\left[\gamma^{-1}\left[(1-q)^{-\frac{\gamma}{\alpha}}-1\right]\right]^{\frac{1}{\beta}}\right\}$,

where $W_0(\ldots)$ is the Lambert function (Corless et al., 1996).

The NMSM random number generator is $X=\frac{\lambda}{\beta}W_0\left\{\frac{\lambda\theta}{\beta}\left[\gamma^{-1}(1-Z)^{-\frac{\gamma}{\alpha}}-\gamma^{-1}\right]^{\frac{1}{\beta}}\right\}$,

where the random variable Z has uniform distribution on $(0,1)$.

3.1 Shapes of the NMSM Density and Hazard Rate Functions

The following graphs show that shapes of NMSM density are positively, negatively skewed and symmetrical. The NMSM distribution has increasing, decreasing and inverted bathtub hazard rate function.

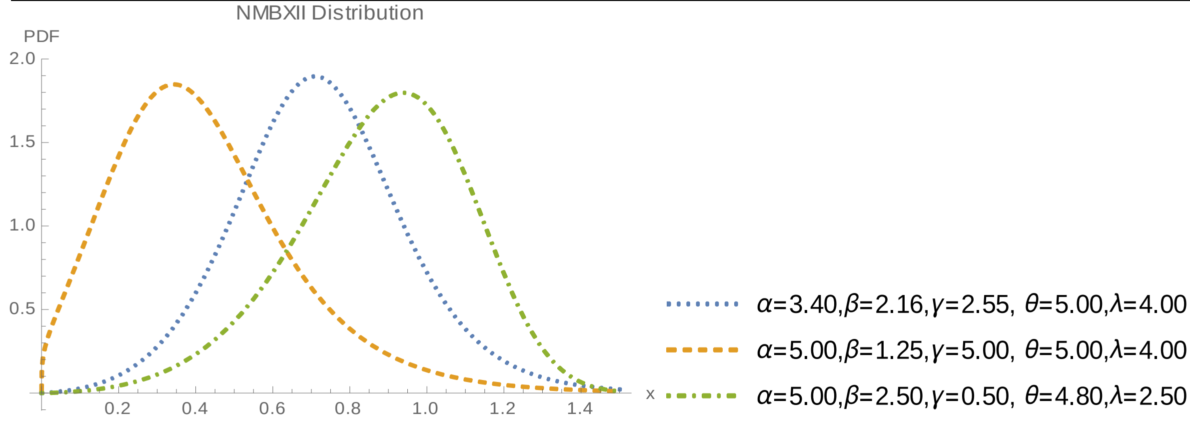

Figure 3.1: Plots of pdf of NMSM Distribution

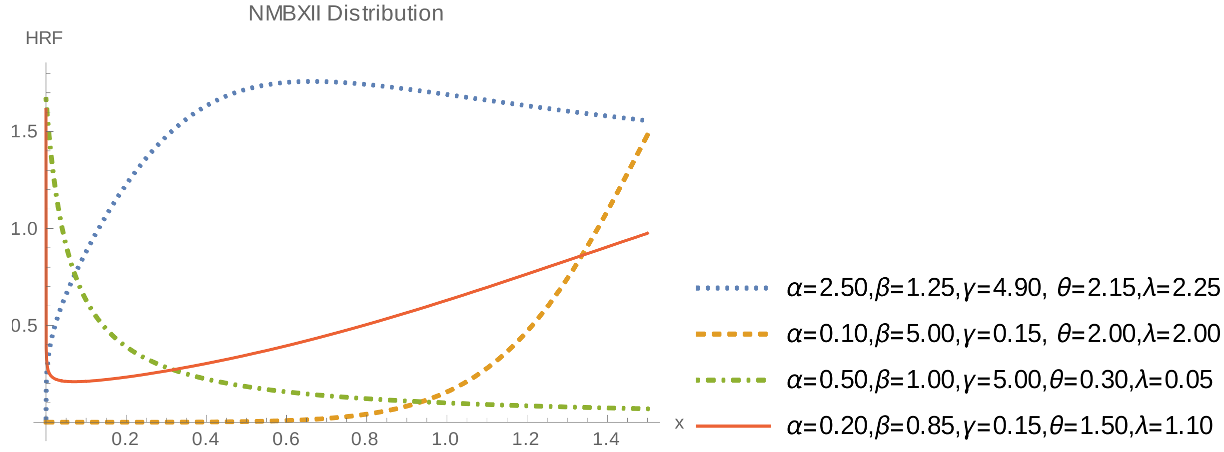

Figure 3.2: Plots of hrf of NMSM Distribution

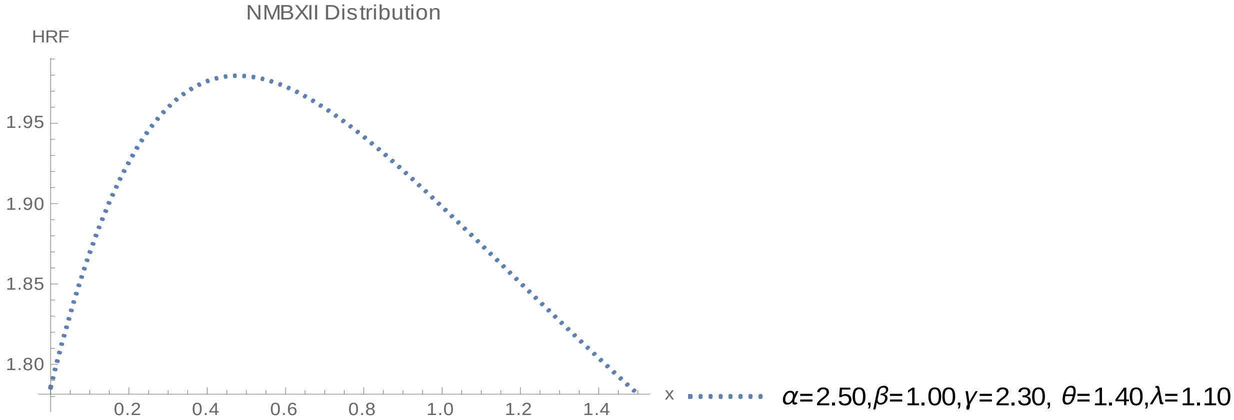

Figure 3.3: Plots of hrf of NMSM Distribution

3.2 Sub-Models

The NMSM distribution has the following sub models.

Table 1:Sub-models of the NMSM distribution

Sr.No.	X	α	β	γ	θ	λ	Name of Distribution
1	X	α	β	γ	θ	λ	New Modified Singh Maddala(NMSM)
2	X	α	β	γ	θ	0	Modified Singh Maddala(MSM)
3	X	α	β	γ	1	λ	New modified Burr XII (NMBXII)
4	X	α	β	γ	1	0	Modified Burr XII (MBXII)
5	X	α	β	1	θ	λ	New Singh Maddala(NSM)
6	X	α	β	1	θ	0	Singh Maddala(SM) (Singh and Maddala; 1976)
7	X	α	β	1	1	0	Burr XII(BXII)
7	X	1	β	1	1	0	Lomax
8	X	1	β	1	1	0	Log-logistic(LL)
9	X	α	β	$\gamma \to 0$	θ	λ	Generalized modified Weibull (GMW)
10	X	α	β	$\gamma \to 0$	θ	0	Generalized Weibull(GW)
11	X	1	β	$\gamma \to 0$	θ	λ	Modified Weibull(GW) (Lie et al.;2003)
12	X	α	β	$\gamma \to 0$	1	0	Weibull(W)
13	$\frac{1}{X}$	α	β	γ	θ	0	Modified Dagum
14	$\frac{1}{X}$	α	β	1	θ	0	Dagum
15	$\frac{1}{X}$	α	β	1	1	0	Burr III (BIII)
16	$\frac{1}{X}$	α	β	$\gamma \to 0$	θ	0	Generalized Frechet (GF)

4. MOMENTS

Moments, incomplete moments, inequality measures, residual and reverse residual life function and some other properties of NMSM distribution are theoretically derived in this section.

4.1 Moments about the Origin

The rth ordinary moment of NMSM distribution is

$$E\left(X^r\right)=\int_0^\infty x^r \alpha \frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^\beta e^{\lambda x}\left[1+\gamma\left(\frac{x}{\theta}\right)^\beta e^{\lambda x}\right]^{-\frac{\alpha}{\gamma}-1} dx.$$

Let $t=\gamma\left(\frac{x}{\theta}\right)^\beta e^{\lambda x}$, then $dt=\gamma \frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^\beta e^{\lambda x} dx$ and using the Lambert function property

$$W_0^r[x]=\sum_{n=r}^{\infty}\frac{(-r)(-n)^{n-r}}{(n-r)!}(x)^n \text{ for } x^r=\frac{\beta^r}{\lambda^r}W_0^r\left[\frac{\lambda\theta}{\beta}\left(\gamma^{-1}t\right)^{\frac{1}{\beta}}\right],$$

we arrive at $$\frac{\beta^r}{\lambda^r}W_0^r\left[\frac{\lambda\theta}{\beta}\left(\gamma^{-1}t\right)^{\frac{1}{\beta}}\right]=\frac{\beta^r}{\lambda^r}\sum_{n=r}^{\infty}\frac{(-r)(-n)^{n-r}}{(n-r)!}\left(\frac{\lambda\theta}{\beta}\left(\gamma^{-1}t\right)^{\frac{1}{\beta}}\right)^n,$$

$$E\left(X^r\right)=\frac{\alpha}{\gamma}\frac{\beta^r}{\lambda^r}\sum_{n=r}^{\infty}\frac{(-r)(-n)^{n-r}}{(n-r)!}\left(\beta^{-1}\lambda\theta\gamma^{-\frac{1}{\beta}}\right)^n\int_0^{\infty}t^{\frac{n}{\beta}+1-1}[1+t]^{-\frac{\alpha}{\gamma}-1}\,dx,$$

$$E\left(X^r\right)=\frac{\alpha}{\gamma}\frac{\beta^r}{\lambda^r}\sum_{n=r}^{\infty}\frac{(-r)(-n)^{n-r}}{(n-r)!}\left(\beta^{-1}\lambda\theta\gamma^{-\frac{1}{\beta}}\right)^n B\left(1+\frac{n}{\beta},\frac{\alpha}{\gamma}-\frac{n}{\beta}\right),\quad r=1,2,3,4,\ldots, \tag{11}$$

where $B(.,.)$ is beta function.

The factorial moments $E[X]_n=\sum_{r=1}^{n}\varphi_r E(X^r)$ of NMSM distribution are

$$E[X]_m=\frac{\alpha}{\gamma}\frac{\beta^r}{\lambda^r}\sum_{r=1}^{m}\varphi_r\frac{\beta^r}{\lambda^r}\sum_{n=r}^{\infty}\frac{(-r)(-n)^{n-r}}{(n-r)!}\left(\frac{\lambda\theta\gamma^{-\frac{1}{\beta}}}{\beta}\right)^n B\left(1+\frac{n}{\beta},\frac{\alpha}{\gamma}-\frac{n}{\beta}\right), \tag{12}$$

where $[X]_i=X(X+1)(X+2)......(X+i-1)$ and φ_r is Stirling number of the first kind.

The Mellin transform helps to determine moments for a probability distribution. The Mellin transform of X with NMSM distribution is

$$\mathrm{M}\{f(x);\mathrm{s}\}=f^*(s)=\int_0^{\infty}f(x)\,x^{s-1}dx.$$

$$\mathrm{M}\{f(x);\mathrm{s}\}=\int_0^{\infty}x^{s-1}\alpha\frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{-\frac{\alpha}{\gamma}-1}dx,$$

$$\mathrm{M}\{f(x);\mathrm{s}\}=E\left[X^{s-1}\right]=\frac{\alpha}{\gamma}\frac{\beta^{s-1}}{\lambda^{s-1}}\sum_{n=s-1}^{\infty}\frac{(1-s)(-n)^{n-s-1}}{(n-s-1)!}\left(\frac{\lambda\theta\gamma^{-\frac{1}{\beta}}}{\beta}\right)^n B\left(1+\frac{n}{\beta},\frac{\alpha}{\gamma}-\frac{n}{\beta}\right). \tag{13}$$

The qth central moments, Pearson's measure of skewness and Kurtosis and cumulants of X with NMSM distribution are determined from the relationships

$$\mu_q=\sum_{v=1}^{q}\binom{q}{v}(-1)^v\mu'_v\,\mu'_{v-q},\quad \gamma_1=\frac{\mu_3}{(\mu_2)^{\frac{3}{2}}},$$

$$\beta_2=\frac{\mu_4}{(\mu_2)^2}\text{ and } k_r=\mu'_r-\sum_{c=1}^{r-1}\binom{r-1}{c-1}k_c\,\mu'_{r-c}.$$

The numerical measure of the mean, variance, skewness and kurtosis of the NMSM distribution for selected values of the parameters to illustrate their effect on these measures.

Table 2: Mean, Variance, skewness and kurtosis of the NMSM distribution

$\alpha,\beta,\gamma,\theta,\lambda$	Mean	Variance	Skewness	Kurtosis
(1,1,1,1,1)	0.7672	0.5218	2.0776	10.2172
(1,0.5,1,0.5,1)	0.5939	0.6304	2.4263	11.6752
(1,2,3,4,5)	0.9259	0.2462	1.9130	9.4936
(5,4,3,2,1)	1.0299	0.0931	0.5588	4.2718
(0.5,0.5,0.5,0.5,0.5)	1.4660	3.0627	2.1072	9.5731
(5,5,5,5,5)	1.1564	0.0373	0.3705	4.3420
(1,5,5,0.5,0.5)	1.3460	5.1237	6.4715	67.3985
(0.5,1,2,4,8)	0.6313	0.2110	1.9665	9.3215
(5,5,0.5,0.5,5)	0.2775	0.0025	-0.3789	3.1452
(5,5,0.5,5,1)	2.1555	0.1390	-0.4957	3.3395

The graphical displays to describe the parameter α and β that controls skewness and kurtosis measures of the NMSM distribution are added.

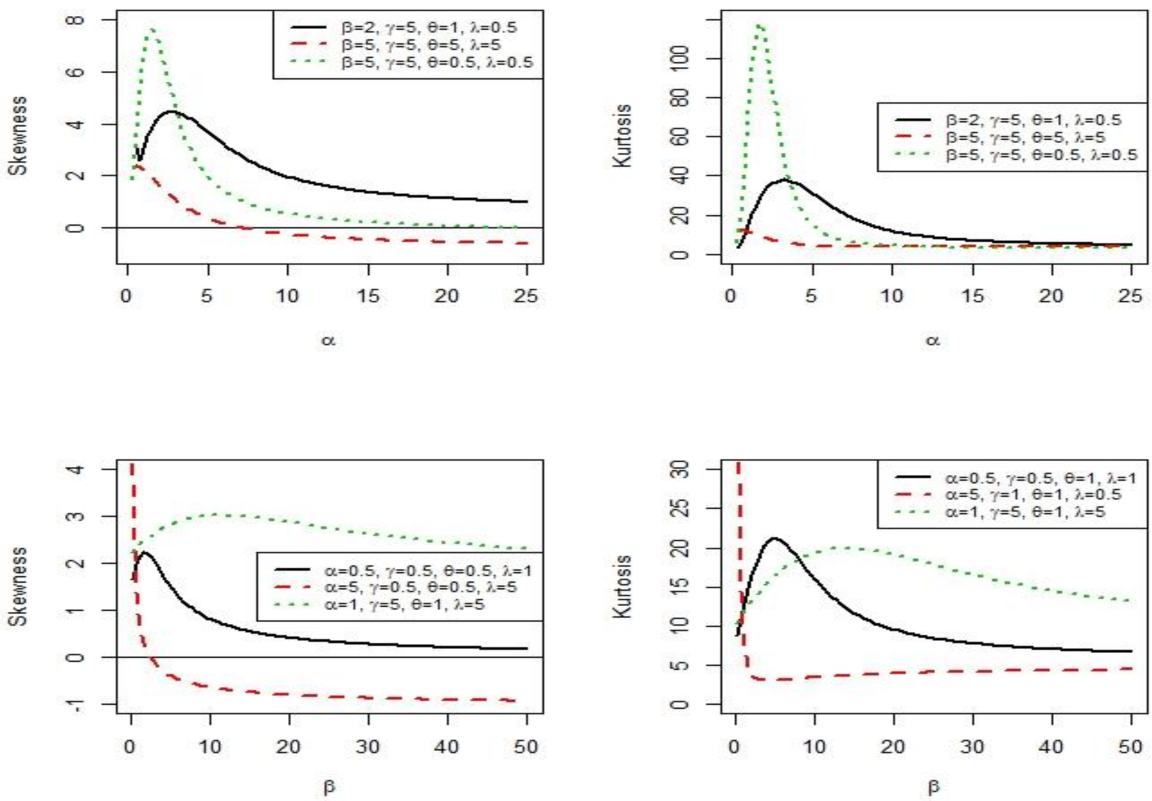

Figure: 4.1: skewness and kurtosis measures of the NMSM distribution

4.2 Moments of Order Statistics

Moments of order statistics have applications in reliability and life testing. Moments of order statistics are designed for replacement policy with the prediction of failure of future items determined from few early failures.

The pdf of $X_{\mathrm{m}:n}$ for NMSM distribution is

$$f\left(x_{\mathrm{m}:n}\right)=\frac{1}{B(\mathrm{m},n-m+1)}\sum_{i=0}^{m-1}(-1)^{i}\binom{m-1}{i}\alpha\frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{-\frac{\alpha}{\gamma}(n-m+1+i)-1}.$$

Moments about the origin of $X_{\mathrm{m}:n}$ for NMSM distribution are

$$E\left(X^{r}{}_{\mathrm{m}:n}\right)=\int_{0}^{\infty}x^{r}f\left(x_{\mathrm{m}:n}\right)dx,$$

$$E\left(X^{r}{}_{\mathrm{m}:n}\right)=\int_{0}^{\infty}x^{r}\frac{1}{B(\mathrm{m},n-m+1)}\sum_{i=0}^{m-1}(-1)^{i}\binom{m-1}{i}\alpha\frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{-\frac{\alpha}{\gamma}(n-m+1+i)-1}dx,$$

$$E\left(X^{r}{}_{\mathrm{m}:n}\right)=\frac{\frac{\alpha}{\gamma}\sum_{i=0}^{m-1}(-1)^{i+k}\binom{m-1}{i}\frac{\beta^{r}}{\lambda^{r}}\sum_{n=r}^{\infty}\frac{(-r)(-n)^{n-r}}{(n-r)!}\left(\beta^{-1}\lambda\theta\gamma^{-\frac{1}{\beta}}\right)^{n}B\left(1+\frac{n}{\beta},\frac{\alpha}{\gamma}(n-m+1+i)-\frac{n}{\beta}\right)}{B(\mathrm{m},n-m+1)},\mathrm{r}=1,2,3\ldots.$$

4.3 Incomplete Moments

Incomplete moments are used in mean inactivity life, mean residual life function, and other inequality measures.

The s[th] incomplete moment of the NMSM distribution is

$$M_{s}^{/}(z)=\int_{0}^{z}x^{s}\alpha\frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{-\frac{\alpha}{\gamma}-1}dx,$$

$$M_{s}^{/}(z)=\frac{\alpha}{\gamma}\frac{\beta^{s}}{\lambda^{s}}\sum_{n=s}^{\infty}\frac{(-s)(-n)^{n-s}}{(n-s)!}\left(\frac{\lambda\theta\gamma^{-\frac{1}{\beta}}}{\beta}\right)^{n}B_{\left(\gamma\theta^{-\beta}x^{-\beta}e^{\lambda x}\right)}\left(1+\frac{n}{\beta},\frac{\alpha}{\gamma}-\frac{n}{\beta}\right),\tag{14}$$

where $B_{z}(.,.)$ is incomplete beta function.

The mean deviation about the mean is $MD_{\bar{X}}=E\left|X-\mu_{1}^{1}\right|=2\mu_{1}^{1}F\left(\mu_{1}^{1}\right)-2\mu_{1}^{1}M_{1}^{/}\left(\mu_{1}^{1}\right)$ and mean deviation about median is $MD_{M}=E\left|X-M\right|=2MF(M)-2MM_{1}^{/}(M)$ where $\mu_{1}'=E(X)\ and\ M=Q(0.5)$. For a specified probability p, Bonferroni and Lorenz curves are computed as $\mathrm{B}(p)=M_{1}'(q)/p\mu_{1}^{1}$ and $L(p)=M_{1}^{/}(q)/\mu_{1}^{1}$ where $q=Q(p)$.

4.4 Residual Life Functions

The residual life is $m_n(z)=E\left[(X-z)^n \middle| X>z\right]$. For the random variable X with NMSM distribution $m_n(z)$ is given by

$$m_n(z)=\frac{1}{S(z)}\int_z^{\infty}(x-z)^s f(x)dx=\frac{1}{S(z)}\sum_{s=0}^{n}\binom{n}{s}(-z)^{n-s}\int_z^{\infty}x^s f(x)dx,$$

$$m_n(z)=\frac{1}{(1-F(z))}\sum_{s=0}^{n}\binom{n}{s}(-z)^{n-s}\frac{\alpha}{\gamma}\frac{\beta^s}{\lambda^s}\sum_{k=s}^{\infty}\frac{(-s)(-k)^{k-s}}{(k-s)!}\left(\frac{\lambda\theta\gamma^{-\frac{1}{\beta}}}{\beta}\right)^k\left[B\left(1+\frac{k}{\beta},\frac{\alpha}{\gamma}-\frac{k}{\beta}\right)-B_{\left(\gamma\theta^{-\beta}x^{-\beta}e^{\lambda x}\right)}\left(1+\frac{k}{\beta},\frac{\alpha}{\gamma}-\frac{k}{\beta}\right)\right].(15)$$

The average remaining lifetime of a component at time z, say $m_1(z)$, or life expectancy called mean residual life (MRL) function is

$$m_1(z)=\frac{1}{(1-F(z))}\sum_{s=0}^{1}\binom{1}{s}(-z)^{1-s}\frac{\alpha}{\gamma}\frac{\beta^s}{\lambda^s}\sum_{k=s}^{\infty}\frac{(-s)(-k)^{k-s}}{(k-s)!}\left(\frac{\lambda\theta\gamma^{-\frac{1}{\beta}}}{\beta}\right)^k\left[B\left(1+\frac{k}{\beta},\frac{\alpha}{\gamma}-\frac{k}{\beta}\right)-B_{\left(\gamma\theta^{-\beta}x^{-\beta}e^{\lambda x}\right)}\left(1+\frac{k}{\beta},\frac{\alpha}{\gamma}-\frac{k}{\beta}\right)\right]. \quad (16)$$

The reverse residual life, say $M_n(z)=E\left[(z-X)^n / X\le z\right]$, of X with NMSM distribution has the following n^th^ moment

$$M_n(z)=\frac{1}{F(z)}\int_0^{z}(z-x)^n f(x)dx=\frac{1}{F(z)}\sum_{s=0}^{n}(-1)^s\binom{n}{s}z^{n-s}M_s^{/}(z),$$

$$M_n(z)=\frac{1}{F(z)}\sum_{s=0}^{n}(-1)^s\binom{n}{s}z^{n-s}\frac{\alpha}{\gamma}\frac{\beta^s}{\lambda^s}\sum_{k=s}^{\infty}\frac{(-s)(-k)^{k-s}}{(k-s)!}\left(\frac{\lambda\theta\gamma^{-\frac{1}{\beta}}}{\beta}\right)^k B_{\left(\gamma\theta^{-\beta}x^{-\beta}e^{\lambda x}\right)}\left(1+\frac{k}{\beta},\frac{\alpha}{\gamma}-\frac{k}{\beta}\right). \quad (17)$$

The waiting time z for the failure of a component has passed with condition that this failure had happened in the interval [0, z] is called mean waiting time (MWT) or mean inactivity time. The waiting time z for failure of a component for NMSM distribution is defined by

$$M_1(z)=\frac{1}{F(z)}\sum_{s=0}^{1}(-1)^s\binom{1}{s}z^{1-s}\frac{\alpha}{\gamma}\frac{\beta^s}{\lambda^s}\sum_{k=s}^{\infty}\frac{(-s)(-k)^{k-s}}{(k-s)!}\left(\frac{\lambda\theta\gamma^{-\frac{1}{\beta}}}{\beta}\right)^k B_{\left(\gamma\theta^{-\beta}x^{-\beta}e^{\lambda x}\right)}\left(1+\frac{k}{\beta},\frac{\alpha}{\gamma}-\frac{k}{\beta}\right). \quad (18)$$

5. RELIABILITY AND UNCERTAINTY MEASURES

In this section, different reliability and uncertainty measures for the NMSM distribution are studied.

5.1 Stress-Strength Reliability of NMSM Distribution

Let $X_1 \sim NMSM(\alpha_1,\beta,\gamma,\theta,\lambda), X_2 \sim NMSM(\alpha_2,\beta,\gamma,\theta,\lambda)$ and X_1 represents strength and X_2 represents stress, then reliability of the component is:

$$R=\Pr(X_2<X_1)=\int_{-\infty}^{\infty}\int_{-\infty}^{x_1}f(x_1,x_2)dx_2dx_1=\int_0^{\infty}f_{x_1}(x)F_{x_2}(x)dx,$$

$$R=\int_0^\infty \alpha_1 \frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^\beta e^{\lambda x}\left[1+\gamma\left(\frac{x}{\theta}\right)^\beta e^{\lambda x}\right]^{-\frac{\alpha_1}{\gamma}-1}\left\{1-\gamma\left[1+\left(\frac{x}{\theta}\right)^\beta e^{\lambda x}\right]^{-\frac{\alpha_2}{\gamma}-1}\right\}dx=\frac{\alpha_2}{(\alpha_1+\alpha_2)}. \quad (19)$$

(i) R is independent of $\beta,\gamma,\theta\, and\, \lambda$.

(ii) $For\, \alpha_1=\alpha_2, R=0.5$, X_1 and X_2 are identically and independently distributed i.e. $\mathrm{R}=\Pr(X_1>X_2)=\Pr(X_1<X_2)$.

5.2 Multicomponent Stress-Strength Reliability Estimator $R_{s,k}$ Based on NMSM Distribution

Suppose a machine has at least "s" components working out of "k" components. The strengths of all the components of the system are $X_1, X_2, \ldots X_k$ and stress Y is applied on the system. Both the strengths $X_1, X_2, \ldots X_k$ are i.i.d. and are independent of stress Y. G is the cdf of Y and F is the cdf of X. The reliability of a machine is the probability that the machine functions properly.

Let $X \sim \mathrm{NMSM}(\alpha_1,\beta,\gamma,\theta,\lambda), Y \sim \mathrm{NMSM}(\alpha_2,\beta,\gamma,\theta,\lambda)$ with common parameters $\beta,\gamma,\theta\, and\, \lambda$ and unknown shape parameters $\alpha_1\, and\, \alpha_2$. The multicomponent stress-strength reliability for NMSM distribution is given by

$$R_{s,k}=P(strengths>stress)=P[at\, least\, "s"\, of\, (X_1,X_2,\ldots X_k)\, exceed\, Y], \quad (20)$$

$$R_{s,k}=\sum_{l=s}^{k}\binom{k}{\ell}\int_0^\infty [1-F(y)]^l\,[F(y)]^{k-l}\,dG(y)$$ (Bhattacharyya and Johnson; 1974).

(21)

$$R_{s,k}=\sum_{l=s}^{k}\binom{k}{l}\int_0^\infty\left(\left[1+\gamma\left(\frac{x}{\theta}\right)^\beta e^{\lambda x}\right]^{-\frac{\alpha_1}{\gamma}}\right)^l\left(1-\left[1+\gamma\left(\frac{x}{\theta}\right)^\beta e^{\lambda x}\right]^{-\frac{\alpha_1}{\gamma}}\right)^{(k-l)} d\left[1+\gamma\left(\frac{x}{\theta}\right)^\beta e^{\lambda x}\right]^{-\frac{\alpha_2}{\gamma}}.$$

Let $t=\left[1+\gamma\left(\frac{x}{\theta}\right)^\beta e^{\lambda x}\right]^{-\frac{\alpha_2}{\gamma}}$, then we obtain

$$R_{s,k}=\sum_{\ell=s}^{k}\binom{k}{\ell}\int_0^1\left(t^\nu\right)^\ell\left(1-t^\nu\right)^{(k-\ell)}dt.$$

Let $z=t^\nu, t=z^{\frac{1}{\nu}}, \quad dt=\frac{1}{\nu}z^{\frac{1}{\nu}-1}dz$, then

$$R_{s,k}=\sum_{\ell=s}^{k}\binom{k}{\ell}\int_0^1 (z)^\ell(1-z)^{(k-\ell)}\frac{1}{\nu}z^{\frac{1}{\nu}-1}dz,$$

$$R_{s,k}=\frac{1}{\nu}\sum_{\ell=s}^{k}\binom{k}{\ell}\int_0^1 (z)^{\ell+\frac{1}{\nu}-1}(1-z)^{(k-\ell)}dz,$$

$$R_{s,k}=\frac{1}{\nu}\sum_{\ell=s}^{k}\binom{k}{\ell}B\left(\ell+\frac{1}{\nu},k-\ell+1\right), where\ \nu=\frac{\alpha_1}{\alpha_2}. \tag{22}$$

The probability $R_{s,k}$ in the (22) is called reliability in a multicomponent stress-strength model.

5.3 Renyi, Havrda and Chavrat, Tsallis and Q-entropy

The uncertainty measures are used to study anomalous diffusion, heartbeat intervals (cardiac autonomic neuropathy (CAN), DNA sequences, daily temperature fluctuations (climatic) and study of information content signals.

Renyi entropy (1961) for NMSM distribution is

$$I_R(v)=\frac{1}{1-v}\log(\int_{-\infty}^{\infty}(f(x))^{v}dx)\ \ v\neq 1, v>0, \tag{23}$$

Let $\int_0^\infty\left[f(x)\right]^{v}dx=\int_0^\infty\left[\alpha\frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{-\frac{\alpha}{\gamma}-1}\right]^{v}dx,$

Let $\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}=y$, $dx=\frac{1}{\gamma}x(\beta+\lambda x)^{-1}\left(\frac{x}{\theta}\right)^{-\beta}e^{-\lambda x}dy,$

$$\int_0^\infty\left[f(x)\right]^{v}dx=\frac{\alpha^{v}}{\gamma^{v}}\sum_{\ell=0}^{v-1}\sum_{n=\ell-2v+2}^{\infty}\binom{v-1}{\ell}\frac{-(\ell-2v+2)(-n)^{n-(\ell-2v+2)-1}}{\left[n-(\ell-2v+2)\right]!}\left(\frac{\lambda}{\gamma}\right)^{n+v-1}\theta^{n}B\left(\frac{n}{\beta}+v,\frac{\alpha v}{\gamma}-\frac{n}{\beta}\right), \tag{24}$$

$$I_R=\frac{1}{1-v}\log\left\{\frac{\alpha^{v}}{\gamma^{v}}\sum_{\ell=0}^{v-1}\sum_{n=\ell-2v+2}^{\infty}\binom{v-1}{\ell}\frac{-(\ell-2v+2)(-n)^{n-(\ell-2v+2)-1}}{\left[n-(\ell-2v+2)\right]!}\left(\frac{\lambda}{\gamma}\right)^{n+v-1}\theta^{n}B\left(\frac{n}{\beta}+v,\frac{\alpha v}{\gamma}-\frac{n}{\beta}\right)\right\}.$$

Renyi entropy tends to Shannon entropy $as\, v\to 1$.

Havrda and Chavrat Entropy (1967) for NMSM distribution is

$$H_{HC}(v)=\frac{1}{v-1}\log(\int_{-\infty}^{\infty}(f(x))^{v}dx)\ \ v\neq 1, v>0. \tag{25}$$

$$H_{HC}(v)=\frac{1}{v-1}\log\left\{\frac{\alpha^{v}}{\gamma^{v}}\sum_{\ell=0}^{v-1}\sum_{n=\ell-2v+2}^{\infty}\binom{v-1}{\ell}\frac{-(\ell-2v+2)(-n)^{n-(\ell-2v+2)-1}}{\left[n-(\ell-2v+2)\right]!}\left(\frac{\lambda}{\gamma}\right)^{n+v-1}\theta^{n}B\left(\frac{n}{\beta}+v,\frac{\alpha v}{\gamma}-\frac{n}{\beta}\right)\right\}. \tag{26}$$

Tsallis and Q-entropies can be obtained in similar way.

6. CHARACTERIZATIONS

In this section, NMSM distribution is characterized via: (i) Conditional expectation; (ii) truncated moment; (iii) Hazard function and (iv) Mills ratio.

6.1 Characterization Based on Conditional Expectation

The NMSM distribution is characterized via conditional expectation.

Proposition 6.1.1: Let $X:\Omega\to(0,\infty)$ be a continuous random variable with cdf $F(x)$ ($0<F(x)<1$ for $x>0$), then for $\alpha>\gamma$, X has cdf (3) if and only if

$$E\left[\left(\frac{X}{\theta}\right)^{\beta} e^{\lambda X}\middle| X>t\right]=\frac{1}{(\alpha-\gamma)}\left\{1+\alpha\left(\frac{t}{\theta}\right)^{\beta} e^{\lambda t}\right\} \quad \textit{for } t>0. \tag{27}$$

Proof. If X has cdf (3), then

$$E\left[\left(\frac{X}{\theta}\right)^{\beta} e^{\lambda X}\middle| X>t\right]=\left[1-F(t)\right]^{-1}\int_{t}^{\infty}\left(\frac{x}{\theta}\right)^{\beta} e^{\lambda x}\alpha\frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^{\beta} e^{\lambda x}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta} e^{\lambda x}\right]^{-\frac{\alpha}{\gamma}-1} dx.$$

Upon integration by parts and simplification, we arrive at

$$E\left[\left(\frac{X}{\theta}\right)^{\beta} e^{\lambda X}\middle| X>t\right]=\frac{1}{(\alpha-\gamma)}\left\{1+\alpha\left(\frac{t}{\theta}\right)^{\beta} e^{\lambda t}\right\} \quad \textit{for } t>0.$$

Conversely if (27) holds, then

$$\int_{t}^{\infty}\left(\frac{x}{\theta}\right)^{\beta} e^{\lambda x} f(x)dx=\frac{(1-F(t))}{(\alpha-\gamma)}\left\{1+\alpha\left(\frac{t}{\theta}\right)^{\beta} e^{\lambda t}\right\}. \tag{28}$$

Differentiating (28) with respect to t, we obtain

$$-\left(\frac{t}{\theta}\right)^{\beta} e^{\lambda t} f(t)=\frac{(1-F(t))}{(\alpha-\gamma)}\left[1+\alpha\left(\frac{t}{\theta}\right)^{\beta} e^{\lambda t}\right]-f(t)\left[\frac{\alpha}{t(\alpha-\gamma)}(\beta+\lambda t)\left(\frac{t}{\theta}\right)^{\beta} e^{\lambda t}\right].$$

After simplification and integration we arrive at $F(t)=1-\left[1+\gamma\left(\frac{t}{\theta}\right)^{\beta} e^{\lambda t}\right]^{-\frac{\alpha}{\gamma}}$ for $t\geq 0$.

6.2 Characterizations Based on Truncated Moment of a Function of the Random Variable

Here we characterize NMSM distribution via relationship between truncated moment of a function of X and another function. This characterization is stable in the sense of weak convergence (Glänzel; 1990).

Proposition 6.2.1 Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $g(x)=\left[1+\left(\frac{x}{\theta}\right)^{\beta} e^{\lambda x}\right]^{-1}$ *for* $x>0$. The random variable X has pdf (4) if and only if the function $h(x)$ defined in Theorem 1 has the form $h(x)=\frac{\alpha}{\alpha+\gamma}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta} e^{\lambda x}\right]^{-1}, \quad x\geq 0$.

Proof Let the random variable X have pdf (4), then

$$(1-F(x))E\left(g(X)\middle|X\geq x\right)=\frac{\alpha}{\alpha+\gamma}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta} e^{\lambda x}\right]^{-(\frac{\alpha}{\gamma}+1)} \quad x>0.$$

or

$$E\left(g(X)\middle|X\geq x\right)=\frac{\alpha}{\alpha+\gamma}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta} e^{\lambda x}\right]^{-1}, x>0.$$

and

$$h(x)-g(x)=-\frac{\gamma}{\alpha+\gamma}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta} e^{\lambda x}\right]^{-1}, \quad x>0$$

Conversely if $h(x)$ is given as above, then

$$h'(x)=-\frac{\alpha}{\alpha+\gamma}\gamma\frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{-2}<0,\ for\ x>0,$$

and

$$s'(x)=\frac{h'(x)}{h(x)-g(x)}=\frac{\frac{\alpha}{\gamma}\gamma\frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}}{\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]},\ x>0,$$

and hence

$$s(x)=\ln\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{\frac{\alpha}{\gamma}},x>0,$$

and

$$e^{-s(x)}=\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{-\frac{\alpha}{\gamma}}\quad x>0.$$

In view of Theorem 1, X has density (4).

Corollary 6.2.1: Let $X:\Omega\to(0,\infty)$ be a continuous random variable. The pdf of X is (4) if and only if there exist functions $h(x)$ *and* $g(x)$ defined in Theorem 1satisfying the differential equation

$$\frac{h'(x)}{h(x)-g(x)}=\frac{\frac{\alpha}{\gamma}\gamma\frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}}{\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]},\ x>0. \tag{29}$$

Remarks:6.2.1 The general solution of (29) in Corollary 6.2.1 is

$$h(x)=\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{\frac{\alpha}{\gamma}}\left[-\int\frac{\frac{\alpha}{\gamma}\gamma\frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}}{\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{\frac{\alpha}{\gamma}+1}}g(x)dx+D\right]$$ where D is a constant.

6.3 Characterization Via Hazard Function

In this sub-section, NMSM distribution is characterized via hazard function.

Definition 6.3.1: Let $X:\Omega\to(0,\infty)$ be a continuous random variable. The pdf of X is (4) if and only if its hazard function $h_F(x)$ satisfies the differentiable equation

$$\frac{d}{dx}\left[\ln f(x)\right]=\frac{h_F'(x)}{h_F(x)}-h_F(x).$$

Proposition 6.3.1 Let $X:\Omega\rightarrow(0,\infty)$ be continuous random variable .The pdf of X is (4) if and only if its hazard function, $h_F(x)$, satisfies the first order differential equation

$$xh_F'(x)+h_F(x)=\left\{\lambda+\gamma\lambda\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}+x^{-1}(\beta+\lambda x)^2\right\}\alpha\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{-2}. \quad (30)$$

Proof. If X has pdf (4), then (30) surly holds. Now if the (30) holds, then

$$\frac{d}{dx}\left\{xh_f(x)\right\}=\alpha\frac{d}{dx}\left\{(\beta+\lambda x)\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{-1}\right\},$$

or
$$h_F(x)=\alpha\frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]^{-1},x>0$$

which is the hazard function of NMSM distribution.

6.4 Characterization Based on Mills Ratio

Definition 6.4.1: Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable having absolutely continuous cdf $F(x)$ an d pdf $f(x)$ .The Mills ratio, $m(x)$, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{d}{dx}\left[\ln f(x)\right]=-\left[\frac{1}{m(x)}+\frac{m'(x)}{m(x)}\right].$$

Proposition 6.4.1: Let $X:\Omega\rightarrow(0,\infty)$ be continuous random variable .The pdf of X is (4) if and only if the Mills ratio satisfies the first order differential equation

$$m_F'(x)x(\beta+\lambda x)+m_F(x)\left[(\beta+\lambda x)^2-\beta\right]=\alpha^{-1}\gamma x(\beta+\lambda x). \quad (31)$$

Proof If X has pdf (4), then the (31) surely holds. Now if the (31) holds, then

$$\frac{d}{dx}\left[m_F(x)\alpha\frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]=\frac{d}{dx}\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right],$$

or
$$m(x)=\frac{\left[1+\gamma\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}\right]}{\alpha\frac{1}{x}(\beta+\lambda x)\left(\frac{x}{\theta}\right)^{\beta}e^{\lambda x}},$$

which is Mills ratio of NMSM distribution.

7. MAXIMUM LIKELIHOOD ESTIMATION

In this section, parameter estimates are derived using maximum likelihood method. The log-likelihood function for the vector of parameters $\Phi=(\alpha,\beta,\gamma,\theta,\lambda)$ of NMSM distribution is

$$\ell(\Phi)=\ln L(\Phi)=n\ln(\alpha)-n\beta\ln(\theta)-\sum_{i=1}^{n}\ln x_i+\sum_{i=1}^{n}\ln(\beta+\lambda x_i)+\beta\sum_{i=1}^{n}\ln x_i+ \lambda\sum_{i=1}^{n}x_i-\left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\ln\left[1+\gamma\theta^{-\beta}x_i^{\beta}e^{\lambda x_i}\right]. \tag{32}$$

In order to compute the estimates of parameters of NMSM distribution, the following nonlinear equations must be solved simultaneously.

$$\frac{\delta}{\delta\alpha}\ell(\Phi)=\frac{n}{\alpha}-\frac{1}{\gamma}\sum_{i=1}^{n}\ln\left[1+\gamma\theta^{-\beta}x_i^{\beta}e^{\lambda x_i}\right]=0, \tag{33}$$

$$\frac{\delta}{\delta\beta}\ell(\Phi)=-n\ln(\theta)+\sum_{i=1}^{n}\ln x_i+\sum_{i=1}^{n}(\beta+\lambda x_i)^{-1}-(\alpha+\gamma)\sum_{i=1}^{n}\frac{(\ln x_i-\ln\theta)\theta^{-\beta}x_i^{\beta}e^{\lambda x_i}}{\left[1+\gamma\theta^{\beta}x_i^{-\beta}e^{-\lambda x_i}\right]}=0, \tag{34}$$

$$\frac{\delta}{\delta\gamma}\ell(\Phi)=\frac{\alpha}{\gamma^2}\sum_{i=1}^{n}\ln\left[1+\gamma\theta^{-\beta}x_i^{\beta}e^{\lambda x_i}\right]-\left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\frac{\left(\theta^{-\beta}x_i^{\beta}e^{\lambda x_i}\right)}{\left[1+\gamma\theta^{-\beta}x_i^{\beta}e^{\lambda x_i}\right]}=0, \tag{35}$$

$$\frac{\delta}{\delta\theta}\ell(\Phi)=-\frac{n\beta}{\theta}+(\alpha+\gamma)\beta\sum_{i=1}^{n}\left[1+\gamma\theta^{\beta}x_i^{-\beta}e^{-\lambda x_i}\right]^{-1}\theta^{-\beta-1}x_i^{\beta}e^{\lambda x_i}=0, \tag{36}$$

$$\frac{\delta}{\delta\lambda}\ell(\Phi)=\sum_{i=1}^{n}x_i(\beta+\lambda x_i)^{-1}-\sum_{i=1}^{n}x_i+(\alpha+\gamma)\sum_{i=1}^{n}\frac{\theta^{-\beta}x_i^{\beta+1}e^{\lambda x_i}}{\left[1+\gamma\theta^{-\beta}x_i^{\beta}e^{\lambda x_i}\right]}=0. \tag{37}$$

8. SIMULATION STUDIES

In this Section, we perform the simulation study to illustrate the performance of MLE's of NMSM distribution. The random number generation is obtained with inverse of its cdf. The MLEs, say $(\hat{\alpha}_i,\hat{\beta}_i,\hat{\gamma}_i,\hat{\theta}_i,\hat{\lambda}_i)$ for i=1,2,...,N, have been obtained by CG routine in R programme.

The simulation study is based on graphical results. We generate N=1000 samples of sizes n=20,25,…,1000 from NMSM distribution and get true values of $\alpha,\beta,\gamma,\theta$ and λ parameters as 5,6,0.5,1 and 2 respectively for this simulation study. We also calculate the mean, standard deviations (sd), bias and mean square error (MSE) of the MLEs. The bias and MSE are calculated by (for $h=\alpha,\beta,\gamma,\theta,\lambda$)

$$Bias_h=\tfrac{1}{N}\sum_{i=1}^{N}\left(\hat{h}_i-h\right) \text{ and } MSE_h=\tfrac{1}{N}\sum_{i=1}^{N}\left(\hat{h}_i-h\right)^2.$$

The results are given by Figure 8.1. Figure 8.1 reveals that the empirical means tend to the true parameter values and that the sds, biases and MSEs decrease when the sample size increases. These results are in agreement with first-order asymptotic theory.

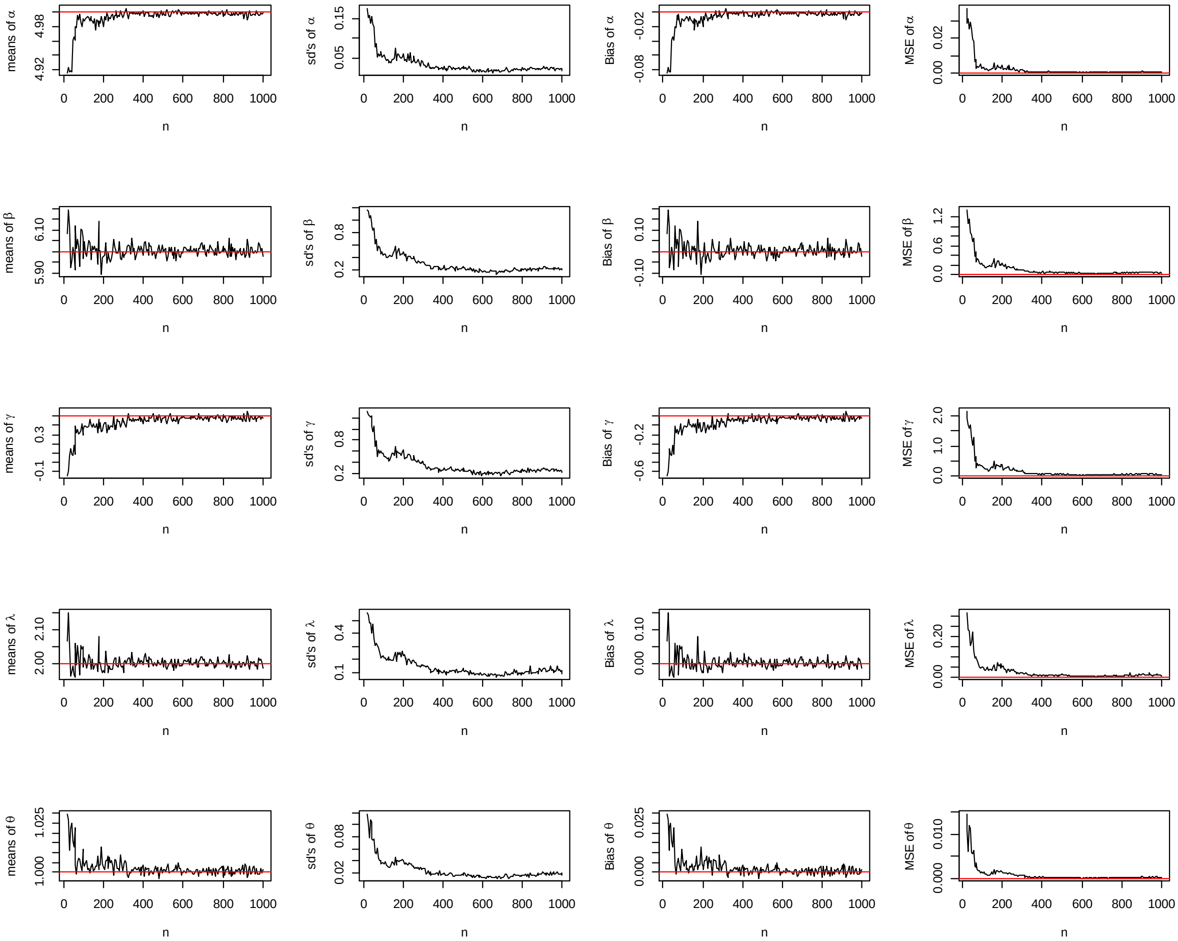

Figure 8.1. Simulation results of the special NMSM distribution

9. APPLICATIONS

The NMSM distribution is compared with MSM, SM, MBXII, BXII, Lomax, Log-logistic distributions. Different goodness fit measures such Cramer-von Mises (W), Anderson Darling (A), Kolmogorov- Smirnov statistics with p-values and likelihood ratio statistics ($-\ell$) are computed using R-package for survival times of guinea pigs and prices of thirty one wooden toys. The better fit corresponds to smaller W, A, K-S and $-\ell$ value. The maximum likelihood estimates (MLEs) of unknown parameters and values of goodness of fit measures are computed for NMSM distribution and its sub-models.

9.1 Survival times of guinea pigs data

Survival times of 72 guinea pigs ill with contagious tubercle bacilli conveyed by Bjerkedal (1960) are : 10, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230,231, 240, 245, 251, 253, 254, 255, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555.

The MLEs (standard errors) and goodness-of-fit statistics like $-\ell$, A, W, K-S with p-values are given in table 3.

Table 3: MLEs (standard errors) and Goodness-of-fit statistics for data set I

Model	α	β	γ	λ	θ	$-\ell$	A	W	K-S (p-value)
NMSM	0.009084599 (0.00667775)	0.738838064 (0.8447382)	0.033090649 (2.466064)	67.342644057 (0.01405517)	0.034432409 (101.7066)	**423.4807**	**0.279015**	**0.04755605**	**0.069985 (0.8723)**
MSM	0.09693605 (0.6082713)	2.57107078 (0.4756781)	0.05652065 (0.3552205)	0	66.57384308 (162.5729073)	425.1213	0.4159587	0.06732784	0.077789 (0.7763)
SM	1.588969 (0.908306)	2.624544 (0.4690821)	1	194.535770 (64.0228996)	1	425.1288	0.4023472	0.06430378	0.076968 (0.7872)
MBXII	0.0145 (0.01612)	12.9000 (5.5069)	0.8079 (0.0814)	0	1	490.2748	0.7577	0.1348	0.4812 (6.55e-15)
BXII	0.08466762 (0.1510882)	2.35980977 (4.2037699)	1	0	1	548.261	1.373439	0.1966047	0.51092 (<2.2e-16)
Lomax	0.1994794 (0.02350829)	1	1	0	1	549.0066	1.267504	0.18015	0.51192 (<2.2e-16)
Log-Logistic	1	0.3072592 (0.02793286)	1	0	1	584.4259	0.8626136	0.119925	0.73405 (< 2.2e-16)

We can perceive that the NMSM distribution is best fitted model than the other sub-models because the values of all criteria of goodness of fit are significantly smaller for NMSM distribution.

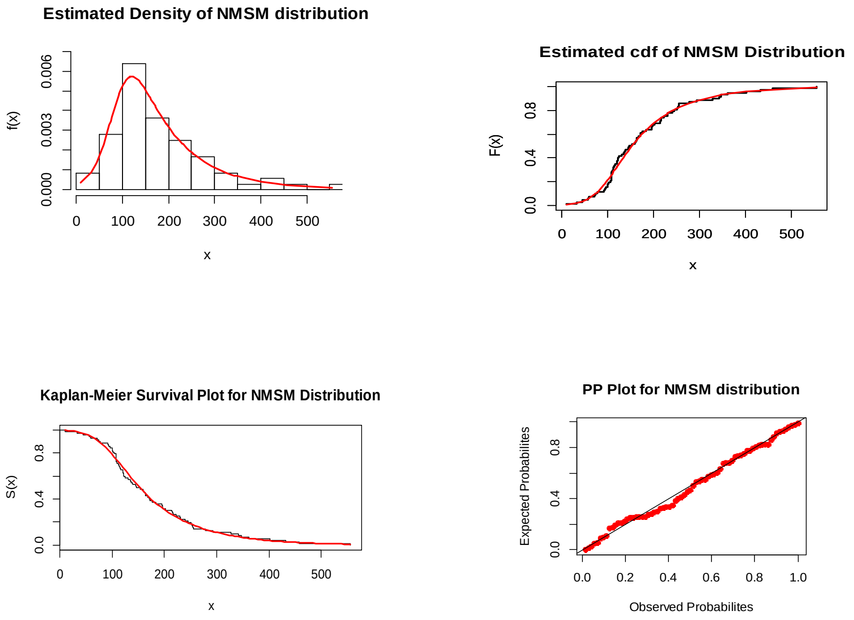

Fig 9.1: plots of fitted pdf, cdf, survival function and the probability-probability (P-P) plot for data set I

We can perceive that the NMSM distribution is best fitted model because all the plots of fitted pdf, cdf, survival function and the probability-probability (P-P) plot for NMSM are closer fit to data.

9.2 Prices of Wooden Toys

The prices of thirty one wooden toys at Suffolk craft shop in 1991 are: 4.2, 1.12, 1.39, 2, 3.99, 2.15, 1.74, 5.81, 1.7, 2.85, 0.5, 0.99, 11.5, 5.12, 0.9, 1.99, 6.24, 2.6, 3, 12.2, 7.36, 4.75, 11.59, 8.69, 9.8, 1.85, 1.99, 1.35, 10, 0.65, 1.45.

The MLEs (standard errors) and goodness-of-fit statistics like $-\ell$, A, W, K-S with p-values are given in table 4.

Table 4: MLEs (standard errors) and Goodness-of-fit statistics for data set II

Model	α	β	γ	λ	θ	$-\ell$	A	W	K-S (p-value)
NMSM	1.325543 (52.352551)	2.320047 (1.723071)	15.087213 (595.612237)	2.357509 (3.886553)	5.487198 (94.768715)	**72.63759**	**0.2769906**	**0.03958655**	**0.094603 (0.9442)**
MSM	53.671554 (2527.555518)	1.567199 (1.425999)	28.835345 (1314.262793)	0	43.542836 (1373.055860)	74.73868	0.5838663	0.09555475	0.13574 (0.6175)
SM	1.669719 (5.285228)	1.607768 (1.263627)	1	0	4.595593 (13.842109)	74.73887	0.5693017	0.09292011	0.13344 (0.639)
MBXII	0.14457996 (0.06701295)	1.58095022 (1.29072818)	0.08069273 (0.26442258)	0	1	74.73863	0.5788114	0.09464383	0.13492 (0.6251)
BXII	0.2276987 (0.07832594)	3.8515546 (1.18745384)	1	1	1	76.40228	0.5154045	0.06597862	0.1312 (0.66)
Lomax	0.6945145 (0.1247382)	1	1	0	1	86.93639	0.374435	0.05615229	0.29516 (0.009021)
Log-Logistic	1	1.214184 (0.1790725)	1	0	1	88.46715	0.3749721	0.05575932	0.42881 (2.238e-05)

We can perceive that the NMSM distribution is best fitted model than the other sub-models because the values of all criteria of goodness of fit are significantly smaller for NMSM distribution.

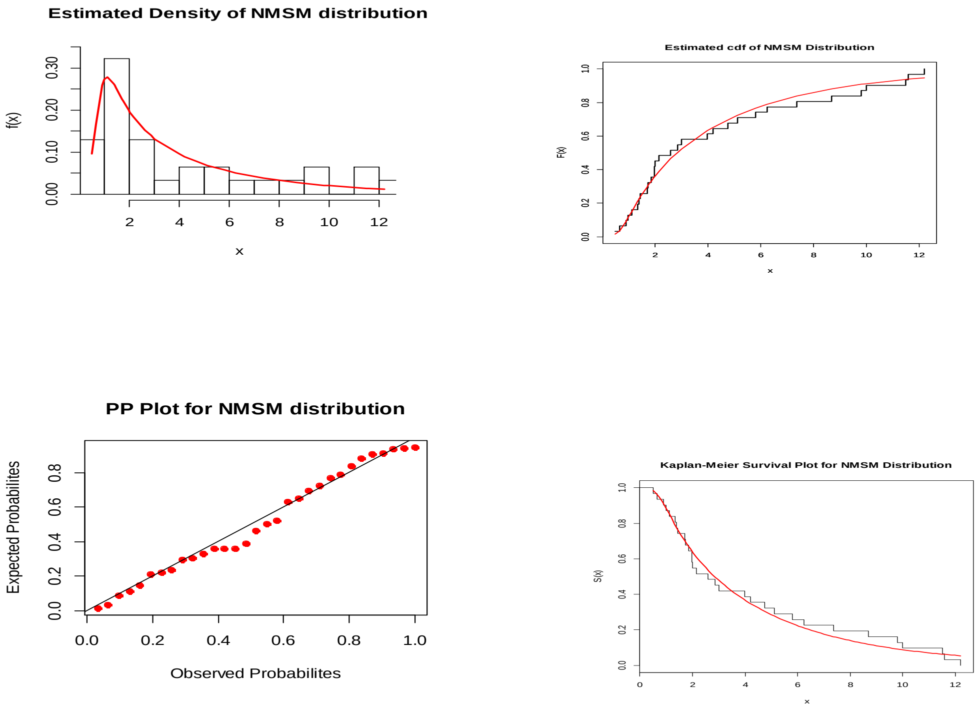

Fig 9.2: plots of fitted pdf, cdf, survival function and the probability-probability (P-P) plot for data set II

We can perceive that the NMSM distribution is best fitted model because all the plots of fitted pdf, cdf, survival function and the probability-probability (P-P) plot for NMSM are closer fit to data.

10. CONCLUDING REMARKS

We have developed NMSM distribution along with some of its properties such as structural properties, descriptive measures based on the quantiles, some plots, sub-models, moments, inequality measures, residual and reverse residual life function, stress-strength reliability, multicomponent stress-strength reliability model and uncertainty measures. The NMSM distribution has been characterized via different techniques. Maximum Likelihood estimates have been computed. The simulation studies have performed on the basis of graphical results to see the performance of maximum likelihood estimates the parameters. Goodness of fit to show that NMSM distribution is a better fit. Applications of the NMSM model to survival times of guinea pigs and prices of wooden toys data illustrated its significance and flexibility. We have proved that NMSM distribution is empirically better for lifetime applications.

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Appendix A

Theorem1. Let $(\Omega, \mathrm{F}, \mathrm{P})$ be given probability space and let $\mathrm{H}=[a_1,a_2]$ an interval with $a_1 < a_2$ ($a_1=-\infty, a_2=\infty$). Let $X:\Omega \to [a_1,a_2]$ be a continuous random variable with distribution function F and Let g be real function defined on $\mathrm{H}=[a_1,a_2]$ such that $E[g(X)|\mathrm{X} \geq \mathrm{x}]=h(x)\ \ x \in H$ is defined with some real function $h(x)$ should be in simple form. Assume that $g(x)\varepsilon C([a_1,a_2]), h(x)\varepsilon C^2([a_1,a_2])$ and F is a continuously differentiable and strictly monotone function on H: To conclude, assume that the equation $g(x)=h(x)$ has no real solution in the interior of H.Then F is obtained from the functions $g(x)\ and\ h(x)$ as $F(x)=\int\limits_a^x k\left|\frac{h'(t)}{h(t)-g(t)}\right|\exp(-s(t))dt$, where $s(t)$ is the solution of equation $s'(t)=\frac{h'(t)}{h(t)-g(t)}$ and k is a constant, chosen to make $\int\limits_{a_1}^{a_2} dF=1.$