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Marshall-Olkin Power Lomax Distribution: Properties and Estimation Based on Complete and Censored Samples

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Abstract

We study a new distribution called the Marshall-Olkin Power Lomax distribution. A comprehensive account of its mathematical properties including explicit expressions for the ordinary moments, moment generating function, order statistics, Renyi entropy, and probability weighted moments are derived. The model parameters are estimated by the method of maximum likelihood. Monte Carlo simulation study is carried out to estimate the parameters and the performance of the estimates is judged via the average biases and mean squared error values. The usefulness of the proposed model is illustrated via real-life data set.

Keywords: lifetime data, Marshall-Olkin family, maximum likelihood estimation, Lomax distribution, moments, order statistic

1. Introduction

The Lomax distribution introduced by Lomax (1954), (also known as Pareto Type II distribution) is one of the well-known distributions used for modeling of actuarial sciences, business failure, size of cities, medical and biological sciences, income and wealth inequality, engineering, lifetime and reliability datasets. Lomax distribution has been considered as a heavy tailed alternative to the exponential, Weibull and gamma distributions (Bryson, 1974). It is also associated with Burr family of distributions (Tadikamalla, 1980). In the lifetime, the Lomax model belongs to the family of decreasing failure rate (Chahkandi & Ganjali, 2009). Lomax distribution has been used for modeling income and wealth data (Atkinson & Harrison, 1978), for firm size data (Corbellini et al. 2010), for reliability and life testing (Harris, 1968), receiver operating characteristic (ROC) curve analysis (Campbell & Ratnaparkhi, 1993) and Hirsch-related statistics (Glänzel, 1987).

Moreover, many authors derived and studied the generalized forms of the Lomax distribution. For example, transmuted exponentiated Lomax (Ashour & Eltehiwy, 2013), McLomax distribution (Lemonte & Cordeiro, 2013), Weibull-Lomax distribution (Tahir et al. 2015), gamma-Lomax distribution (Cordeiro et al. 2015), transmuted Weibull Lomax distribution (Afify et al. 2015), weighted Lomax distribution (Kilany, 2016) and the power Lomax (PLx) distribution (Rady et al. 2016).

The cumulative distribution function of PLx distribution is given by

$$G(x) = 1 - \lambda^{\alpha}\left(\lambda + x^{\beta}\right)^{-\alpha}, x \ge 0, \alpha, \lambda, \beta > 0. \tag{1}$$

The corresponding probability density function is

$$g(x) = \alpha\beta\lambda^{\alpha} x^{\beta-1}\left(\lambda + x^{\beta}\right)^{-\alpha-1}, \tag{2}$$

In this study, we proposed a more flexible extension of the Power Lomax distribution by considering the distribution above as baseline model in Marshall and Olkin family (Marshall & Olkin, 1997), hereafter referred as Marshall-Olkin

Power Lomax (MOPLx). Additionally, we also discuss its theoretical properties. The maximum likelihood method is considered to estimate the parameters using complete and type II censored sampling. We considered real-life data to illustrate the usefulness of the proposed model. The real data application shows that the suggested distribution performs superior when compared with other important distributions.

The cumulative distribution function (cdf) of Marshall and Olkin (MO) family is defined by

$$F(x)=\frac{G(x)}{1-(1-\gamma)(1-G(x))},\ x\in\mathbb{R}, \tag{3}$$

with probability density function

$$f(x)=\frac{\gamma g(x)}{\left[1-(1-\gamma)(1-G(x))\right]^{2}},\ x\in\mathbb{R}, \tag{4}$$

Where $G(x)$ and $g(x)$ are cdf and pdf of the baseline distribution. Using this approach an additional shape parameter (γ) is added which is responsible for the skewness, kurtosis and tail weights. Moreover, this new model can be used as an alternative to gamma and Weibull distributions.

Many authors have proposed distributions using MO scheme. Marshall and Olkin (1997) developed Marshall-Olkin Exponential and Weibull distributions, Marshall–Olkin extended Lomax distribution by (Rao et al. 2009), Marshall-Olkin extended uniform distribution by (Jose & Krishna, 2011), Marshall-Olkin Fréchet distribution by (Krishna et al. 2013), Marshall-Olkin exponential Weibull distribution by (Pogány et al. 2015), Marshall-Olkin extended inverted Kumaraswamy distribution by (Usman & Haq, 2018), Marshall-Olkin length biased exponential distribution by (Haq et al. 2017) and Marshall-Olkin logistic-exponential distribution by (Mansoor et al. 2018).

2. The MOPLx Distribution

We define the four parameter Marshall-Olkin Power Lomax distribution by inserting (1) into (3) obtaining the cdf of the MOPLx distribution

$$F(x)=\frac{1-\lambda^{\alpha}\left(\lambda+x^{\beta}\right)^{-\alpha}}{1-(1-\gamma)\left(\lambda^{\alpha}\left(\lambda+x^{\beta}\right)^{-\alpha}\right)},\ x\geq 0. \tag{5}$$

The corresponding pdf is

$$f(x)=\frac{\gamma\alpha\beta\lambda^{\alpha}x^{\beta-1}\left(\lambda+x^{\beta}\right)^{-\alpha-1}}{\left[1-(1-\gamma)\lambda^{\alpha}\left(\lambda+x^{\beta}\right)^{-\alpha}\right]^{2}},\quad x\geq 0, \tag{6}$$

where $\alpha,\beta,\gamma,\lambda$ are positive parameters.

3. Mathematical Properties of MOPLx Distribution

The mathematical properties of MOPLx distribution including shapes of the pdf and hrf, linear representations of the cdf and pdf, quintile function (qf), random number generator, ordinary moments, incomplete moments, moment generating function, mean residual life, probability weighted moments and reversed residual life are investigated in this section.

3.1 Shape Characteristics of the Pdf and Hrf of MOPLx Distribution

In this subsection, the limiting behavior of the pdf and hrf of MOPLx distribution at the origin are calculated.

Fact: *As x approaches the origin, limits of the MOPLx pdf and hrf are as follows*

$$\lim_{x\to 0} f(x)=\lim_{x\to 0} h(x)=\begin{cases}\infty & for \quad \beta<1\\ \dfrac{\alpha}{\gamma\lambda} & for \quad \beta=1\\ 0 & for \quad \beta>1\end{cases}$$

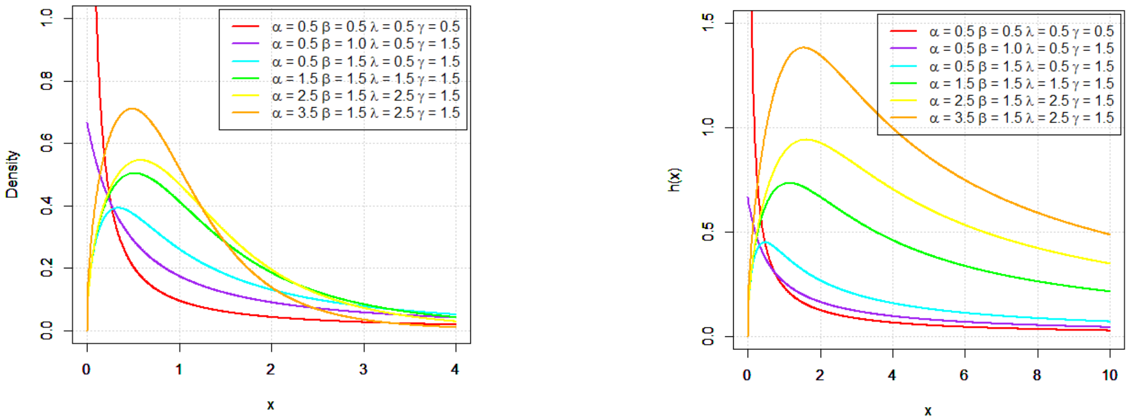

Figure 1. Pdf and hrf curves for selected parameter values

It is observed from the Fact above and Figure 1:

i. The pdf and hrf are decreasing if $\alpha > 0, \beta \le 1, \lambda > 0, \gamma > 0$.

ii. The pdf and hrf are unimodal if $\alpha > 0, \beta > 0, \lambda > 0, \gamma > 0$.

3.2 Useful Representations

Here we present two linear representations of the pdf and cdf of MOPLx distribution. Consider the following well-known binomial expansions (for $0 < \alpha < 1$),

$$(1-a)^{-n} = \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{\Gamma(n)\Gamma(k+1)} a^k. \tag{7}$$

Using (7), the numerator of (6) can be expressed as

$$\left[1-(1-\gamma)\lambda^{\alpha}\left(\lambda+x^{\beta}\right)^{-\alpha}\right]^{-2} = \sum_{k=0}^{\infty} \frac{\Gamma(2+k)}{\Gamma(2)\Gamma(k+1)} (1-\gamma)^k \lambda^{\alpha k} \left(\lambda+x^{\beta}\right)^{-\alpha k}. \tag{8}$$

Therefore, from (6) and (8) the pdf of MOPLx can be expressed as

$$f(x) = \frac{\gamma\alpha\beta}{\lambda} \sum_{k=0}^{\infty} (k+1)(1-\gamma)^k x^{\beta-1} \left(1+\frac{x^{\beta}}{\lambda}\right)^{-\alpha(k+1)-1}. \tag{9}$$

3.3 Quantile Function and Random Number Generator

The quantile function, say $Q(u) = F^{-1}(u)$, of X is given by

$$u = \frac{1-\lambda^{\alpha}\left(\lambda+Q(u)^{\beta}\right)^{-\alpha}}{1-(1-\gamma)\left(\lambda^{\alpha}\left(\lambda+Q(u)^{\beta}\right)^{-\alpha}\right)}.$$

Upon some simplifications, it reduces to the following form

$$Q(u) = \left(\lambda\left(\left(\frac{1-u}{1-(1-\gamma)u}\right)^{\frac{-1}{\alpha}} - 1\right)\right)^{\frac{1}{\beta}}, \tag{10}$$

where $u$is obtained from a uniform random variable on the unit interval $(0,1)$.

3.4 Moments

In this subsection, we derive the expressions for ordinary moments, incomplete moments and the moment generating function.

3.4.1 Ordinary Moments

If X is a continuous random variable with the MOPLx distribution, then the rth moment of X is given by

$$\mu'_r = \alpha\gamma \sum_{k=0}^{\infty} (k+1)(1-\gamma)^k \lambda^{\frac{r}{\beta}} B\left[\frac{r}{\beta}+1, \left(\alpha(k+1)-\frac{r}{\beta}\right)\right], \tag{11}$$

Thus,

$$\mu'_r = \frac{\gamma\alpha\beta}{\lambda} \sum_{k=0}^{\infty} (k+1)(1-\gamma)^k \int_0^{\infty} x^{r+\beta-1}\left(1+\frac{x^\beta}{\lambda}\right)^{-\alpha(k+1)-1} dx.$$

Using the following transformation, $y=\frac{x^\beta}{\lambda} \Rightarrow dx=\lambda^{\frac{1}{\beta}}\frac{1}{\beta}y^{\frac{1}{\beta}-1}dy$, we arrive at

$$\mu'_r = \gamma\alpha \sum_{k=0}^{\infty} (k+1)(1-\gamma)^k \lambda^{\frac{r}{\beta}} \int_0^{\infty} y^{\frac{r}{\beta}} (1+y)^{-\alpha(k+1)-1} dy,$$

Again using the transformation, $y=\frac{w}{1-w} \Rightarrow dy=\frac{dw}{(1-w)^2}$, we have

$$\mu'_r = \gamma\alpha \sum_{k=0}^{\infty} (k+1)(1-\gamma)^k \lambda^{\frac{r}{\beta}} \int_0^{1} w^{\frac{r}{\beta}} (1-w)^{\alpha(k+1)-\frac{r}{\beta}-1} dw$$

and the result follows.

3.4.2 Moment Generating Function

The moment generating function (mgf) of the MOPLx distribution is

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \alpha\gamma \sum_{k=0}^{\infty} (k+1)(1-\gamma)^k \lambda^{\frac{r}{\beta}} B\left[\frac{r}{\beta}+1, \alpha(k+1)-\frac{r}{\beta}\right]$$

The cumulants (k_s) of the MOPLx distribution are attained from the above expression as

$$k_s = \mu'_s - \sum_{k=1}^{s-1} \binom{s-1}{k-1} k_k \mu'_{s-k}, \mu_s = \sum_{k=0}^{s} \binom{s}{k} (-1)^k \mu_1'^k \mu'_{s-k}$$

where $\mu'_1 = k.$ Thus,

$$k_2 = \mu'_2 - (\mu'_1)^2,$$

$$k_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3,$$

$$k_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3(\mu'_2)^2 + 12\mu'_2(\mu'_1)^2 - 6(\mu'_1)^4.$$

The coefficients of skewness and kurtosis for the MOPLx distribution can also be obtained from the third and the fourth standardized cumulants by using formulae $\gamma_1 = \frac{k_3}{k_4^{3/2}}$ and $\gamma_2 = \frac{k_4}{k_2^2}$ respectively.

3.4.3 Incomplete Moments

If X is a continuous random variable with the MOPLx distribution, then the incomplete moment of X is given by

$$\varphi(\mathrm{x}) = \gamma\alpha\lambda^{\frac{r}{\beta}} \sum_{k=0}^{\infty} (k+1)(1-\gamma)^k B\left[\frac{x^\beta}{\lambda+x^\beta}; \frac{r}{\beta}+1, \alpha(k+1)-\frac{r}{\beta}\right].$$

4. Rényi Entropy

If X is a continuous random variable with the MOPLx distribution, then the Rényi Entropy of X is given by

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[\frac{(\gamma\alpha\beta\lambda^\alpha)^\delta \lambda^{\delta\left(1-\frac{1}{\beta}\right)+\frac{1}{\beta}-1}}{\beta} \sum_{k=0}^{\infty} \varphi_k B\left[\delta\left(1-\frac{1}{\beta}\right)+\frac{1}{\beta}, \delta\left(\alpha+\frac{1}{\beta}\right)-\frac{1}{\beta}\right]\right].$$

To obtain such result note that

$$I(\delta)=\int_0^\infty \left(\gamma\alpha\beta\lambda^{\alpha}\right)^{\delta} x^{\delta(\beta-1)}\left(\lambda+x^{\beta}\right)^{-\delta(\alpha+1)}\left[1-(1-\gamma)\lambda^{\alpha}\left(\lambda+x^{\beta}\right)^{-\alpha}\right]^{-2\delta} dx.$$

After simplification, the expression becomes

$$I(\delta)=\left(\gamma\alpha\beta\lambda^{\alpha}\right)^{\delta}\sum_{i,j=0}^{\infty}\varphi_{i,j}\int_0^\infty x^{\delta(\beta-1)}\left(1+\frac{x^{\beta}}{\lambda}\right)^{-\delta(\alpha+1)-\alpha k} dx,$$

Where $$\varphi_k=\frac{\Gamma(k+2\delta)}{\Gamma(2\delta)\Gamma(k+1)}\frac{(1-\lambda)^k\lambda^{\alpha k}}{\lambda^{-\delta(\alpha+1)-\alpha k}}.$$

Hence, we have that

$$I(\delta)=\frac{\left(\gamma\alpha\beta\lambda^{\alpha}\right)^{\delta}\lambda^{\delta\left(1-\frac{1}{\beta}\right)+\frac{1}{\beta}-1}}{\beta}\sum_{k=0}^{\infty}\varphi_k\int_0^1 w^{\delta\left(1-\frac{1}{\beta}\right)+\frac{1}{\beta}-1}(1-w)^{\delta\left(\alpha+\frac{1}{\beta}\right)-\frac{1}{\beta}-1} dw,$$

after some algebra, we have that

$$I(\delta)=\frac{\left(\gamma\alpha\beta\lambda^{\alpha}\right)^{\delta}\lambda^{\delta\left(1-\frac{1}{\beta}\right)+\frac{1}{\beta}-1}}{\beta}\sum_{k=0}^{\infty}\varphi_k B\left[\delta\left(1-\frac{1}{\beta}\right)+\frac{1}{\beta},\delta\left(\alpha+\frac{1}{\beta}\right)-\frac{1}{\beta}\right].$$

Substitution completes the proof.

5. The Probability Weighted Moments

The probability weighted moments can be obtained from the following relation

$$\tau_{r,s}=E[X^rF(x)^s]=\int_{-\infty}^{\infty}x^r f(x)(F(x))^s dx. \tag{12}$$

Substituting (9) and (10) into (12) and replacing h with s, leads to:

$$\tau_{r,s}=\frac{\gamma\alpha\beta}{\lambda}\sum_{i,k=0}^{\infty}w_i(k+1)(1-\gamma)^k\int_0^\infty x^{r+\beta-1}\left(1+\frac{x^{\beta}}{\lambda}\right)^{-\alpha(i+j+k+1)-1} dx.$$

Hence, the PWM of MOPLx distribution takes the following form

$$\tau_{r,s}=\alpha\gamma\sum_{k=0}^{\infty}(k+1)(1-\gamma)^k\lambda^{\frac{r}{\beta}}B\left[\frac{r}{\beta}+1,\alpha(i+j+k+1)-\frac{r}{\beta}\right].$$

5.1 Stress Strength Reliability

In this subsection, we derive the stress strength reliability R when X_1 and X_2 are independent random variables, X_1 follows MOPLx(α_1, λ, β, γ_1) and X_2 follows MOPL(α_2, λ, β, γ_2), then $R=P(X_1<X_2)$ is the measured as;

$$R=P(X_2<X_1)=\int_0^\infty f_1(x)F_2(x)\,dx.$$

Then,

$$R=\int_0^\infty\frac{\gamma 1\alpha 1\beta x^{\beta-1}\left(1+\frac{x^{\beta}}{\lambda}\right)^{-\alpha 1-1}}{\left[1-(1-\gamma 1)\left(1+\frac{x^{\beta}}{\lambda}\right)^{-\alpha 1}\right]^2}\frac{1-\left(1+\frac{x^{\beta}}{\lambda}\right)^{-\alpha 2}}{1-(1-\gamma 2)\left(1+\frac{x^{\beta}}{\lambda}\right)^{-\alpha 2}}dx.$$

Applying the binomial theory, we can rewrite the previous equation as

$$R=\sum_{i,k=0}^{\infty}C_i\int_0^{\infty}\left(x^{\beta-1}\left(1+\frac{x^{\beta}}{\lambda}\right)^{-\alpha 1(k+1)-\alpha 2i-1} ?-x^{\beta-1}\left(\quad+\frac{x^{\beta}}{\lambda}\right)^{-\alpha 1(k+1)-\alpha 2(i+1)-1}\right)dx,$$

where $C_i=\gamma 1\alpha 1\beta i(k+1)(1-\gamma 2)^i(1-\gamma 1)^k$. Then, for the form of R we obtain

$$R=\sum_{i,k=0}^{\infty}\lambda C_i\left(\frac{1}{\alpha 1(k+1)+\alpha 2(i+1)}-\frac{1}{\alpha 1(k+1)+\alpha 2i}\right).$$

6. Order Statistics

The kth order statistics of MOPLx distribution is

$$f_{k:n}(x)=\frac{1}{B(r,n-k+1)}f(x)F(x)^{k-1}[1-F(x)]^{n-k}.$$

The binomial expansion yields

$$f_{k:n}(x)=\frac{f(x)}{B(k,n-k+1)}\sum_{v=0}^{n-k}(-1)^v\binom{n-k}{v}F(x)^{v+k-1}, \tag{13}$$

where, $B(.,.)$ is the beta function. Substituting (9) and (10) in (13) and replacing h with $v+k-1,$ leads to

$$f_{k:n}(x)=\frac{\gamma\alpha\beta}{\lambda B(k,n-k+1)}\sum_{i,k=0}^{\infty}\eta(k+1)(1-\gamma)^k x^{r+\beta-1}\left(1+\frac{x^{\beta}}{\lambda}\right)^{-\alpha(i+j+k+1)-1}, \tag{14}$$

Where $\eta=(-1)^v\binom{n-k}{v}w_i$.

Setting $k=1$ and $k=n,$ in (14), we obtain the pdfs of the first and largest order statistics of the MOPLx distribution. Further, the r^{th} moment of k^{th} order statistics for MOPLx distribution is defined by:

$$E\left(X_{k:n}{}^{r}\right)=\frac{\gamma\alpha}{B(k,n-k+1)}\sum_{k=0}^{\infty}(k+1)(1-\gamma)^k\lambda^{\frac{r}{\beta}}B\left[\frac{r}{\beta}+1,\alpha(i+j+k+1)-\frac{r}{\beta}\right].$$

Lorenz and Bonferroni curves are the most widely used inequality measures in income and wealth distribution (see Kleiber 1999). The Lorenz, Bonferroni and Zenga curves are obtained, respectively, as follows

$$L_F(x)=\frac{\int_0^t xf(x)dx}{E(X)}=\frac{\gamma\alpha\lambda^{\frac{1}{\beta}}\sum_{k=0}^{\infty}(k+1)(1-\gamma)^k B\left[\frac{x^{\beta}}{\lambda+x^{\beta}};\frac{1}{\beta}+1,\alpha(k+1)-\frac{1}{\beta}\right]}{\alpha\gamma\sum_{k=0}^{\infty}(k+1)(1-\gamma)^k\lambda^{\frac{1}{\beta}}B\left[\frac{1}{\beta}+1,\alpha(k+1)-\frac{1}{\beta}\right]},$$

and

$$B_F(x)=\frac{\int_0^t xf(x)dx}{E(X)F(x)}=\frac{L_F(x)}{F(x)},$$

after some algebraic manipulation, we have that

$$B_F(x)=\frac{\gamma\alpha\lambda^{\frac{1}{\beta}}\sum_{k=0}^{\infty}(k+1)(1-\gamma)^k B\left[\frac{x^{\beta}}{\lambda+x^{\beta}};\frac{1}{\beta}+1,\alpha(k+1)-\frac{1}{\beta}\right]}{\alpha\gamma\sum_{k=0}^{\infty}(k+1)(1-\gamma)^k\lambda^{\frac{1}{\beta}}B\left[\frac{1}{\beta}+1,\alpha(k+1)-\frac{1}{\beta}\right]\left(\frac{1-\lambda^{\alpha}\left(\lambda+x^{\beta}\right)^{-\alpha}}{1-(1-\gamma)\left(\lambda^{\alpha}\left(\lambda+x^{\beta}\right)^{-\alpha}\right)}\right)}.$$

7. Characterization

This section deals with the characterizations of the MOPLx distribution based on: (i) a simple relation between two truncated moments; (ii) hazard function and (iii) reverse hazard function. It is worth mentioning that the characterization (i) can be employed when the cdf does not have a closed form. We present the characterizations (i) - (iii) in three subsections.

7.1 Characterizations (i)

This subsection deals with the characterizations of MOPLx distribution based on two truncated moments. For the first characterization we use Theorem 1 of (Glänzel, 1987). The result holds when the interval H is not closed.

Proposition 7.1.1. Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let, $q_1(x)=\left[1-(1-\gamma)\lambda^{\alpha}\left(\lambda+x^{\beta}\right)^{-\alpha}\right]^2$ and $q_2(x)=q_1(x)\left(\lambda+x^{\beta}\right)^{-1}$ for $x>0$. The random variable X has the pdf (6) if and only if the function ζ defined in Theorem 7.1.1. has the form

$$\zeta(x)=\frac{\alpha}{\alpha+1}\left(\lambda+x^{\beta}\right)^{-1},\quad x>0.$$

Proof: Let X be a random variable with pdf (6), then

$$\left(1-F(x)\right)E\left[q_1(x)\middle|X\ge x\right]=\gamma\lambda^{\alpha}\left(\lambda+x^{\beta}\right)^{-\alpha},\ x>0,$$

and

$$\left(1-F(x)\right)E\left[q_2(x)\middle|X\ge x\right]=\frac{\alpha\gamma\lambda^{\alpha}}{\alpha+1}\left(\lambda+x^{\beta}\right)^{-\alpha-1},\ x>0,$$

and

$$\xi(x)q_1(x)-q_2(x)=-\frac{1}{\alpha+1}q_1(x)\left(\lambda+x^{\beta}\right)^{-1}<0,\ for\ x>0.$$

Conversely, if ξ is as above, then

$$s'(x)=\frac{\xi'(x)q_1(x)}{\xi(x)q_1(x)-q_2(x)}=\alpha\beta x^{\beta-1}\left(\lambda+x^{\beta}\right)^{-1},\ x>0,$$

and hence

$$s(x)=-\log\left\{\left(\lambda+x^{\beta}\right)^{-\alpha}\right\},\quad x>0.$$

Now, in view of Theorem 1 of (Glänzel, 1987), X has density (6).

Corollary 7.1.1. The continuous random variable $X:\Omega\to(0,\infty)$ has the pdf (6) if and only if there exist function q_2 and ζ defined in theorem 1 of (Glänzel, 1987)satisfying the following differential equation

$$\frac{\xi'(x)q_1(x)}{\xi(x)q_1(x)-q_2(x)}=\alpha\beta x^{\beta-1}\left(\lambda+x^{\beta}\right)^{-1},\ x>0,$$

The general solution of the differential equation in Corollary 9.1.1 is

$$\xi(x)=x^{\beta-1}\left(\lambda+x^{\beta}\right)^{\alpha}\left[-\int\alpha\beta x^{\beta-1}\left(\lambda+x^{\beta}\right)^{-\alpha-1}\left(q_1(x)\right)^{-1}q_2(x)+D\right],$$

in where D is a constant. For the functions given in Proposition 7.1.1 with D = 0.

7.2 Characterization (ii)

For the hazard function, h_F, of a twice differentiable distribution function, F, we have

$$\frac{f'(x)}{f(x)}=\frac{h'_F(x)}{h_F(x)}-h_F(x).$$

The following proposition presents a non-trivial characterization of MOPL distribution based on the hazard function.

Proposition 7.2.1. The continuous random variable $X:\Omega\to(0,\infty)$ has density on the hazard function.

$$h'_F(x)-\frac{\beta-1}{x}h_F(x)=\alpha\beta x^{\beta-1}\frac{d}{dx}\left\{\frac{\left(\lambda+x^{\beta}\right)^{-1}}{1-(1-\lambda)\lambda^{\alpha}\alpha\beta x^{\beta-1}\left(\lambda+x^{\beta}\right)^{-\alpha}}\right\},\ x>0,$$

with the initial condition $h_F(0)=0$, for $\beta>1$.

Proof. Is straightforward and hence omitted.

Remark 7.2.1. For $\gamma = 1$, the above differential equation has the following simple form

$$h'_F(x)-\frac{\beta-1}{x}h_F(x)=-\alpha\beta^2 x^{2(\beta-1)}\left(\lambda+x^{\beta}\right)^{-2},\ x>0.$$

7.3 Characterization (iii)

The reverse hazard function, r_F, of a twice differentiable distribution function, F, is defined as

$$r_F(x)=\frac{f(x)}{F(x)},\qquad x\in \text{support of F}.$$

Hence, a characterization of the MOPL distribution, for $\gamma=1$, based on the reverse hazard function is stated.

Proposition 7.3.1. The continuous random variable $X:\Omega\rightarrow(0,\infty)$ has density (6) if and only if, for γ = 1, its reverse hazard function $r_F(x)$ is a solution of

$$r'_F(x)-\frac{\beta-1}{x}r_F(x)=\alpha\beta\lambda^{\alpha}x^{\beta-1}\frac{d}{dx}\left\{\frac{\left(\lambda+x^{\beta}\right)^{-\alpha-1}}{1-\lambda^{\alpha}\left(\lambda+x^{\beta}\right)^{-\alpha}}\right\},\ x>0,$$

with the initial condition $r_F(0)=0$, for $\beta>1$.

8. Maximum Likelihood Estimation

8.1 The ML Estimators for Complete Samples

Here, we discuss the estimation of the unknown parameters of the MOPLx distribution by the maximum likelihood method. The log-likelihood function for the vector of parameters $\Theta=(\alpha,\beta,\lambda,\gamma)^T$ can be expressed as

$$l=n\log\left(\gamma\alpha\beta\lambda^{\alpha}\right)+(\beta-1)\sum_{i=1}^{n}\log\left[x_i\right]-(\alpha+1)\sum_{i=1}^{n}\log\left[\lambda+x_i^{\beta}\right]-2\sum_{i=1}^{n}\log\left[1-(1-\gamma)\lambda^{\alpha}\left(\lambda+x_i^{\beta}\right)^{-\alpha}\right].$$

Now differentiate the log-likelihood function with respect to the unknown parameters

$$\frac{\partial\ln L}{\alpha}=\frac{n}{\alpha}+n\log(\lambda)-\sum_{i=1}^{n}\log\left[\lambda+x_i^{\beta}\right]-2\sum_{i=1}^{n}\frac{-(1-\gamma)\lambda^{\alpha}\log[\lambda]\left(\lambda+x_i^{\beta}\right)^{-\alpha}+(1-\gamma)\lambda^{\alpha}\log\left[\lambda+x_i^{\beta}\right]\left(\lambda+x_i^{\beta}\right)^{-\alpha}}{1-(1-\gamma)\lambda^{\alpha}\left(\lambda+x_i^{\beta}\right)^{-\alpha}},$$

$$\frac{\partial\ln L}{\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log\left[x_i\right]-(1+\alpha)\sum_{i=1}^{n}\frac{\log\left[x_i\right]x_i^{\beta}}{\lambda+x_i^{\beta}}-2\sum_{i=1}^{n}\frac{\alpha(1-\gamma)\lambda^{\alpha}\log\left[x_i\right]x_i^{\beta}\left(\lambda+x_i^{\beta}\right)^{-1-\alpha}}{1-(1-\gamma)\lambda^{\alpha}\left(\lambda+x_i^{\beta}\right)^{-\alpha}},$$

$$\frac{\partial\ln L}{\gamma}=\frac{n}{\gamma}-2\sum_{i=1}^{n}\frac{\lambda^{\alpha}\left(\lambda+x_i^{\beta}\right)^{-\alpha}}{1-(1-\gamma)\lambda^{\alpha}\left(\lambda+x_i^{\beta}\right)^{-\alpha}},$$

$$\frac{\partial\ln L}{\lambda}=\frac{n\alpha}{\lambda}-(1+\alpha)\sum_{i=1}^{n}\frac{1}{\lambda+x_i^{\beta}}-2\sum_{i=1}^{n}\frac{\alpha(1-\gamma)\lambda^{\alpha}\left(\lambda+x_i^{\beta}\right)^{-1-\alpha}-\alpha(1-\gamma)\lambda^{-1+\alpha}\left(\lambda+x_i^{\beta}\right)^{-\alpha}}{1-(1-\gamma)\lambda^{\alpha}\left(\lambda+x_i^{\beta}\right)^{-\alpha}}.$$

The estimates of the unknown parameters $\hat{\alpha},\hat{\beta},\hat{\gamma}$ and $\hat{\lambda}$ can be estimated by equating the derived simultaneous equations to zero. Since the structure of the equations is complex therefore, it is hard to get unique solution without iterative procedure. Therefore, we use statistical software to get the estimates numerically and used Newton-Raphson algorithm for this purpose.

8.2 The ML Estimators for Type-II Censored Samples

The occurrence of type-II censoring is very common in many applications in survival analysis, for instance, the lifetime of electronic devices are usually finished after a fixed number of failures r, hence, $n - r$ are be the number of censored data. In this case the logarithm of the likelihood function is given by

$$ln\, L = ln\left(\frac{n!}{(n-r)!}\right) + nln\gamma + rln\alpha + rln\beta + r\alpha ln\lambda + (\beta - 1)\sum_{i=0}^{r} lnx_i - (\alpha+1)\sum_{i=0}^{r} ln\left(\lambda + x_i{}^{\beta}\right)$$
$$-2\sum_{i=0}^{r} ln(1-(1-\gamma)T_i{}^{-\alpha}) \quad -\alpha(n-r)T_r - (n-r)ln(1-(1-\gamma)T_r{}^{-\alpha})\,.$$

Where $T_i = \left(1 + \frac{x_i}{\lambda}\right)$ and $T_r = \left(1 + \frac{x_r}{\lambda}\right)$.

Now differentiate the log-likelihood function with respect to the unknown parameters the likelihood equations are respectively given by

$$U_\alpha = \frac{r}{\alpha} + rln\lambda - \sum_{i=0}^{r} ln\left(\lambda + x_i{}^{\beta}\right) - 2(1-\gamma)\sum_{i=0}^{r}\frac{T_i{}^{-\alpha} lnT_i}{1-(1-\gamma)T_i{}^{-\alpha}} - (n-r)lnT_r - (n-r)(1-\gamma)\frac{T_r{}^{-\alpha} lnT_r}{1-(1-\gamma)T_r{}^{-\alpha}},$$

$$U_\beta = \frac{r}{\beta} + \sum_{i=0}^{r} lnx_i - (\alpha+1)\sum_{i=0}^{r}\frac{x_i{}^{\beta} lnx_i}{\lambda + x_i{}^{\beta}} - 2\alpha(1-\gamma)\sum_{i=0}^{r}\frac{T_i{}^{-\alpha-1} x_i{}^{\beta} lnx_i}{\lambda(1-(1-\gamma)T_i{}^{-\alpha})} - \alpha(n-r)\frac{x_r{}^{\beta} lnx_r}{\lambda T_r}$$
$$-\alpha(n-r)(1-\gamma)\frac{T_r{}^{-\alpha-1} x_r{}^{\beta} lnx_r}{\lambda(1-(1-\gamma)T_r{}^{-\alpha})},$$

$$U_\lambda = \frac{r\alpha}{\lambda} - (\alpha+1)\sum_{i=0}^{r}\frac{1}{\lambda + x_i{}^{\beta}} + 2\alpha(1-\gamma)\sum_{i=0}^{r}\frac{T_i{}^{-\alpha-1} x_i{}^{\beta}}{\lambda^2(1-(1-\gamma)T_i{}^{-\alpha})} - \alpha(n-r)\frac{x_r{}^{\beta}}{\lambda^2 T_r} - \alpha(n-r)(1-\gamma)\frac{T_r{}^{-\alpha-1} x_r{}^{\beta} lnx_r}{\lambda^2(1-(1-\gamma)T_r{}^{-\alpha})},$$

$$U_\gamma = \frac{n}{\gamma} - 2\sum_{i=0}^{r}\frac{T_i{}^{-\alpha}}{1-(1-\gamma)T_i{}^{-\alpha}} - (n-r)\frac{T_r{}^{-\alpha} x_r{}^{\beta} lnx_r}{1-(1-\gamma)T_r{}^{-\alpha}}.$$

8.3 Simulation Study

In this section, we present a simulation study for the purpose of estimation of unknown parameters of MOPLx distribution via maximum likelihood method for complete and censored samples. The algorithm for estimation used here is designed as follows;

- Random samples of sizes $n = 30, 50, 100$ and 300 are generated form MOPLx distribution under complete and type II censored samples.
- We generate the samples using the following parametric space;
  - $\alpha = 0.5, \beta = .5, \lambda = .5, \gamma = .5$
  - $\alpha = 0.5, \beta = .0, \lambda = .5, \gamma = .5$
  - $\alpha = 1.5, \beta = .0, \lambda = .5, \gamma = .5$
  - $\alpha = 1.5, \beta = .0, \lambda = .5, \gamma = .5$
- For type-II censoring we censored 20% and 50% observations.
- We repeat this process 10,000 times and estimate the parameters.
- Simulation results are presented in Table 1-3.

The evaluation is done based on two criteria: the empirical bias and the estimated *Mean Square Error (MSE) of the estimated parameters computed from the maximum likelihood estimation. The estimated results are shown in Table 1-3.*

It should be noted that the performance of estimators get better and converge to the true parameter values as sample size increases. Overall the estimates under uncensored data are better than those under censored samples because the MSE, for same sample size, under uncensored data set has minimum values than MSE under censored data.

Table 1. Average Estimates, absolute Bias and MSE

$(\alpha=0.5,\beta=0.5,\lambda=0.5,\gamma=0.5)$						$(\alpha=0.5,\beta=0.5,\lambda=0.5,\gamma=1.5)$					
Scheme			Estimates	Bias	MSE	Scheme			Estimates	Bias	MSE
30	0%	$\hat{\alpha}$	0.51104	0.01104	0.00682	30	0%	$\hat{\alpha}$	0.52176	0.02176	0.01292
		$\hat{\beta}$	0.52757	0.02757	0.01744			$\hat{\beta}$	0.52560	0.02560	0.01561
		$\hat{\gamma}$	0.50687	0.00687	0.00314			$\hat{\gamma}$	1.51789	0.01789	0.02833
		$\hat{\lambda}$	0.51906	0.01906	0.01042			$\hat{\lambda}$	0.51305	0.01305	0.00732
	20%	$\hat{\alpha}$	0.58941	0.08941	0.01737		20%	$\hat{\alpha}$	0.60239	0.10239	0.02801
		$\hat{\beta}$	0.67353	0.17353	0.07105			$\hat{\beta}$	0.66298	0.16298	0.05698
		$\hat{\gamma}$	0.63695	0.13695	0.02647			$\hat{\gamma}$	1.90651	0.40651	0.23559
		$\hat{\lambda}$	0.66147	0.16147	0.05523			$\hat{\lambda}$	0.64915	0.14915	0.04157
	50%	$\hat{\alpha}$	0.76504	0.26504	0.08828		50%	$\hat{\alpha}$	0.77421	0.27421	0.10141
		$\hat{\beta}$	1.18924	0.68924	0.93011			$\hat{\beta}$	1.09802	0.59802	0.49866
		$\hat{\gamma}$	1.04534	0.54534	0.35863			$\hat{\gamma}$	3.13534	1.63534	3.22188
		$\hat{\lambda}$	1.18111	0.68111	2.18177			$\hat{\lambda}$	1.13235	0.63235	0.87580
50	0%	$\hat{\alpha}$	0.50731	0.00731	0.00383	50	0%	$\hat{\alpha}$	0.51311	0.01311	0.00691
		$\hat{\beta}$	0.51583	0.01583	0.00893			$\hat{\beta}$	0.51497	0.01497	0.00825
		$\hat{\gamma}$	0.50346	0.00346	0.00175			$\hat{\gamma}$	1.50887	0.00887	0.01584
		$\hat{\lambda}$	0.50989	0.00989	0.00541			$\hat{\lambda}$	0.50680	0.00680	0.00399
	20%	$\hat{\alpha}$	0.58393	0.08393	0.01243		20%	$\hat{\alpha}$	0.58862	0.08862	0.01714
		$\hat{\beta}$	0.65602	0.15602	0.04478			$\hat{\beta}$	0.64646	0.14646	0.03735
		$\hat{\gamma}$	0.63225	0.13225	0.02194			$\hat{\gamma}$	1.89812	0.39812	0.19821
		$\hat{\lambda}$	0.64640	0.14640	0.03605			$\hat{\lambda}$	0.64137	0.14137	0.03041
	50%	$\hat{\alpha}$	0.75701	0.25701	0.07651		50%	$\hat{\alpha}$	0.76321	0.26321	0.08484
		$\hat{\beta}$	1.09547	0.59547	0.49762			$\hat{\beta}$	1.05882	0.55882	0.38101
		$\hat{\gamma}$	1.02495	0.52495	0.30662			$\hat{\gamma}$	3.07409	1.57409	2.74599
		$\hat{\lambda}$	1.09619	0.59619	0.57072			$\hat{\lambda}$	1.06604	0.56604	0.41768
100	0%	$\hat{\alpha}$	0.50374	0.00374	0.00187	100	0%	$\hat{\alpha}$	0.50702	0.00702	0.00335
		$\hat{\beta}$	0.50660	0.00660	0.00412			$\hat{\beta}$	0.50826	0.00826	0.00385
		$\hat{\gamma}$	0.50153	0.00153	0.00087			$\hat{\gamma}$	1.50478	0.00478	0.00778
		$\hat{\lambda}$	0.50446	0.00446	0.00258			$\hat{\lambda}$	0.50366	0.00366	0.00191
	20%	$\hat{\alpha}$	0.57929	0.07929	0.00890		20%	$\hat{\alpha}$	0.58309	0.08309	0.01142
		$\hat{\beta}$	0.63821	0.13821	0.02781			$\hat{\beta}$	0.63689	0.13689	0.02630
		$\hat{\gamma}$	0.62859	0.12859	0.01869			$\hat{\gamma}$	1.88447	0.38447	0.16639
		$\hat{\lambda}$	0.63509	0.13509	0.02485			$\hat{\lambda}$	0.63222	0.13222	0.02217
	50%	$\hat{\alpha}$	0.74932	0.24932	0.06723		50%	$\hat{\alpha}$	0.75336	0.25336	0.07168
		$\hat{\beta}$	1.04236	0.54236	0.34493			$\hat{\beta}$	1.02857	0.52857	0.30949
		$\hat{\gamma}$	1.01218	0.51218	0.27551			$\hat{\gamma}$	3.03357	1.53357	2.46947
		$\hat{\lambda}$	1.04074	0.54074	0.34341			$\hat{\lambda}$	1.02917	0.52917	0.31544
300	0%	$\hat{\alpha}$	0.50131	0.00131	0.00061	300	0%	$\hat{\alpha}$	0.50213	0.00213	0.00108
		$\hat{\beta}$	0.50226	0.00226	0.00126			$\hat{\beta}$	0.50244	0.00244	0.00118
		$\hat{\gamma}$	0.50041	0.00041	0.00028			$\hat{\gamma}$	1.50173	0.00173	0.00261
		$\hat{\lambda}$	0.50128	0.00128	0.00079			$\hat{\lambda}$	0.50124	0.00124	0.00063
	20%	$\hat{\alpha}$	0.57664	0.07664	0.00673		20%	$\hat{\alpha}$	0.57800	0.07800	0.00759
		$\hat{\beta}$	0.62969	0.12969	0.01959			$\hat{\beta}$	0.62934	0.12934	0.01913
		$\hat{\gamma}$	0.62613	0.12613	0.01660			$\hat{\gamma}$	1.87799	0.37799	0.14915
		$\hat{\lambda}$	0.62825	0.12825	0.01845			$\hat{\lambda}$	0.62730	0.12730	0.01774
	50%	$\hat{\alpha}$	0.74506	0.24506	0.06170		50%	$\hat{\alpha}$	0.74656	0.24656	0.06331
		$\hat{\beta}$	1.01485	0.51485	0.27982			$\hat{\beta}$	1.01013	0.51013	0.26962
		$\hat{\gamma}$	1.00372	0.50372	0.25784			$\hat{\gamma}$	3.00988	1.50988	2.31722
		$\hat{\lambda}$	1.01235	0.51235	0.27614			$\hat{\lambda}$	1.00868	0.50868	0.26902

Table 2. Average Estimates, absolute Bias and MSE

$(\alpha = 0.5, \beta = 1.0, \lambda = 0.5, \gamma = 0.5)$						$(\alpha = 1.5, \beta = 1.0, \lambda = 1.5, \gamma = 1.5)$					
Scheme			Estimates	Bias	MSE	Scheme			Estimates	Bias	MSE
30	0%	$\hat{\alpha}$	0.51225	0.01225	0.00680	30	0%	$\hat{\alpha}$	1.56196	0.06196	0.11520
		$\hat{\beta}$	1.05171	0.05171	0.06839			$\hat{\beta}$	1.05357	0.05357	0.07075
		$\hat{\gamma}$	0.50608	0.00607	0.00310			$\hat{\gamma}$	1.51819	0.01819	0.02798
		$\hat{\lambda}$	0.51751	0.01751	0.01026			$\hat{\lambda}$	1.51122	0.01122	0.01640
	20%	$\hat{\alpha}$	0.58962	0.08961	0.01726		20%	$\hat{\alpha}$	2.38103	0.88103	1.27093
		$\hat{\beta}$	1.34498	0.34498	0.27219			$\hat{\beta}$	1.33381	0.33381	0.24225
		$\hat{\gamma}$	0.63684	0.13684	0.02640			$\hat{\gamma}$	1.90953	0.40953	0.23717
		$\hat{\lambda}$	0.66133	0.16133	0.05541			$\hat{\lambda}$	1.89541	0.39541	0.19562
	50%	$\hat{\alpha}$	0.76623	0.26623	0.08878		50%	$\hat{\alpha}$	5.30389	3.80389	96248.5
		$\hat{\beta}$	2.31203	1.31203	38.9622			$\hat{\beta}$	2.19571	1.19571	1.95728
		$\hat{\gamma}$	1.04455	0.54455	0.36125			$\hat{\gamma}$	3.14016	1.64016	3.22548
		$\hat{\lambda}$	1.26053	0.76053	37.6338			$\hat{\lambda}$	3.07383	1.57383	2.72584
50	0%	$\hat{\alpha}$	0.50713	0.00713	0.00381	50	0%	$\hat{\alpha}$	1.54138	0.04138	0.06352
		$\hat{\beta}$	1.03129	0.03129	0.03601			$\hat{\beta}$	1.03368	0.03368	0.03704
		$\hat{\gamma}$	0.50344	0.00344	0.00179			$\hat{\gamma}$	1.50895	0.00895	0.01595
		$\hat{\lambda}$	0.50999	0.00998	0.00553			$\hat{\lambda}$	1.50506	0.00506	0.00945
	20%	$\hat{\alpha}$	0.58380	0.08380	0.01242		20%	$\hat{\alpha}$	2.31791	0.81791	0.92839
		$\hat{\beta}$	1.30591	0.30591	0.17276			$\hat{\beta}$	1.29438	0.29438	0.15477
		$\hat{\gamma}$	0.63207	0.13207	0.02183			$\hat{\gamma}$	1.89062	0.39062	0.19174
		$\hat{\lambda}$	0.64545	0.14545	0.03540			$\hat{\lambda}$	1.88372	0.38372	0.16977
	50%	$\hat{\alpha}$	0.75515	0.25515	0.07529		50%	$\hat{\alpha}$	9.34680	7.84680	67797.3
		$\hat{\beta}$	2.19218	1.19218	1.98401			$\hat{\beta}$	2.10481	1.10481	1.47961
		$\hat{\gamma}$	1.02631	0.52631	0.30645			$\hat{\gamma}$	3.07059	1.57059	2.73022
		$\hat{\lambda}$	1.10933	0.60933	1.29192			$\hat{\lambda}$	3.03693	1.53693	2.49545
100	0%	$\hat{\alpha}$	0.50256	0.00256	0.00184	100	0%	$\hat{\alpha}$	1.52058	0.02058	0.03020
		$\hat{\beta}$	1.01408	0.01408	0.01586			$\hat{\beta}$	1.01870	0.01870	0.01773
		$\hat{\gamma}$	0.50232	0.00232	0.00086			$\hat{\gamma}$	1.50509	0.00509	0.00776
		$\hat{\lambda}$	0.50568	0.00568	0.00252			$\hat{\lambda}$	1.50286	0.00286	0.00463
	20%	$\hat{\alpha}$	0.57954	0.07954	0.00896		20%	$\hat{\alpha}$	2.25810	0.75810	0.69367
		$\hat{\beta}$	1.27344	0.27344	0.11030			$\hat{\beta}$	1.27511	0.27511	0.10697
		$\hat{\gamma}$	0.62823	0.12823	0.01855			$\hat{\gamma}$	1.88701	0.38701	0.16888
		$\hat{\lambda}$	0.63416	0.13416	0.02434			$\hat{\lambda}$	1.88233	0.38233	0.15742
	50%	$\hat{\alpha}$	0.74888	0.24888	0.06685		50%	$\hat{\alpha}$	7.71486	6.21486	72813.5
		$\hat{\beta}$	2.08498	1.08498	1.38057			$\hat{\beta}$	2.04946	1.04946	1.21119
		$\hat{\gamma}$	1.01227	0.51227	0.27532			$\hat{\gamma}$	3.03594	1.53594	2.47640
		$\hat{\lambda}$	1.04042	0.54042	0.34251			$\hat{\lambda}$	3.01962	1.51962	2.37152
300	0%	$\hat{\alpha}$	0.50114	0.00114	0.00061	300	0%	$\hat{\alpha}$	1.50672	0.00672	0.00971
		$\hat{\beta}$	1.00499	0.00499	0.00522			$\hat{\beta}$	1.00554	0.00554	0.00529
		$\hat{\gamma}$	0.50056	0.00056	0.00028			$\hat{\gamma}$	1.50159	0.00159	0.00261
		$\hat{\lambda}$	0.50160	0.00160	0.00080			$\hat{\lambda}$	1.50091	0.00091	0.00157
	20%	$\hat{\alpha}$	0.57668	0.07668	0.00674		20%	$\hat{\alpha}$	2.23018	0.73018	0.56911
		$\hat{\beta}$	1.25879	0.25879	0.07854			$\hat{\beta}$	1.25824	0.25824	0.07630
		$\hat{\gamma}$	0.62602	0.12602	0.01657			$\hat{\gamma}$	1.87703	0.37703	0.14826
		$\hat{\lambda}$	0.62798	0.12798	0.01837			$\hat{\lambda}$	1.87596	0.37596	0.14494
	50%	$\hat{\alpha}$	0.74507	0.24507	0.06169		50%	$\hat{\alpha}$	8.16368	6.66368	49.4472
		$\hat{\beta}$	2.02545	1.02545	1.11010			$\hat{\beta}$	2.02014	1.02014	1.07601
		$\hat{\gamma}$	1.00353	0.50353	0.25758			$\hat{\gamma}$	3.00986	1.50986	2.31658
		$\hat{\lambda}$	1.01175	0.51176	0.27519			$\hat{\lambda}$	3.00474	1.50474	2.28431

Table 3. Average Estimates, absolute Bias and MSE

$(\alpha=1.5, \beta=1.5, \lambda=1.5, \gamma=1.5)$						$(\alpha=1.5, \beta=2.0, \lambda=1.5, \gamma=1.5)$					
Scheme			Estimates	Bias	MSE	Scheme			Estimates	Bias	MSE
30	0%	$\hat{\alpha}$	1.56381	0.06381	0.11292	30	0%	$\hat{\alpha}$	1.56765	0.06765	0.11651
		$\hat{\beta}$	1.59316	0.09316	0.16978			$\hat{\beta}$	2.11379	0.11379	0.28091
		$\hat{\gamma}$	1.51821	0.01821	0.02750			$\hat{\gamma}$	1.51621	0.01621	0.02790
		$\hat{\lambda}$	1.51082	0.01082	0.01601			$\hat{\lambda}$	1.50942	0.00942	0.01625
	20%	$\hat{\alpha}$	2.39566	0.89566	1.33681		20%	$\hat{\alpha}$	2.38617	0.88617	1.30414
		$\hat{\beta}$	1.99582	0.49582	0.52410			$\hat{\beta}$	2.65726	0.65726	0.93033
		$\hat{\gamma}$	1.90875	0.40875	0.23893			$\hat{\gamma}$	1.90867	0.40867	0.23742
		$\hat{\lambda}$	1.89455	0.39455	0.19633			$\hat{\lambda}$	1.89483	0.39483	0.19567
	50%	$\hat{\alpha}$	3.13762	1.63762	604013.		50%	$\hat{\alpha}$	4.32377	2.82377	478724.
		$\hat{\beta}$	3.26336	1.76336	4.21266			$\hat{\beta}$	4.38404	2.38404	7.72803
		$\hat{\gamma}$	3.13020	1.63020	3.20415			$\hat{\gamma}$	3.13539	1.63539	3.22149
		$\hat{\lambda}$	3.06706	1.56706	2.70376			$\hat{\lambda}$	3.07043	1.57043	2.71411
50	0%	$\hat{\alpha}$	1.53479	0.03479	0.06471	50	0%	$\hat{\alpha}$	1.53968	0.03968	0.06462
		$\hat{\beta}$	1.54467	0.04467	0.08378			$\hat{\beta}$	2.06516	0.06516	0.15165
		$\hat{\gamma}$	1.51228	0.01228	0.01640			$\hat{\gamma}$	1.50905	0.00905	0.01595
		$\hat{\lambda}$	1.50785	0.00785	0.00972			$\hat{\lambda}$	1.50529	0.00529	0.00947
	20%	$\hat{\alpha}$	2.30656	0.80656	0.91384		20%	$\hat{\alpha}$	2.31206	0.81206	0.91818
		$\hat{\beta}$	1.94187	0.44187	0.34426			$\hat{\beta}$	2.59418	0.59418	0.61660
		$\hat{\gamma}$	1.89642	0.39642	0.19647			$\hat{\gamma}$	1.89426	0.39426	0.19410
		$\hat{\lambda}$	1.88810	0.38811	0.17335			$\hat{\lambda}$	1.88623	0.38623	0.17148
	50%	$\hat{\alpha}$	5.73117	4.23117	812208.		50%	$\hat{\alpha}$	9.38975	7.88975	61973.1
		$\hat{\beta}$	3.17665	1.67665	3.40718			$\hat{\beta}$	4.20188	2.20188	5.84133
		$\hat{\gamma}$	3.07196	1.57196	2.74207			$\hat{\gamma}$	3.07362	1.57362	2.74766
		$\hat{\lambda}$	3.03704	1.53704	2.49827			$\hat{\lambda}$	3.03882	1.53882	2.50424
100	0%	$\hat{\alpha}$	1.51818	0.01818	0.02955	100	0%	$\hat{\alpha}$	1.51892	0.01892	0.02999
		$\hat{\beta}$	1.52328	0.02328	0.03903			$\hat{\beta}$	2.03008	0.03008	0.06864
		$\hat{\gamma}$	1.50482	0.00482	0.00766			$\hat{\gamma}$	1.50432	0.00432	0.00765
		$\hat{\lambda}$	1.50291	0.00291	0.00457			$\hat{\lambda}$	1.50259	0.00259	0.00457
	20%	$\hat{\alpha}$	2.26037	0.76037	0.69579		20%	$\hat{\alpha}$	2.25829	0.75830	0.69192
		$\hat{\beta}$	1.91292	0.41292	0.24076			$\hat{\beta}$	2.54684	0.54684	0.42569
		$\hat{\gamma}$	1.88582	0.38582	0.16789			$\hat{\gamma}$	1.88593	0.38593	0.16793
		$\hat{\lambda}$	1.88139	0.38140	0.15661			$\hat{\lambda}$	1.88163	0.38163	0.15677
	50%	$\hat{\alpha}$	11.4003	9.90029	35052.1		50%	$\hat{\alpha}$	10.3484	8.84842	8712.00
		$\hat{\beta}$	3.08015	1.58015	2.75358			$\hat{\beta}$	4.10132	2.10132	4.86114
		$\hat{\gamma}$	3.03257	1.53257	2.46513			$\hat{\gamma}$	3.03101	1.53101	2.45707
		$\hat{\lambda}$	3.01673	1.51673	2.36215			$\hat{\lambda}$	3.01604	1.51604	2.35867
300	0%	$\hat{\alpha}$	1.50776	0.00776	0.00949	300	0%	$\hat{\alpha}$	1.50586	0.00586	0.00949
		$\hat{\beta}$	1.50910	0.00910	0.01209			$\hat{\beta}$	2.00992	0.00992	0.02134
		$\hat{\gamma}$	1.50109	0.00109	0.00255			$\hat{\gamma}$	1.50181	0.00181	0.00254
		$\hat{\lambda}$	1.50051	0.00051	0.00153			$\hat{\lambda}$	1.50114	0.00114	0.00153
	20%	$\hat{\alpha}$	2.22586	0.72586	0.56362		20%	$\hat{\alpha}$	2.22774	0.72774	0.56558
		$\hat{\beta}$	1.88789	0.38789	0.17285			$\hat{\beta}$	2.51489	0.51489	0.30425
		$\hat{\gamma}$	1.87937	0.37937	0.15014			$\hat{\gamma}$	1.87795	0.37795	0.14893
		$\hat{\lambda}$	1.87771	0.37771	0.14634			$\hat{\lambda}$	1.87670	0.37670	0.14550
	50%	$\hat{\alpha}$	8.12542	6.62542	49.0549		50%	$\hat{\alpha}$	8.12523	6.62523	48.5332
		$\hat{\beta}$	3.01747	1.51747	2.37920			$\hat{\beta}$	4.03240	2.03240	4.26856
		$\hat{\gamma}$	3.01117	1.51117	2.32077			$\hat{\gamma}$	3.00918	1.50918	2.31264
		$\hat{\lambda}$	3.00620	1.50620	2.28885			$\hat{\lambda}$	3.00462	1.50462	2.28297

9. Applications

In this section, a real life data set is used to illustrate the usefulness of the derived model. For this purpose, we utilize the data set belongs to the remission time of cancer patients. The data consist of 128 observations and earlier studied by Lee and Wang (2003).

We utilize the MLE method to observe the goodness of fit for the MOPLx distribution and compare the proposed distribution with Weibull-Lomax (WLx), transmuted Lomax (TLx), exponential-Lomax (ELx), beta-Lomax (BLx), Lomax (Lx), beta-exponential (BE), Marshall-Olkin length exponential (MOLBE) and Marshall-Olkin extended exponential (MOEE) distribution. The selection of model is based on AIC (Akaike information criterion), the BIC (Bayesian information criterion). Furthermore, we also consider Anderson and Darling (A*), Cramér-von Mises (W*) and Kolmogorov–Smirnov (D) statistics. In general the smaller values of these statistics indicate the better fit to the data. The test statistics are

$$AIC = -2L\left(\hat{\theta}\right) + 2q \text{ and } BIC = -2L\left(\hat{\theta}\right) + 2q\log(n)$$

$$A^2 = -n - \sum_{i=1}^{n} \frac{2i-1}{n}\left[\ln\left(F\left(x_i, \hat{\gamma}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}\right)\right) + \ln\left(1 - F\left(x_i, \hat{\gamma}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}\right)\right)\right]$$

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left(F\left(x_i, \hat{\gamma}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}\right) - \frac{2i-1}{n}\right)^2$$

where L is log-likelihood function and q is the number of parameters.

The Maximum Likelihood Estimates along with the goodness of fit measures are presented in Table (5). The numerical results are computed using R software.

Table 4. Basic Descriptive measures of the data set

X_{min}	Q_1	Median	Mean	Q_3	X_{max}	S^2
0.080	3.348	6.395	9.366	11.838	79.050	110.425

The descriptive statistics are presented in Table 4 which shows that data set is positively skewed. The MLEs and goodness of fit for considered distributions are reported in Table 5. The MOPLx distribution provided the best fit among the chosen models. Figure 2 provides the estimated pdf and cdf superimposed on the histogram of dataset. This fitted densities support the findings presented in Table 5. Figure 3 illustrate the plots of profile-log likelihood functions of the fitted MOPLx distribution.

Table 5. The MLEs of the model parameters and Goodness of fit Measures for data set

Model	MLE	-logL	AIC	BIC	A^*	W^*
MOPLx	$\hat{\alpha} = 2.16340$ $\hat{\beta} = 1.05562$ $\hat{\gamma} = 17.9449$ $\hat{\lambda} = 2.41605$	**409.537**	**827.075**	**832.483**	**0.09337**	**0.01517**
WLx	$\hat{a} = 1343.77$ $\hat{b} = 1.51086$ $\hat{\alpha} = 0.01216$ $\hat{\beta} = 8.66117$	409.992	827.983	839.391	0.17834	0.02643
TLx	$\hat{\alpha} = 4.02162$ $\hat{\lambda} = 19.0931$ $\hat{\theta} = -0.8443$	410.826	830.252	838.808	0.14592	0.19505
Lx	$\hat{\alpha} = 13.9385$ $\hat{\lambda} = 121.023$	413.833	831.666	837.37	1.37678	0.21259
ELx	$\hat{\alpha} = 4.58570$ $\hat{\lambda} = 24.7414$ $\hat{\theta} = 1.58619$	410.072	826.144	834.70	0.17979	0.02625
BLx	$\hat{a} = 1.58582$ $\hat{b} = 24.2398$ $\hat{\alpha} = 0.19390$ $\hat{\beta} = 20.5900$	410.081	824.163	829.867	0.18184	0.02660
BE	$\hat{\lambda} = 0.64554$ $\hat{a} = 1.44850$ $\hat{b} = 0.17919$	412.344	830.688	839.244	0.55805	0.09945
MOLBE	$\hat{\alpha} = 0.08458$ $\hat{\beta} = 13.2996$	411.957	827.915	833.619	0.41090	0.03132
MOEE	$\hat{\beta} = 0.10986$ $\hat{\gamma} = 1.30301$	414.326	832.652	838.356	1.11321	0.17009

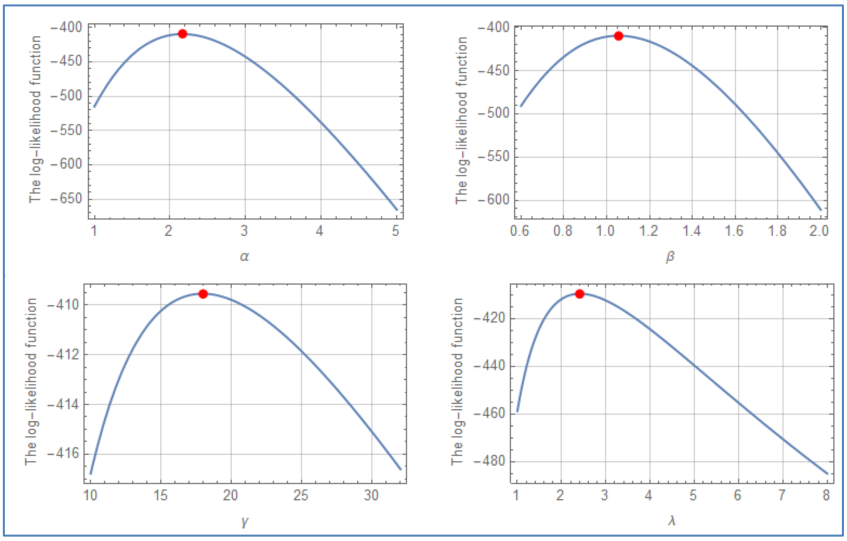

Figure 2. Profile log-likelihood function of the MOPLx for data set

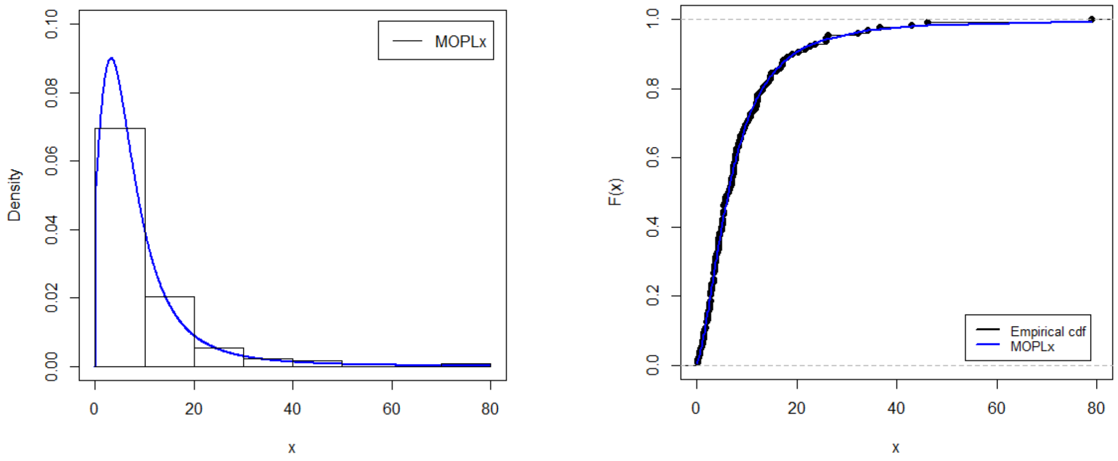

Figure 3. Fitted PDF and CDF of the MOPLx for the data set

10. Conclusion

In this work, we derive and study a four parameter distribution called Marshall-Olkin Power Lomax (MOPLx) distribution. The proposed distribution is derived using the generator approach by Marshall and Olkin (1997). We study some of its statistical properties including, moments, moment generating function, incomplete moments, mean residual life, mean activity time, expressions of the order statistics. The estimation of parameters is computed by the method of maximum likelihood for complete and type II censored data. A comprehensive simulation study is used to evaluate the proposed estimators. A real life data set is used to illustrate the usefulness of the proposed distribution. In conclusion, the MOPLx distribution provides a better fit and can be consider a good model for skewed dataset.

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