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Research article

On the Burr XII-Power Cauchy distribution: Properties and applications

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Abstract: We propose a new four-parameter lifetime model with flexible hazard rate called the Burr XII Power Cauchy (BXII-PC) distribution. We derive the BXII-PC distribution via (i) the T-X family technique and (ii) nexus between the exponential and gamma variables. The new proposed distribution is flexible as it has famous sub-models such as Burr XII-half Cauchy, Lomax-power Cauchy, Lomax-half Cauchy, Log-logistic-power Cauchy, log-logistic-half Cauchy. The failure rate function for the BXII-PC distribution is flexible as it can accommodate various shapes such as the modified bathtub, inverted bathtub, increasing, decreasing; increasing-decreasing and decreasing-increasing-decreasing. Its density function can take shapes such as exponential, J, reverse-J, left-skewed, right-skewed and symmetrical. To illustrate the importance of the BXII-PC distribution, we establish various mathematical properties such as random number generator, moments, inequality measures, reliability measures and characterization. Six estimation methods are used to estimate the unknown parameters of the proposed distribution. We perform a simulation study on the basis of the graphical results to demonstrate the performance of the maximum likelihood, maximum product spacings, least squares, weighted least squares, Cramer-von Mises and Anderson-Darling estimators of the parameters of the BXII-PC distribution. We consider an application to a real data set to prove empirically the potentiality of the proposed model.

Keywords: moments; inequality measures; residual life functions; reliability; maximum likelihood estimation
Mathematics Subject Classification: 60E05, 62E15, 62F10

1. Introduction

Data analysis is imperative in every aspect of statistical analysis. The statistical characteristics such as skewness, kurtosis, bimodality, monotonic and non-monotonic failure rates are obtained from datasets. The selection of a suitable model for data analysis is challenging task because it depends on the nature of the dataset. However, if a wrong model is applied to analyze the dataset it leads to loss of information and invalid inferences. It is necessary to search and identify the most suitable model for the given dataset.

In the recent decade, many continuous distributions have been introduced in statistical literature. Some of these distributions, however, are not flexible enough for data sets from survival analysis, life testing, reliability, finance, environmental sciences, biometry, hydrology, ecology and geology. Hence, the applications of the generalized models to these fields are clear requisite. The generalization techniques such as either inserting one or more shape parameters or transform of the parent distribution are useful to (i) increase the applicability of a parent distribution; (ii) explore skewness and tail properties and (iii) improve the goodness-of-fit of the generalized distributions. The Cauchy distribution is the ratio of two independent normal variables if the denominator variable has mean zero. The Cauchy distribution has wide applications in stochastic modeling of decreasing failure rate life components, clinical trials and finance risks.

During the recent years, the Cauchy distribution has been shown great interest in literature such generalized Cauchy [1], truncated Cauchy [2–4], beta-Cauchy [5–7], Marshall-Olkin half Cauchy [8], beta-half Cauchy [9], Kumaraswamy-half Cauchy [10], Weibull power Cauchy [11] and modified skew-normal-Cauchy distribution [12].

The Burr-XII (BXII) distribution among Burr family [13] is widely applied to model insurance data and failure time data. Many generalizations of the BXII distributions are available in the literature such as Burr XII power series [14], generalized Burr XII power series [15], Burr XII system of densities [16], Burr XII inverse Rayleigh [17] and Burr XII- moment exponential [18].

The idea is to incorporate Cauchy distribution into a larger family through an application of the Burr XII cdf. In fact, based on the T-X transform defined by [19], we construct the BXII-PC distribution. The new model has flexible shapes to model various lifetime data sets. The moments of the Cauchy distribution do not exist, but the BXII-PC distribution has moments. Additionally, its special models produce better fits than other well-known models.

This study is based on the following motivations: (i) to generate distributions with symmetrical, left-skewed, right-skewed, J and reverse-J shaped as well as high kurtosis; (ii) to have monotone and non-monotone failure rate function; (iii) to derive mathematical properties such as ordinary moments, incomplete moments, inequality measures, conditional moments, reliability measures and characterization; (iv) to estimate the precision of the maximum likelihood, maximum product spacings, least squares,weighted least squares, Cramer-von Mises and Anderson-Darling estimators by means of Monte Carlo simulations; (v) to reveal the potentiality of the BXII-PC model; (vi) to deliver better fits than other models and (vii) to infer empirically.

The content of the article is structured as follows. Section 2 derives the BXII-PC model from (i) the T-X family technique and (ii) linking the exponential and gamma variables. We study basic structural properties, random number generator and sub-models for the BXII-PC model. Section 3 presents certain mathematical properties such as ordinary moments, incomplete moments, inequality measures, conditional moments, reliability measures and characterization. Section 4 is devoted to parameter estimation methods. Section 5 presents simulation studies on the basis of graphical results to see the performance of maximum likelihood, maximum product spacings, least squares, weighted least squares, Cramer-von Mises and Anderson-Darling estimators of the BXII-PC distribution. In Section 6, we consider an application to illustrate the potentiality and utility of the BXII-PC model. We test the competency of the BXII-PC model via various model selection criteria. In Section 7, we offer some conclusions.

2. The BXII-PC distribution

We derive the BXII-PC distribution from the T-X family technique. We also obtain this model by linking the exponential and gamma variables. We discuss basic structural properties. We highlight the shapes of the density and failure rate functions.

2.1. T-X family technique

The cumulative distribution function (cdf) and probability density function (pdf) of the Power-Cauchy distribution [20] are given, respectively, by

$$G(x) = \frac{2}{\pi}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right), \kappa, \theta > 0, x \geq 0$$

and

$$g(x) = \frac{2}{\pi}\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[1+\left(\frac{x}{\theta}\right)^{2\kappa}\right]^{-1}, x > 0.$$

The cumulative hazard rate function of the Power Cauchy distribution is

$$W[G(x)] = -\log[1-G(x)] = -\log\left[1-\frac{2}{\pi}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right].$$

The cdf of the T-X family [19] of distributions has the form

$$F(x) = \int_{a}^{W[G(x;\xi)]} r(t)dt, \qquad x \in \mathbb{R}, \tag{2.1}$$

where $r(t)$ is the pdf of the random variable (rv) T, where $T \in [a, \mathrm{b}]$ for $-\infty \leq a < \mathrm{b} < \infty$ and $W[G(x;\xi)]$ is a function of the baseline cdf of a rv X with the vector parameter ξ, which satisfies the conditions:

i) $W[G(x;\xi)] \in [a, \mathrm{b}]$, ii) $W[G(x;\xi)]$ is differentiable and monotonically non-decreasing and iii) $\lim_{x\to-\infty} W[G(x;\xi)] = a$ and $\lim_{x\to\infty} W[G(x;\xi)] = b$.

The pdf of the T-X family can be expressed as

$$f(x) = \left\{\frac{\partial}{\partial x}W[G(x;\xi)]\right\} r\{W[G(x;\xi)]\}, \; x \in \mathbb{R}. \tag{2.2}$$

We derive the cdf of the BXII-PC distribution from the T-X family technique by setting

$$r(t) = \alpha\beta t^{\beta-1}\left(1 + t^{\beta}\right)^{-\alpha-1}, \quad t > 0, \alpha > 0, \beta > 0$$

and

$$W[G(x)] = -\log\left[1 - \frac{2}{\pi}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right].$$

The cdf of the BXII-PC distribution takes the form

$$F(x) = 1 - \left(1 + \left\{-\log\left[1 - \frac{2}{\pi}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta}\right)^{-\alpha}, \; x \geqslant 0, \tag{2.3}$$

where α , $\beta, \kappa, \theta > 0$ are the parameters.

The BXII-PC density can be expressed as

$$f(x) = \frac{2\alpha\beta\kappa}{\theta\pi}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[1 + \left(\frac{x}{\theta}\right)^{2\kappa}\right]^{-1}\left[1 - 2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]^{-1} \times$$
$$\left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta-1}\left(1 + \left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta}\right)^{-\alpha-1}, \; x > 0. \tag{2.4}$$

Hereafter, a rv with pdf (2.4) is denoted by X ~BXII-PC$(\alpha,\beta,\kappa,\theta)$. (i) For $\kappa = 1$, the BXII-PC distribution reduces to Burr XII Half Cauchy (BXII-HC) distribution; (ii) For $\beta = 1$, the BXII-PC distribution reduces to the Lomax Power Cauchy (Lomax-PC) distribution; (iii) For $\kappa = \beta = 1$, the BXII-PC distribution reduces to the Lomax Half Cauchy (Lomax-HC) distribution; (iv) For $\alpha = 1$, the BXII-PC distribution reduces to the log-logistic Power Cauchy (Log-Log-PC) distribution and (v) For $\alpha = \kappa = 1$, the BXII-PC distribution reduces to the log-logistic half Cauchy (Log-Log-HC) distribution.

2.2. Nexus between the exponential and gamma variables

We derive the BXII-PC distribution from nexus between the exponential and gamma variables.

Lemma 2.2.1. *If* $W_1 \sim \exp(1)$ *and* $W_2 \sim gamma(\alpha, 1)$ *are independent, then for* $W_1 = \left\{-\log\left[1 - \frac{2}{\pi}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta} W_2$, *we have that X has the density* (2.4).

Proof. If $W_1 \sim \exp(1)$, i.e. $f(w_1) = e^{-w_1}$, $w_1 > 0$,

$W_2 \sim gamma(\alpha, 1)$, i.e. $f(w_2) = \frac{w_2{}^{\alpha-1}e^{-w_2}}{\Gamma(\alpha)}$, $w_2 > 0$,

then, the joint distribution of the two rvs is $f(w_1, w_2) = \frac{w_2{}^{\alpha-1}e^{-w_2}e^{-w_1}}{\Gamma(\alpha)}$, $w_1 > 0$, $w_2 > 0$.

Let $W_1 = \left\{-\log\left[1 - \frac{2}{\pi}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\}^{\beta} W_2$, the joint density of the rvs X and W_2 has the form

$$f(x, w_2) = \frac{w_2{}^{\alpha-1}e^{-w_2}e^{-\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\left(\frac{x_1}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta}w_2}}{\Gamma(\alpha)}\frac{2\beta\kappa}{\theta\pi}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[1 + \left(\frac{x}{\theta}\right)^{2\kappa}\right]^{-1} \times$$
$$\left[1 - 2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]^{-1}\left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta-1} w_2, \;\; x > 0, \; w_2 > 0.$$

The marginal density of X takes the form

$$f(x) = \frac{2\beta\kappa}{\theta\pi}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[1+\left(\frac{x}{\theta}\right)^{2\kappa}\right]^{-1}\left[1-2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]^{-1}\times$$
$$\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta-1}\frac{1}{\Gamma(\alpha)}\int_0^{\infty} w_2{}^{\alpha} e^{-\left(1+\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\left(\frac{x_1}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta}\right)w_2}\, dw_2.$$

After simplification, we arrive at

$$f(x) = \frac{2\alpha\beta\kappa}{\theta\pi}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[1+\left(\frac{x_1}{\theta}\right)^{2\kappa}\right]^{-1}\left[1-2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]^{-1}\times$$
$$\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta-1}\left(1+\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta}\right)^{-\alpha-1},\ x>0,$$

which is the BXII-PC density. □

The survival, failure rate and cumulative failure rate functions of X are given, respectively, by (for x>0)

$$S(x) = \left(1+\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta}\right)^{-\alpha},$$
$$h(x) = \frac{d}{dx}\left[-\ln\left(1+\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta}\right)^{-\alpha}\right],$$

and

$$\lambda(x) = -\ln\left\{1+\left[-\log\left(1-2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right)\right]^{\beta}\right\}^{-\alpha}.$$

The quantile function of X (for $0 < q < 1$) follows from

$$x_q = \theta\left\{\tan\left[\frac{\pi}{2}\left(1-\exp\left\{-\left[(1-q)^{-\frac{1}{\alpha}}-1\right]^{\frac{1}{\beta}}\right\}\right)\right]\right\}^{\frac{1}{\kappa}},$$

and its random number generator with $Z \sim$ Uniform (0,1) is the solution of the nonlinear equation

$$X = \theta\left\{\tan\left\{\frac{\pi}{2}\left[1-\exp\left(-\left((1-Z)^{-\frac{1}{\alpha}}-1\right)^{\frac{1}{\beta}}\right)\right]\right\}\right\}^{\frac{1}{\kappa}}.$$

2.3. Shapes of the BXII-PC density and hazard rate functions

We plot the density and failure rate functions of the BXII-PC distribution for selected parameter values. The BXII-PC density can display numerous shapes such as symmetrical, right-skewed, left-skewed, J, reverse-J and exponential (as Figure 1). The failure rate function can highlight shapes as modified bathtub, inverted bathtub, increasing, decreasing; increasing-decreasing and decreasing-increasing-decreasing (as Figure 2). Therefore, the BXII-PC distribution is quite flexible and can be applied to numerous data sets.

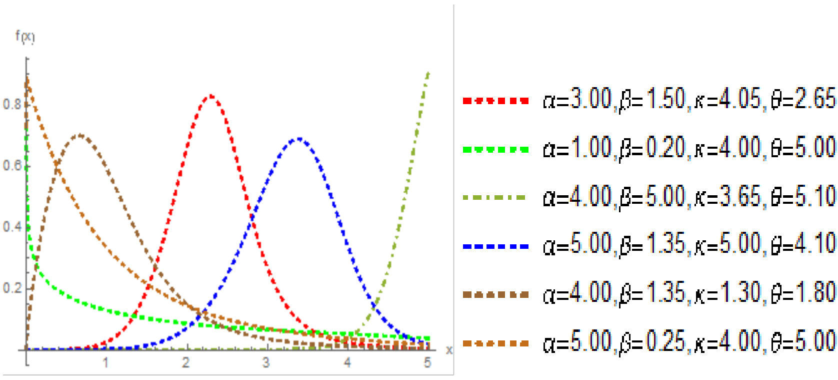

Figure 1. Plots of the BXII-PC density.

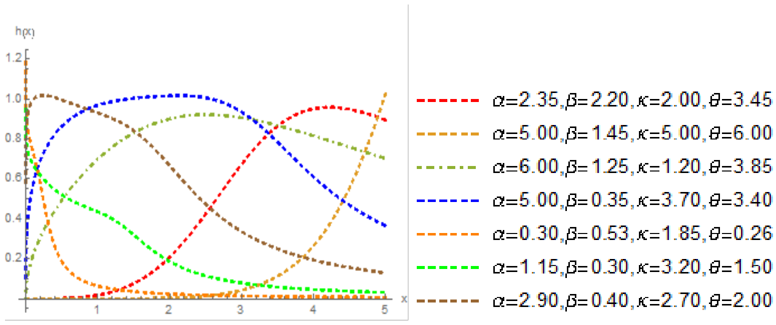

Figure 2. Plots of the BXII-PC hazard rate.

3. Mathematical properties

Here, we present certain mathematical and statistical properties such as the ordinary moments, incomplete moments, inequality measures, conditional moments, reliability measures and characterization.

3.1. Moments

The moments are significant tools for statistical analysis in pragmatic sciences. The r^{th} ordinary moment of X, say $\mu_r' = E(X^r)$, can be expressed from (2.4) as

$$E(X^r) = \int_0^\infty x^r \frac{\alpha\beta\kappa}{\theta} 2\pi^{-1} \left(\frac{x}{\theta}\right)^{\kappa-1} \left[1 + \left(\frac{x}{\theta}\right)^{2\kappa}\right]^{-1} \left(1 - 2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right)^{-1} \times$$
$$\left[-\log\left(1 - 2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right)\right]^{\beta-1} \left\{1 + \left[-\log\left(1 - 2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right)\right]^{\beta}\right\}^{-\alpha-1} dx.$$

Letting $w = \left[-\log\left(1 - 2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right)\right]$, we have

$$x^r = \theta^r \left(\cot\left\{\frac{\pi}{2}\left[1 - \exp(-w)\right]\right\}\right)^{\frac{r}{\kappa}} = \theta^r \left(\cot\left\{\frac{\pi}{2}\left[\exp(-w)\right]\right\}\right)^{\frac{r}{\kappa}}.$$

The following power series ([21]) can be obtained from Mathematica

$$\cot(x)^s = \sum_{i=0}^{\infty} a_i(s)\, x^{2i-s},$$

where $a_0(s) = 1, a_1(s) = -s/3,\ \ a_2(s) = s(5s-7)/90$, etc.

Using this expression we have

$$\left(\cot\left\{\frac{\pi}{2}\left[\exp(-w)\right]\right\}\right)^{\frac{r}{\kappa}} = \sum_{i=0}^{\infty} a_i\left(\frac{r}{\kappa}\right)\left(\frac{\pi}{2}\right)^{2i-\frac{r}{\kappa}} \exp\left[\left(\frac{r}{\kappa}-2i\right)w\right],$$

where $\exp\left[\left(\frac{r}{\kappa}-2i\right)w\right] = \sum_{j=0}^{\infty}\frac{\left(\frac{r}{\kappa}-2i\right)^j}{j!}w^j$.

The rth ordinary moment of X with BXII-PC distribution is

$$E(X^r) = \alpha\beta\int_0^{\infty}\sum_{i=0}^{\infty} a_i\left(\frac{r}{\kappa}\right)\left(\frac{\pi}{2}\right)^{2i-\frac{r}{\kappa}}\sum_{j=0}^{\infty}\frac{\left(\frac{r}{\kappa}-2i\right)^j}{j!}w^{j+\beta-1}\left\{1+w^{\beta}\right\}^{-\alpha-1}dw.$$

Letting $w^{\beta} = y, w = y^{\frac{1}{\beta}},\ dw = \frac{1}{\beta}y^{\frac{1}{\beta}-1}dy$, we have

$$E(X^r) = \alpha\int_0^{\infty}\sum_{i=0}^{\infty} a_i\left(\frac{r}{\kappa}\right)\left(\frac{\pi}{2}\right)^{2i-\frac{r}{\kappa}}\sum_{j=0}^{\infty}\frac{\left(\frac{r}{\kappa}-2i\right)^j}{j!}\ y^{\frac{j}{\beta}}\left\{1+y\right\}^{-\alpha-1}dy,$$

$$\mu_r' = E(X^r) = \alpha\sum_{i,j=0}^{\infty} a_i\left(\frac{r}{\kappa}\right)\left(\frac{\pi}{2}\right)^{2i-\frac{r}{\kappa}}\left(\frac{r}{\kappa}-2i\right)^j\frac{B\left(\frac{j}{\beta}+1,\alpha-\frac{j}{\beta}\right)}{j!}, r = 1,2,3,..., \tag{3.1}$$

where $\alpha\beta > j$ and $B(.,.)$ is the beta function.

The rth central moment (μ_r), coefficients of skewness (γ_1) and kurtosis (γ_2) of X are $\mu_r = \sum_{\ell=1}^{r}(-1)^{\ell}\binom{r}{\ell}\mu_{\ell}'\mu_{r-\ell}'$, $\gamma_1 = \mu_3/\sqrt[3]{\mu_2}$ and $\beta_2 = \mu_4/(\mu_2)^2$.

The numerical values for the mean $\left(\mu_1'\right)$, median $(\tilde{\mu})$, standard deviation (σ), skewness (γ_1) and kurtosis (γ_2) of the BXII-PC distribution for selected values of $\alpha, \beta, \kappa, \theta$ are listed in Table 1.

Table 1. Quantities μ_1' , $\tilde{\mu}$, σ, γ_1 and γ_2 for the BXII-PC distribution.

$\alpha,\beta,\kappa,\theta$	μ_1'	$\tilde{\mu}$	σ	γ_1	γ_2
3,3,3,3	2.9284	2.9115	0.4935	0.7169	16.3785
4,3,3,3	2.8058	2.8057	0.4347	0.1143	4.0651
5,3,3,3	2.7214	2.7296	0.4033	-0.0687	3.5402
6,3,3,3	2.6575	2.6707	0.3828	-0.1730	3.4238
10,3,3,3	2.4962	2.5176	0.3407	-0.3407	3.3782
3,4,3,3	3.0350	3.0319	0.3877	0.1800	4.3310
3,5,3,3	3.1055	3.1086	0.3222	0.0096	3.7844
3,6,3,3	3.1556	3.1619	0.2766	-0.0972	3.6498
3,10,3,3	3.2649	3.2743	0.1783	-0.3192	3.6605
3,3,4,3	2.9383	2.9934	0.3707	0.3424	7.2468
3,3,5,3	2.9468	2.9466	0.2976	0.1800	5.5220
3,3,6,3	2.9535	2.9554	0.2488	0.0851	4.9608
3,3,7.25,3	2.9600	2.9631	0.0062	0.2066	4.6374
3,3,10,3	2.9695	2.9732	0.1505	-0.0909	4.4444
3,3,3,5	4.8806	4.8525	0.8225	0.7479	18.3789
3.5,3.5,3.5,3	2.9350	2.9377	0.3500	0.0386	3.9623
3.5,3.7,3.5,3	2.9550	2.9589	0.3349	0.0006	3.8412
20,2.6,2,1.5	0.9376	0.9464	0.2048	-0.2035	3.0003
20,2.5,2,1.5	0.9151	0.9233	0.2071	-0.1822	2.9775
20,2.25,2,1.5	0.8536	0.8600	0.2124	-0.1189	2.9221
20,2,2,2	1.0444	1.0494	0.2893	-0.0392	2.8715
15,2,2,2	1.1224	1.1255	0.3141	0.0092	2.9339
15,2.5,2,2	1.2933	1.3029	0.2965	-0.1347	3.0037
15,2.25,2,2	1.2139	1.2209	0.3058	-0.0716	2.9588
12,2,2,2	1.1881	1.1887	0.3365	0.0640	3.0175

3.2. Conditional moments

Life expectancy, mean waiting time and inequality measures can be obtained from the incomplete moments. The sth incomplete moment for the BXII-PC distribution is

$$M_s'(z) = \int_0^z x^s \frac{\alpha\beta\kappa}{\theta} 2\pi^{-1} \left(\frac{x}{\theta}\right)^{\kappa-1} \left[1 + \left(\frac{x}{\theta}\right)^{2\kappa}\right]^{-1} \left(1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right)^{-1} \times$$
$$\left[-\log\left(1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]^{\beta-1} \left\{1 + \left[-\log\left(1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]^{\beta}\right\}^{-\alpha-1} dx.$$

Setting $w^{\beta} = y, w = y^{\frac{1}{\beta}},\ dw = \frac{1}{\beta} y^{\frac{1}{\beta}-1} dy$, we have

$$M_s'(z) = \alpha \sum_{i,j=0}^{\infty} a_i \left(\frac{s}{\kappa}\right)\left(\frac{\pi}{2}\right)^{2i-\frac{s}{\kappa}} \frac{\left(\frac{s}{\kappa} - 2i\right)^j}{j!} \int_0^z y^{\frac{j}{\beta}} \{1 + y\}^{-\alpha-1} dy,$$

$$M'_s(z) = \alpha \sum_{i,j=0}^{\infty} a_i \left(\frac{s}{\kappa}\right)\left(\frac{\pi}{2}\right)^{2i-\frac{s}{\kappa}} \left(\frac{s}{\kappa} - 2i\right)^j \frac{B_z\left(\frac{j}{\beta}+1, \alpha - \frac{j}{\beta}\right)}{j!}, \tag{3.2}$$

where $B_z(.,.)$ is incomplete beta function.

The mean deviation about the mean ($\delta_1 = E|X-\mu|$) and about the median ($\delta_2 = E|X-\tilde{\mu}|$) can be written as $\delta_1 = 2\mu F(\mu) - 2\mu M'_1(\mu)$ and $\delta_2 = \mu - 2M'_1(\tilde{\mu})$, respectively, where $\mu = E(X)$ and $\tilde{\mu} = x_{0.5}$. The quantities $M'_1(\mu)$ and $M'_1(\tilde{\mu})$ can be obtained from (3.2). For specific probability p, Lorenz and Bonferroni curves are computed as $L(p) = \frac{M'_1(q)}{\mu'}$ and $B(p) = L(p)|\, p$ where $q = Q(p)$.

The rth conditional moment $E(X^r|\, X > z)$ is

$$E(X^r|\, X > z) = \frac{1}{S(z)}\left[\mu'_r - E_{X\le z}(X^r)\right].$$

$$E(X^r|\, X > z) = \frac{1}{S(z)}\alpha \sum_{i,j=0}^{\infty} a_i \left(\frac{r}{\kappa}\right)\left(\frac{\pi}{2}\right)^{2i-\frac{r}{\kappa}} \frac{\left(\frac{r}{\kappa} - 2i\right)^j \left[B\left(\frac{j}{\beta}+1, \alpha - \frac{j}{\beta}\right) - B_z\left(\frac{j}{\beta}+1, \alpha - \frac{j}{\beta}\right)\right]}{j!}.$$

The rth reversed conditional moment $E(X^r|\, X \le z)$ is

$$E(X^r|\, X \le z) = \frac{\alpha}{F(z)} \sum_{i,j=0}^{\infty} a_i \left(\frac{r}{\kappa}\right)\left(\frac{\pi}{2}\right)^{2i-\frac{r}{\kappa}} \frac{\left(\frac{r}{\kappa} - 2i\right)^j \left[B_z\left(\frac{j}{\beta}+1, \alpha - \frac{j}{\beta}\right)\right]}{j!}.$$

3.3. Stochastic ordering

Stochastic orders are widely used in reliability, survival analysis, economics and operations research for judging the comparative behavior of distributions. Here, we present a result on the stochastic order for the BXII-PC distribution with β, κ, θ as common parameters. A random variable X_1 with a pfd denoted by $f_1(\alpha_1, \beta, \kappa, \theta)$ is said to be smaller than another random variable X_2 with a pfd denoted by $f_2(\alpha_2, \beta, \kappa, \theta)$ in likelihood ratio order, denoted by $X_1 \le_{lr} X_2$, if $\frac{f_1(\alpha_1,\, \beta, \kappa, \theta)}{f_2(\alpha_2, \beta, \kappa, \theta)} \le 0$.

Proposition 3.3.1. *Let* $X_1 \sim \text{BXII} - \text{PC}(\alpha_1, \beta, \kappa, \theta)$ *and* $X_2 \sim \text{BXII} - \text{PC}(\alpha_2, \beta, \kappa, \theta)$. *If* $\alpha_2 \le \alpha_1$, *then the* BXII − PC *distribution is ordered according to likelihood ratio ordering.*

Proof. For $X_1 \sim \text{BXII} - \text{PC}(\alpha_1, \beta, \kappa, \theta)$,

$$f_1(\alpha_1, \beta, \kappa, \theta) = \frac{2\alpha_1\beta\kappa}{\theta\pi}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[1+\left(\frac{x}{\theta}\right)^{2\kappa}\right]^{-1}\left[1 - 2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]^{-1} \times$$
$$\left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta-1}\left(1 + \left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta}\right)^{-\alpha_1-1}, \; x > 0,$$

and for $X_2 \sim \text{BXII} - \text{PC}(\alpha_2, \beta, \kappa, \theta)$,

$$f_2(\alpha_2, \beta, \kappa, \theta) = \frac{2\alpha_2\beta\kappa}{\theta\pi}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[1+\left(\frac{x}{\theta}\right)^{2\kappa}\right]^{-1}\left[1 - 2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]^{-1} \times$$
$$\left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta-1}\left(1 + \left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta}\right)^{-\alpha_2-1}, \; x > 0.$$

Thus

$$\frac{f_1(\alpha_1,\beta,\kappa,\theta)}{f_2(\alpha_2,\beta,\kappa,\theta)}=\frac{\alpha_1\left(1+\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta}\right)^{-\alpha_1-1}}{\alpha_2\left(1+\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta}\right)^{-\alpha_2-1}},$$

$$\frac{f_1(\alpha_1,\beta,\kappa,\theta)}{f_2(\alpha_2,\beta,\kappa,\theta)}=\frac{\alpha_1}{\alpha_2}\left(1+\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta}\right)^{-\alpha_1+\alpha_2},$$

$$\frac{\partial}{\partial x}\frac{f_1(\alpha_1,\beta,\kappa,\theta)}{f_2(\alpha_2,\beta,\kappa,\theta)}=(\alpha_2-\alpha_1)\,\frac{\alpha_1}{\alpha_2}\,\frac{2\beta\kappa}{\theta\pi}\,\frac{\left(\frac{x}{\theta}\right)^{\kappa-1}\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta-1}}{\left[1+\left(\frac{x}{\theta}\right)^{2\kappa}\right]\left[1-2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]}\times$$
$$\left(1+\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\left(\frac{x}{\theta}\right)^{\kappa}\right)\right]\right\}^{\beta}\right)^{\alpha_2-\alpha_1-1}.$$

If $\alpha_2 \leq \alpha_1$, we have $X_1 \leq_{lr} X_2$, i.e., $\frac{f_1(\alpha_1,\beta,\kappa,\theta)}{f_2(\alpha_2,\beta,\kappa,\theta)} \leq 0$ is decreasing. Therefore for the BXII-PC distribution, random variable X_1 is said to be smaller than a random variable X_2 in likelihood ratio order $X_1 \leq_{lr} X_2$, since $\frac{f_1(\alpha_1,\beta,\kappa,\theta)}{f_2(\alpha_2,\beta,\kappa,\theta)} \leq 0$. □

3.4. *Reliability estimation of multicomponent stress-strength model*

Consider a system that has m identical components out of which s components are functioning. The strengths of m components are $X_i, i = 1,2,...,m$ with common cdf F while, the stress Y imposed on the components has cdf G. The strengths $X_i, i = 1,2,...,m$ and stress Y are i.i.d. The probability that the system operates properly is reliability of the system i.e.

$$R_{s,m} = P[strengths\,(X_i, i = 1,2,...,m) > stress\,(Y)] = P[\text{at the minimum “}s\text{” } of\,(X_i, i = 1,2,...,m)\,exceed\,Y],$$

$$R_{s,m}=\sum_{l=s}^{m}\binom{m}{\ell}\int_0^{\infty}[1-F(y)]^{l}\,[F(y)]^{m-l}dG(y),\qquad ([22]) \tag{3.3}$$

Let X ~ BXII − PC $(\alpha_1,\beta,\kappa,\theta)$ and Y ~ BXII − PC $(\alpha_2,\beta,\kappa,\theta)$ with common parameters β,κ,θ and unknown shape parameters α_1 and α_2. The reliability that the system operates properly in multicomponent stress- strength for the BXII-PC distribution is

$$R_{s,m}=\sum_{\ell=s}^{m}\binom{m}{\ell}\int_0^{1}(u^{\nu})^{\ell}\,(1-u^{\nu})^{(m-\ell)}\,du,$$

where $\nu = \frac{\alpha_1}{\alpha_2}$ and $u = \left\{1+\left[-\log\left(1-2\pi^{-1}\tan^{-1}\left(\frac{y}{\theta}\right)^{\kappa}\right)\right]^{\beta}\right\}^{-\alpha_2}$.

Letting $u^{\nu} = w$, we have

$$R_{s,m}=\sum_{\ell=s}^{m}\binom{m}{\ell}\int_0^{1}w^{\ell}\,(1-w)^{(m-\ell)}\,\frac{1}{\nu}w^{\frac{1}{\nu}-1}dw,$$
$$R_{s,m}=\frac{1}{\nu}\sum_{\ell=s}^{m}\binom{m}{\ell}B\left(\ell+\frac{1}{\nu},m-\ell+1\right), \tag{3.4}$$

where $B(.,.)$ is the beta function. The probability in (3.4) is known as the reliability of multicomponent stress-strength model. For s= m=1, the multicomponent stress-strength model reduces to the stress-strength model ([23]) as

$$R_{1,1} = Pr(\mathrm{Y} < \mathrm{X}) = \frac{\alpha_2}{(\alpha_1 + \alpha_2)}, \text{ where } \alpha_1 + \alpha_2 > 0,$$

which is independent of the parameters β, κ and θ.

3.5. Characterizations based on truncated moment of a function of the random variable

In this subsection, we first present a characterization of the BXII-PC distribution in terms of a simple relationship between truncated moment of a function of X and another function. This characterization result employs a version of a theorem due to [24]; see Theorem 7.1 of Appendix A. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As shown in [25], this characterization is stable in the sense of weak convergence.

Proposition 3.5.1. *Let* $\mathrm{X} : \Omega \to (0, \infty)$ *be a continuous rv and let* $q(x) = \left(1 + \left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\}^{\beta}\right)^{-1}$, $x > 0$. *The rv X has pdf* (2.4) *if and only if the function* η *defined in Theorem 7.1 has the form*

$$\eta(x) = \frac{\alpha}{\alpha + 1}\left(1 + \left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\}^{\beta}\right)^{-1}, \quad x > 0.$$

Proof. If X has pdf (2.4), then for (x>0),

$$(1 - F(x))\,E(q(x)|\,X \geq x) = \frac{\alpha}{\alpha + 1}\left(1 + \left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\}^{\beta}\right)^{-(\alpha+1)}, \; x > 0,$$

or

$$E(q(x)|\,X \geq x) = \frac{\alpha}{\alpha + 1}\left(1 + \left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\}^{\beta}\right)^{-1}, \; x > 0,$$

and

$$\eta(x) - q(x) = -\frac{1}{\alpha + 1}\left(1 + \left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\}^{\beta}\right)^{-1}, x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)}{\eta(x) - q(x)}$$

$$= \frac{\frac{\alpha\beta\kappa}{\theta}2\pi^{-1}\left(\frac{t}{\theta}\right)^{\kappa-1}\left[1 + \left(\frac{x}{\theta}\right)^{2\kappa}\right]^{-1}\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]^{-1}\left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\}^{\beta-1}}{\left(1 + \left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\}^{\beta}\right)}, \; x > 0,$$

and hence

$$s(x) = \ln\left(1 + \left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\}^{\beta}\right)^{\alpha}, \; x > 0,$$

and

$$e^{-s(x)} = \left(1 + \left\{- \log\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\}^{\beta}\right)^{-\alpha}, \ x > 0.$$

In view of Theorem 7.1, X has density (2.4). □

Corollary 3.5.1. *Let* $\mathrm{X} : \Omega \to (0, \infty)$ *be a continuous rv. The pdf of X is* (2.4) *if and only if there exist functions* $\eta(x)$ *and* $q(x)$ *defined in Theorem 7.1 satisfying the differential equation*

$$\frac{\eta'(x)}{\eta(x) - q(x)} = \frac{\frac{\alpha\beta\kappa}{\theta}2\pi^{-1}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[1+\left(\frac{x}{\theta}\right)^{2\kappa}\right]^{-1}\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]^{-1}\left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\}^{\beta-1}}{\left(1 + \left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\}^{\beta}\right)}, \ x > 0.$$

Remark 3.5.1. *The general solution of the differential equation in Corollary 3.5.1 is*

$$\eta(x) = \left(1 + \left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\}^{\beta}\right)^{\alpha} \times$$
$$\left[-\int \frac{\frac{\alpha\beta\kappa}{\theta}2\pi^{-1}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[1+\left(\frac{x}{\theta}\right)^{2\kappa}\right]^{-1}\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]^{-1}\left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\}^{\beta-1}}{\left(1 + \left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\}^{\beta}\right)^{\alpha+1}} q(x)\,dx + D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 3.5.1 with D=0. However, it should also be noted that there are other pairs (η, q) *satisfying conditions of Theorem 7.1.*

4. Different estimation methods

In this section, we propose various estimators for estimating the unknown parameters of the BXII-PC distribution. We discuss maximum likelihood, maximum product spacings, least squares, weighted least squares, Cramer-von Mises and Anderson-Darling estimation methods and compare their performances on the basis of a simulated sample from the BXII-PC distribution. The details are as follows.

4.1. Maximum likelihood estimation

We address parameters estimation using maximum likelihood method. The log-likelihood function for the vector of parameters $\xi = (\alpha, \beta, \kappa, \theta)$ of the BXII-PC distribution is

$$\begin{aligned}\ell =& \ell(\xi) = n\ln\left(\frac{2}{\pi}\right) + n\ln(\alpha) + n\ln(\beta) + n\ln(\kappa) - n\ln(\theta) + (\kappa - 1)\sum_{i=1}^{n}\ln\left(\frac{x_i}{\theta}\right) - \sum_{i=1}^{n}\ln\left[1 + \left(\frac{x}{\theta}\right)^{2\kappa}\right] \\ &- \sum_{i=1}^{n}\ln\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right] + (\beta - 1)\sum_{i=1}^{n}\ln\left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\} \\ &- (\alpha + 1)\sum_{i=1}^{n}\ln\left(1 + \left\{-\log\left[1 - 2\pi^{-1}\tan^{-1}\left(\frac{x}{\theta}\right)^{\kappa}\right]\right\}^{\beta}\right). \qquad (4.1)\end{aligned}$$

4.2. Maximum product spacing estimates

The maximum product spacing (MPS) method is an alternative method to MLE for parameter estimation. This method was proposed by [26, 27] as well as independently developed by [28] as an approximation to the Kullback-Leibler measure of information. This method is based on the idea that differences (spacings) between the values of the cdf at consecutive data points should be identically distributed. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be ordered sample of size n from the BXII-PC distribution. The geometric mean of the differences is given by

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i},$$

where, the difference D_i is defined as

$$D_i = \int_{x_{(i-1)}}^{x_{(i)}} f(x)\,dx; \quad i = 1, 2, \ldots, n+1. \tag{4.2}$$

The maximum product spacing (MPS) estimates, say $\hat{\alpha}_{MPS}, \hat{\beta}_{MPS}, \hat{\theta}_{MPS}$ and $\hat{\kappa}_{MPS}$, of α, β, θ and κ are obtained by maximizing the geometric mean of the differences. Substituting cdf of BXII-PC distribution in Eq (4.2) and taking logarithm of the above expression, we have

$$MPS(\xi) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log\left[F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right)\right], \quad i = 1, 2, \ldots, n+1, \tag{4.3}$$

where, $F\left(x_{(0)}\right) = 0$ and $F\left(x_{(n+1)}\right) = 1$. The MPSEs $\hat{\alpha}_{MPS}, \hat{\beta}_{MPS}, \hat{\theta}_{MPS}$ and $\hat{\kappa}_{MPS}$ are obtained by maximizing $MPS(\xi)$.

4.3. Least squares estimates

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be ordered sample of size n from the BXII-PC distribution. Then, the expectation of the empirical cumulative distribution function is defined as

$$E\left[F\left(x_{(i)}\right)\right] = \frac{i}{n+1}; i = 1, 2, \ldots, n.$$

The least square estimates (LSEs) say, $\hat{\alpha}_{\mathrm{LSE}}, \hat{\beta}_{LSE}, \hat{\theta}_{LSE}$ and $\hat{\kappa}_{LSE}$, of α, β, θ and κ are obtained by minimizing

$$QLSE(\xi) = \sum_{i=1}^{n} \left(F\left(x_{(i)}\right) - \frac{i}{n+1}\right)^2. \tag{4.4}$$

4.4. Weighted least squares estimates

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be ordered sample of size n from the BXII-PC distribution. The variance of the empirical cumulative distribution function is defined as

$$V\left[F\left(x_{(i)}\right)\right] = \frac{i(n-i+1)}{(n+2)(n+1)^2}; i = 1, 2, \ldots, n.$$

Then, the weighted least square estimates (WLSEs) say, $\hat{\alpha}_{\text{WLSE}}$, $\hat{\beta}_{WLSE}$, $\hat{\theta}_{WLSE}$ and $\hat{\kappa}_{WLSE}$, of α , β , θ and κ are obtained by minimizing

$$QWLSE(\xi) = \sum_{i=1}^{n} \frac{\left(F(x_{(i)}) - \frac{i}{n+1}\right)^2}{V\left[F(x_{(i)})\right]}. \tag{4.5}$$

4.5. Anderson-Darling estimation

This estimator is based on Anderson-Darling goodness-of-fits statistics which was introduced by [29]. The Anderson-Darling (AD) minimum distance estimates, $\hat{\alpha}_{\text{AD}}$, $\hat{\beta}_{\text{AD}}$, $\hat{\theta}_{\text{AD}}$ and $\hat{\kappa}_{AD}$, of α , β , θ and κ are obtained by minimizing

$$AD(\xi) = -n - \sum_{i=1}^{n} \frac{2i-1}{n} \left[\log F(x_{(i)}) + \log\left\{1 - F(x_{(n+1-i)})\right\}\right]. \tag{4.6}$$

4.6. The Cramer-von mises estimations

The Cramer-von Mises (CVM) minimum distance estimates, $\hat{\alpha}_{\text{CVM}}$, $\hat{\beta}_{\text{CVM}}$, $\hat{\theta}_{\text{CVM}}$ and $\hat{\kappa}_{\text{CVM}}$, of α, β, θ and κ are obtained by minimizing

$$CVM(\xi) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{(i)}) - \frac{2i-1}{2n}\right]^2. \tag{4.7}$$

We refer the interested readers to [30] for AD and CVM goodness-of-fits statistics. To solve the above equations, Eqs (4.1)–(4.6) can be optimized either directly by using the R (optim and maxLik functions), SAS (PROC NLMIXED) and Ox package (sub-routine Max BFGS) or the non-linear optimization methods such as the quasi-Newton procedure to numerically optimize $\ell(\xi)$, $MPS(\xi)$, $QLSE(\xi)$, $QWLSE(\xi)$, $AD(\xi)$ and $CVM(\xi)$ functions.

5. Simulation experiments

In this Section, we perform the simulation studies by using the BXII-PC to see the performance of the above estimators corresponding to this distribution and obtain the graphical results. We generate N=1000 samples of size n=20, 30, . . . , 800 from the BXII-PC distribution with true parameter values $\alpha = 15, \beta = 5, \theta = 0.5$ and $\kappa = 2$. The random numbers generation is obtained by its quantile function. In this simulation study, we calculate the empirical mean, bias and mean square errors (MSEs) and the mean relative estimates (MREs) of all estimators to compare in terms of their biases, MSEs and MREs with varying sample size. The empirical bias, MSE and MRE are calculated by (for $h = \alpha, \beta, \theta, \kappa$)

$$\widehat{Bias}_h = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{h}_i - h\right),$$

$$\widehat{MSE}_h = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{h}_i - h\right)^2$$

and

$$\widehat{MRE}_h = \frac{1}{N}\sum_{i=1}^{N} \hat{h}_i/h$$

respectively. We expect that the empirical means are close to true values. MREs are closer to one when the MSEs and biases are near zero. All results related to estimations were obtained using optim-CG routine in the R programme.

The results of this simulation study are shown in Figures 3–6. These figures show that all estimators are to be consistent, since the MSE and biases decrease with increasing sample size and the values of MREs tend to one as expected. It is clear that the estimates of parameters are asymptotically unbiased. For all parameters estimations, the performances of all estimators are close except the MPS method.

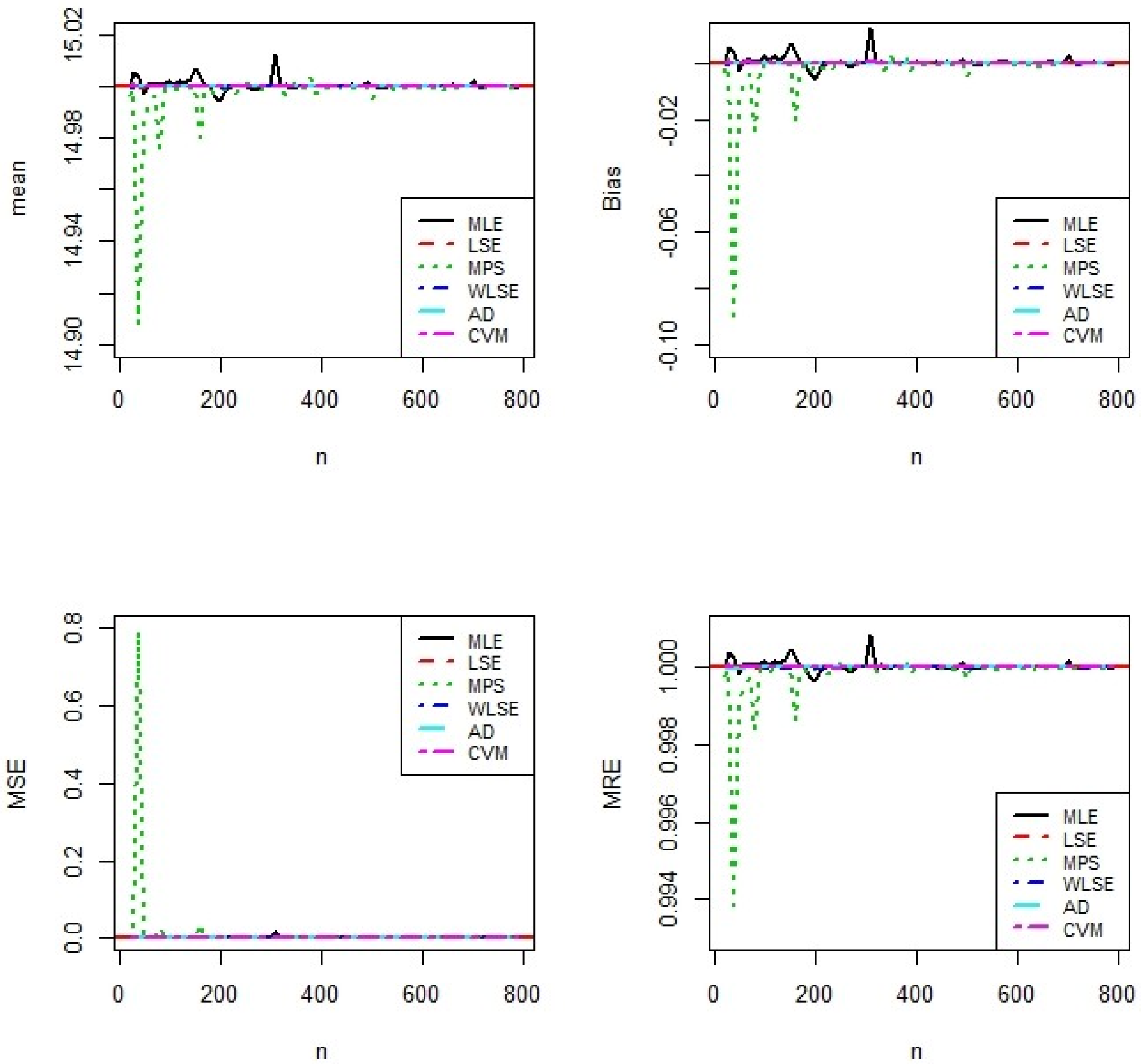

Figure 3. Simulation results of α.

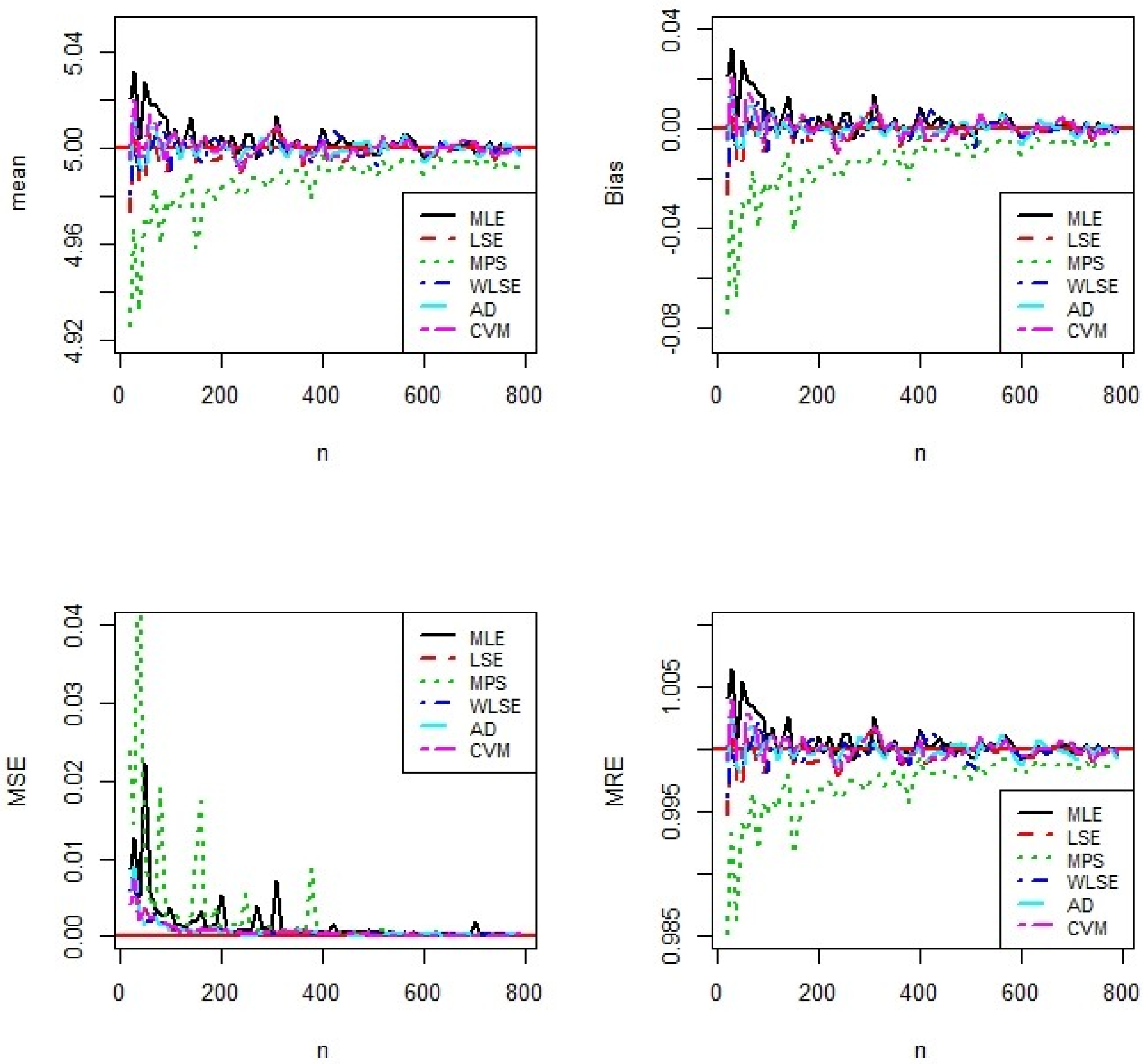

Figure 4. Simulation results of β.

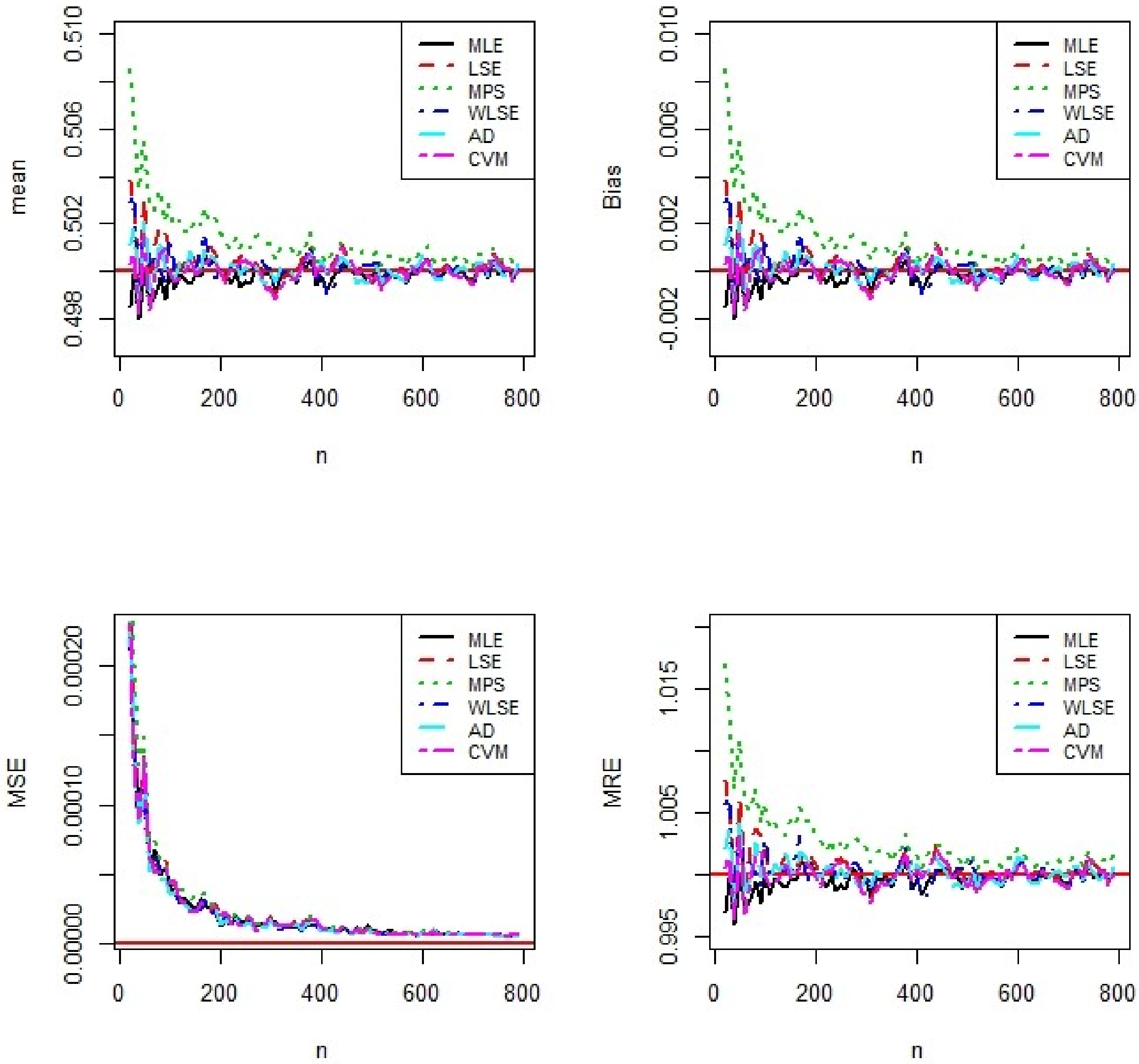

Figure 5. Simulation results of θ.

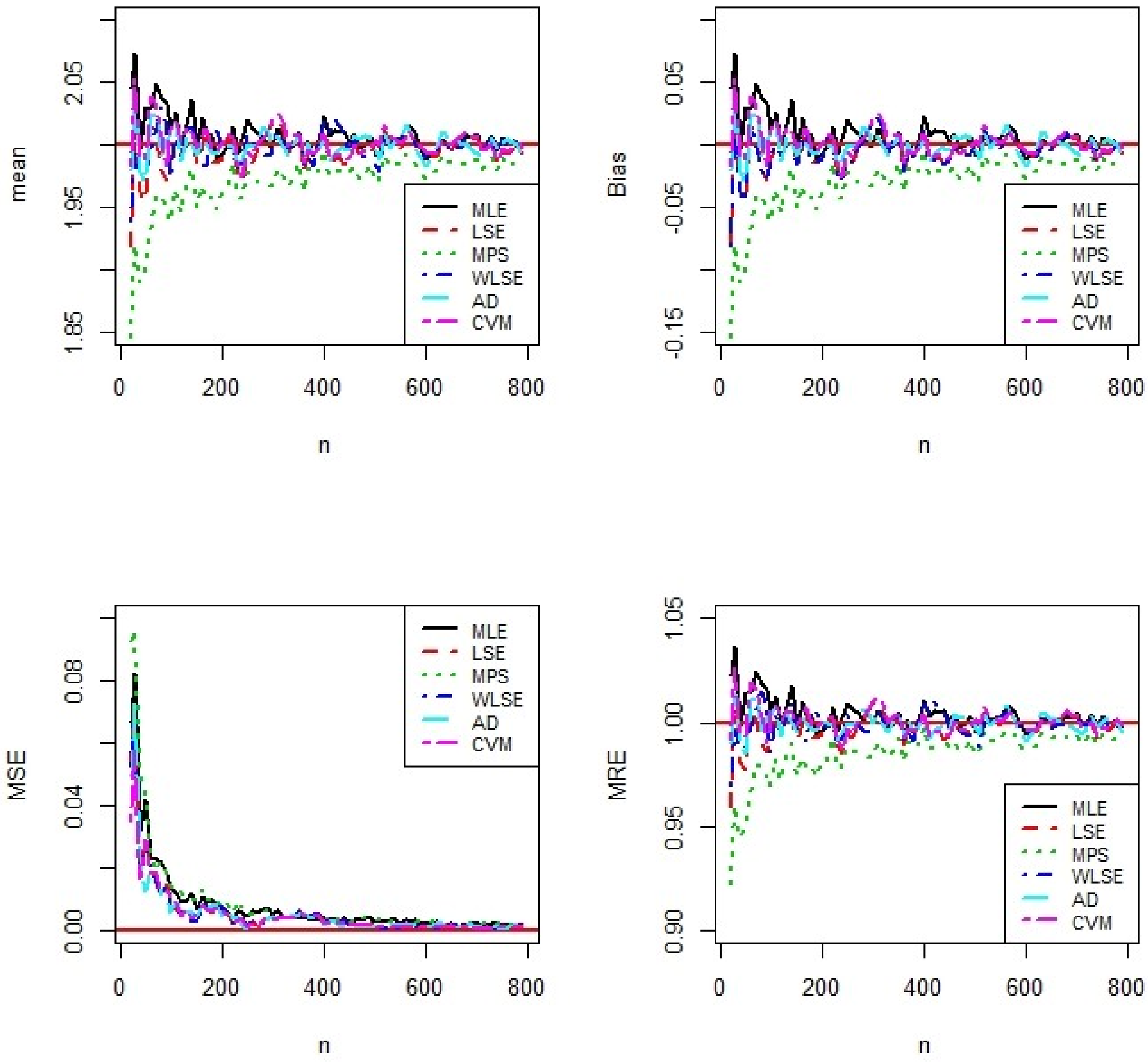

Figure 6. Simulation results of κ.

6. Application of the BXII-PC distribution

We consider an application to successive failures of the air conditioning system [31] for authentication the flexibility, utility and potentiality of the BXII-PC distribution. For this data set, we compare the BXII-PC distribution with BXII-HC, L-PC, LL-PC, Burr III power Cauchy (BIII-PC), Burr III half Cauchy (BIII-HC), Kumaraswamy half Cauchy (K-HC), beta half Cauchy (B-HC), Marshal Olkin power Cauchy (M-PC), Marshal Olkin half Cauchy (M-HC), BXII, PC and HC distributions. For selection of the optimum distribution, we compute the estimate of "likelihood ratio statistics ($-2\hat{\ell}$), Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Cramer-von

Mises (W*), Anderson Darling (A*), and Kolmogorov-Smirnov statistic with p-values [K-S (p-values] statistics" for all competing and sub distributions. We compute the MLEs and their standard errors (in parentheses). Table 2 reports the MLEs, their standard errors (in parentheses) and goodness of fit statistics such as W*, A*, KS (p-values). Table 3 displays the values of $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC.

We infer from the Tables 2 and 3 that BXII-PC distribution is best model, with the smallest values for all criteria of goodness of fit statistics (except BIC).

Figure 7 infers that the BXII-PC distribution is best fitted to empirical data.

Table 2. MLEs (standard errors) and W*, A*, KS (p-values) for successive failures data.

Model	α	β	θ	κ	W*	A*	K-S (p-value)
BXII-PC	4.3914 (3.4537)	10.4378 (12.1928)	7.8493 (33.5290)	0.1169 (0.1245)	0.0722	0.4833	0.0468 (0.8286)
BXII-HC	8.5509 (7.0532)	0.9461 (0.0784)	497.9669 (527.6631)	1	0.1371	0.8572	0.0508 (0.7457)
L-PC	8.1813 (5.9633)	1	462.3083 (425.9378)	0.9487 (0.0766)	0.1352	0.8477	0.0499 (0.7648)
LL-PC	1	28.7435 (8.2993)	0.04533 (0.08912)	0.0614 (0.0174)	0.1031	0.7285	0.0509 (0.7428)
BIII-PC	0.4988 (0.2522)	12.3673 (14.8340)	9.4187 (32.3260)	0.1859 (0.2492)	0.0956	0.6667	0.0517 (0.7257)
BIII-HC	0.3067 (0.0717)	2.7522 (0.3926)	85.1320 (15.6944)	1	0.1916	1.2809	0.0669 (0.3999)
K-HC	0.9862 (0.1415)	2.0882 (0.8367)	116.3634 (72.3209)	-	0.1252	0.8205	0.0549 (0.6529)
B-HC	2.0192 (0.7054)	1.0053 (0.1487)	109.2613 (57.3038)	-	61.9577	358.9412	0.9886 (<2.2e-16)
M-PC	0.0263 (0.0604)	-	4.2916 (6.4311)	1.3353 (0.1215)	61.3057	358.4714	0.9974 (<2.2e-16)
M-HC	3.2167 (2.6159)	-	127.3189 (86.8808)	1	61.9431	359.0706	0.9841 (<2.2e-16)
BXII	0.0337 (0.0564)	7.7318 (12.9583)	-	-	0.5910	4.0335	0.3695 (<2.2e-16)
PC	-	-	48.8403 (4.6197)	1.16519 (0.0776)	0.1377	0.9365	0.0586 (0.5713)
HC	-	-	-	48.6910 (5.0669)	0.1177	0.8127	0.0609 (0.521)

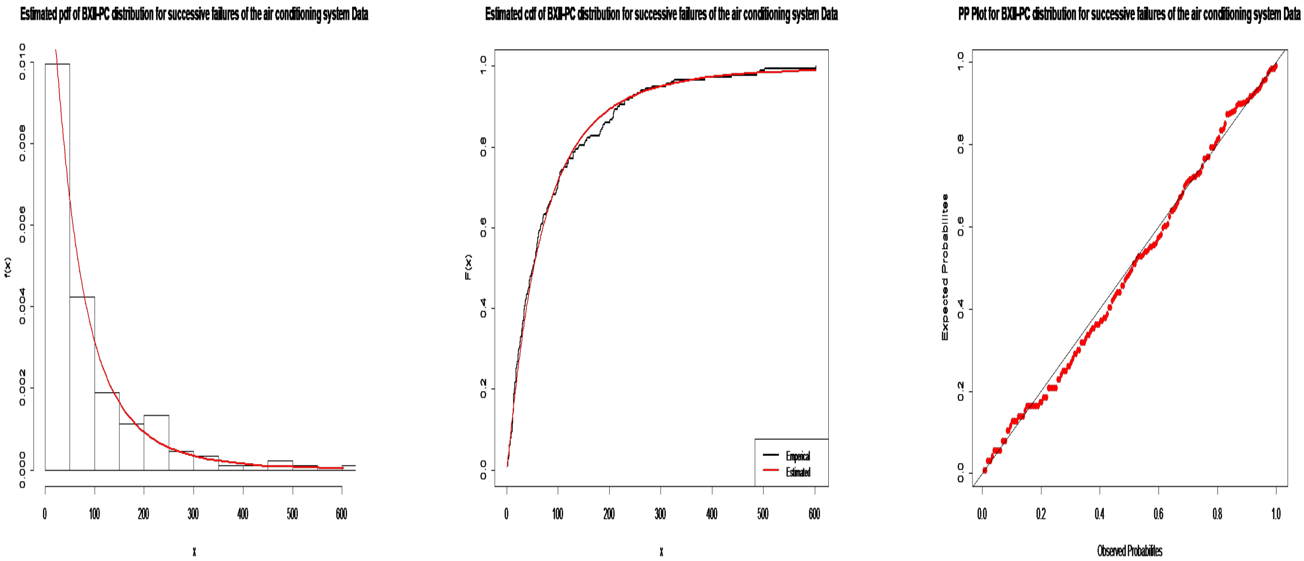

Figure 7. Fitted pdf (left), cdf (center) and PP (right) plots of the BXII-PC distribution for successive failures data.

Table 3. $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC for successive failures data.

Model	$-2\hat{\ell}$	AIC	CAIC	BIC	HQIC
BXII-PC	1957.812	1965.812	1966.042	1978.562	1970.982
BXII-HC	1961.65	1967.65	1967.787	1977.212	1971.527
L-PC	1961.652	1967.652	1967.789	1977.214	1971.529
LL-PC	1965.289	1971.289	1971.426	1980.851	1975.167
BIII-PC	1963.536	1971.536	1971.766	1984.286	1976.706
BIII-HC	1974.491	1980.491	1980.628	1990.053	1984.368
K-HC	1963.48	1969.48	1969.617	1979.042	1973.357
B-HC	1963.488	1969.488	1969.625	1979.05	1973.365
M-PC	1965.343	1971.343	1971.48	1980.905	1975.22
M-HC	1972.826	1976.826	1976.894	1983.201	1979.411
BXII	2216.964	2220.963	2221.032	2227.338	2223.548
PC	1969.137	973.137	1973.205	1979.511	1975.722
HC	1974.15	1976.15	1976.173	1979.337	1977.443

7. Conclusions

We propose a new probability distribution, named BXII-PC distribution, based on Cauchy and Burr XII distribution via T-X family method. Its pdf and hrf shapes are seen as very flexible forms. To illustrate the importance of the BXII-PC distribution, we establish various mathematical properties such as random number generator, sub-models, moments related properties, inequality measures, reliability measures and characterizations. We estimate the model parameters by six different methods. We perform a simulation study on the basis of graphical results to evaluate the performance of maximum likelihood, maximum product spacings, least squares, weighted least squares, Cramer-von Mises and Anderson-Darling estimators of the BXII-PC distribution. We demonstrate the potentiality and utility of the BXII-PC distribution by considering an application to successive failures of the air conditioning system. We apply various model selection criteria and graphical tools to examine the adequacy of the proposed distribution. We infer that the BXII-PC model is empirically suitable for the lifetime applications (successive failures analysis). Therefore, the BXII-PC model is a flexible, reasonable and parsimonious to other existing distributions. Hence it should be included in the distribution theory to assist the researchers. Further, as perspective of future projects, we may consider several intensive subjects (i) unit BXII-PC; (ii) Burr III-PC; (iii) log-Burr XII-Power Cauchy regression; (iv) various characteristics of the bivariate and the multivariate extensions of the BXII-PC; (v) Bayesian estimation of the BXII-PC parameters via complete and censored samples under different loss functions and (vi) the study of the complexity of the BXII-PC via Bayesian methods.

Appendix A

Theorem 7.1. *Let* (Ω, F, P) *be a given probability space and let* $H = [a_1, a_2]$ *be an interval with* $a_1 < a_2$ $(a_1 = -\infty,\ a_2 = \infty)$*. Let* $X : \Omega \to [a_1, a_2]$ *be a continuous random variable with distribution function F and Let* $g(x)$ *be a real function defined on* $H = [a_1, a_2]$ *such that* $E[g(X)|\, X \geq x] = h(x)$ *for* $x \in H$ *is defined with some real function* $h(x)$ *should be in simple form. Assume that* $g(x)\ \varepsilon\, C([a_1, a_2])$, $h(x)\, \varepsilon\, C^2([a_1, a_2])$ *and F is twofold continuously differentiable and strictly monotone function on the set* $[a_1, a_2]$*. We conclude, assuming that the equation* $g(x) = h(x)$ *has no real solution in the inside of* $[a_1, a_2]$*. Then F is obtained from the functions* $g(x)$ *and* $h(x)$ *as* $F(x) = \int_a^x k \left| \frac{h'(t)}{h(t)-g(t)} \right| \exp(-s(t))\, dt$, *where* $s(t)$ *is the solution of equation* $s'(t) = \frac{h'(t)}{h(t)-g(t)}$ *and k is a constant, chosen to make* $\int_{a_1}^{a_2} dF = 1$.

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Conflicts of interest

The authors declare no conflict of interest.

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