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Poisson Transmuted-G Family of Distributions: Its Properties and Applications

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Abstract

In this article, an extension of the transmuted-G family is proposed, in the so-called Poison transmuted-G family of distributions. Some of its statistical properties including quantile function, moment generating function, order statistics, probability weighted moment, stress-strength reliability, residual lifetime, reversed residual lifetime, Rényi entropy and mean deviation are derived. A few important special models of the proposed family are listed. Stochastic characterizations of the proposed family based on truncated moments, hazard function and reverse hazard function, are also studied. The family parameters are estimated via the maximum likelihood approach. A simulation study is carried out to examine the bias and mean square error of the maximum likelihood estimators. The advantage of the proposed family in data fitting is illustrated by means of two applications to failure time data sets.

Key Words: Transmuted-G family; hazard rate function; Maximum likelihood technique; Truncated moments; Simulation.

Mathematical Subject Classification: 60E05, 62E15

1. Introduction

In the last decades, many generalized families of continuous models have been introduced by extending classical probability models and applied to model various phenomena. However, there is a clear need for extended forms of the well-known models by adding one or more parameter(s) in order to obtain greater flexibility for modelling and evaluation different types of data. Shaw and Buckley (2007) proposed the transmuted-G (T-G) family of distributions. The cumulative distribution function (cdf, for short) and probability density function (pdf, for short) of the T-G family can be expressed respectively as follows

$$G^{\text{T-G}}(x;\alpha) = G(x)[1+\alpha-\alpha G(x)] \tag{1}$$

and

$$g^{\text{T-G}}(x;\alpha) = g(x)[1+\alpha-2\alpha G(x)], \tag{2}$$

where $G(x)$ and $g(x)$ are the baseline cdf and pdf, respectively. For $\alpha = 0, (|\alpha| \le 1)$, Eq. (1) gives the baseline distribution. Chakraborty *et al.* (2020) introduced the Kumaraswamy Poisson-G family, where generalized the Poisson-G (P-G) family of distributions. The cdf of the P-G family can be formulated as follows

$$F^{\text{P-G}}(x;\beta) = \frac{1-e^{-\beta F(x)}}{1-e^{-\beta}}, \beta \in R-\{0\}. \tag{3}$$

The pdf corresponding to Eq. (3) is given by

$$f^{\text{P-G}}(x;\beta)=\frac{\beta f(x)e^{-\beta F(x)}}{1-e^{-\beta}},\ \beta\in R-\{0\};-\infty<x<\infty. \tag{4}$$

In this paper, we propose a new extension of the T-G model having two parameters α and β by considering the T-G as the baseline distributions in the P-G family of distributions, in the so-called Poison transmuted-G (PT-G) family of distributions. The pdf and cdf of the PT-G family can be expressed respectively as follows

$$f^{\text{PT-G}}(x;\alpha,\beta)=\frac{\beta\, g(x)\,[1+\alpha-2\alpha\, G(x)]\exp\,[-\beta\, G(x)\{1+\alpha-\alpha\, G(x)\}]}{1-e^{-\beta}} \tag{5}$$

$$F^{\text{PT-G}}(x;\alpha,\beta)=\frac{1-\exp\,[-\beta\, G(x)\{1+\alpha-\alpha\, G(x)\}]}{1-e^{-\beta}} \tag{6}$$

The hazard rate function (hrf) corresponding to Eq. (5) is

$$h^{\text{PT-G}}(x;\alpha,\beta)=\frac{\beta\, g(x)\,[1+\alpha-2\alpha\, G(x)]\exp\,[-\beta\, G(x)\{1+\alpha-\alpha\, G(x)\}]}{\exp\,[-\beta\, G(x)\{1+\alpha-\alpha\, G(x)\}]-e^{-\beta}}$$

where $|\alpha|\le 1,\beta>0,x\in R$ and $G(x)$ is the baseline distribution with the corresponding pdf $g(x)$. We refer to this distribution as the Poisson transmuted-G family, in short as $PT-G(\alpha,\beta)$. The quantile function (qf) of a random variable X with distribution$PT-G(\alpha,\beta)$ say $Q(u)=F^{-1}(u)$, can be obtained by inverting (6) numerically and it is given by

$$Q(u)=G^{-1}\left[\frac{1+\alpha-\sqrt{(1+\alpha)^2-4\alpha R}}{2\alpha}\right],$$

where $R=-1/\beta\, log[\,1-(1-e^{-\beta})u]$. Some well-known G-families recently introduced in the literature are, Poisson-G family (Abouelmagd *et al*., 2017), beta-G Poisson family (Gokarna *et al*., 2018), Marshall-Olkin Kumaraswamy-G family (Handique *et al*., 2017), Generalized Marshall-Olkin Kumaraswamy-G family (Chakraborty and Handique 2018), Exponentiated generalized-G Poisson family (Gokarna and Haitham, 2017), beta Kumaraswamy-G family (Handique *et al*., 2017), beta generated Kumaraswamy Marshall-Olkin-G family (Handique and Chakraborty, 2017a), beta generalized Marshall-Olkin Kumaraswamy-G family (Handique and Chakraborty, 2017b), beta generated Marshall-Olkin-Kumaraswamy-G (Chakraborty *et al*., 2018), exponentiated generalized Marshall-Olkin-G family by (Handique *et al*., 2018), Kumaraswamy generalized Marshall-Olkin-G family (Chakraborty and Handique, 2018), odd modified exponential generalized family (Ahsan *et al*., 2018), Zografos-Balakrishnan Burr XII family (Emrah *et al.*, 2018), new Zero truncated Poisson family (Abouelmagd *et al*., 2019), Extended generalized Gompertz family (Thiago *et al*., 2019), odd flexible Weibull-H family (El-Morshedy and Eliwa, 2019), odd Half-Cauchy family (Chakraborty *et al*., 2020), odd log-logistic Lindley-G family (Alizadeh *et al*., 2020), generalized modified exponential-G family (Handique *et al*., 2020), "odd Chen-G, exponentiated odd Chen-G and discrete Gompertz-G" families (Eliwa *et al.*, 2020, 2020a and 2020b), beta Poisson-G family (Handique *et al*., 2020), among others. Our motivations for using the $PT-G$ family are the following:

- To generate distributions those are right-skewed, left-skewed and symmetric shaped.
- To provide consistently better fit than other generated models under the same baseline distribution.
- To make the kurtosis and the skewness more flexible compared to the baseline model.
- To define special models with all types of the hrf.
- To propose an extended class of distributions that contains the P-G and T-G distributions, which covers some important distributions as special and related cases.

1.1 Important sub models

Here we provide some special cases of the PT-G family of distributions and list their main distributional characteristics.

- **The PT- exponential (PT-E) distribution**

Consider the exponential model with scale parameter $\lambda > 0$, $g(x) = \lambda e^{-\lambda x}$ and $G(x) = 1 - e^{-\lambda x}$, $x > 0$, then for the PT-E model, the pdf and hrf respectively are

$$f^{\text{PT-E}}(x;\alpha,\beta,\lambda) = \frac{\beta\lambda e^{-\lambda x}[1+\alpha-2\alpha(1-e^{-\lambda x})]exp[-\beta(1-e^{-\lambda x})\{1+\alpha-\alpha(1-e^{-\lambda x})\}]}{1-e^{-\beta}}$$

and

$$h^{\text{PT-E}}(x;\alpha,\beta,\lambda) = \frac{\beta\lambda e^{-\lambda x}\left[1+\alpha-2\alpha\left(1-e^{-\lambda x}\right)\right]exp\left[-\beta\left(1-e^{-\lambda x}\right)\left\{1+\alpha-\alpha\left(1-e^{-\lambda x}\right)\right\}\right]}{exp[-\beta(1-e^{-\lambda x})\{1+\alpha-\alpha(1-e^{-\lambda x})\}]-e^{-\beta}}.$$

➢ **The PT- Weibull (PT-W) distribution**

Consider the Weibull distribution (Weibull, 1951) with parameters$\lambda > 0$ and $\theta > 0$ having pdf and cdf $g(x) = \lambda\theta x^{\theta-1}e^{-\lambda x^{\theta}}$ and $(x) = 1 - e^{-\lambda x^{\theta}}$, $x > 0$, respectively. The pdf and hrf of the PT-W distribution respectively are

$$f^{\text{PT-W}}(x;\alpha,\beta,\lambda,\theta) = \frac{\beta\lambda\theta x^{\theta-1}e^{-\lambda x^{\theta}}[1+\alpha-2\alpha(1-e^{-\lambda x^{\theta}})]exp[-\beta(1-e^{-\lambda x^{\theta}})\{1+\alpha-\alpha(1-e^{-\lambda x^{\theta}})\}]}{1-e^{-\beta}}$$

and

$$h^{\text{PT-W}}(x;\alpha,\beta,\lambda,\theta) = \frac{\beta\lambda\theta x^{\theta-1}e^{-\lambda x^{\theta}}\left[1+\alpha-2\alpha\left(1-e^{-\lambda x^{\theta}}\right)\right]exp\left[-\beta\left(1-e^{-\lambda x^{\theta}}\right)\left\{1+\alpha-\alpha\left(1-e^{-\lambda x^{\theta}}\right)\right\}\right]}{exp\left[-\beta\left(1-e^{-\lambda x^{\theta}}\right)\left\{1+\alpha-\alpha\left(1-e^{-\lambda x^{\theta}}\right)\right\}\right]-e^{-\beta}}.$$

Figures 1 and 2 show the pdf and cdf plots for PT-E and PT-W models under selected parameter values. It is found that the proposed family can be generated various models which able to model and evaluations various types of data sets. As we see in Figure 1, the generated models can be used to analyse positive and negative skewness data as well as symmetric data sets. Figure 2 shows that the shape of the hrf can be increasing, decreasing, unimodal and unimodal-bathtub.

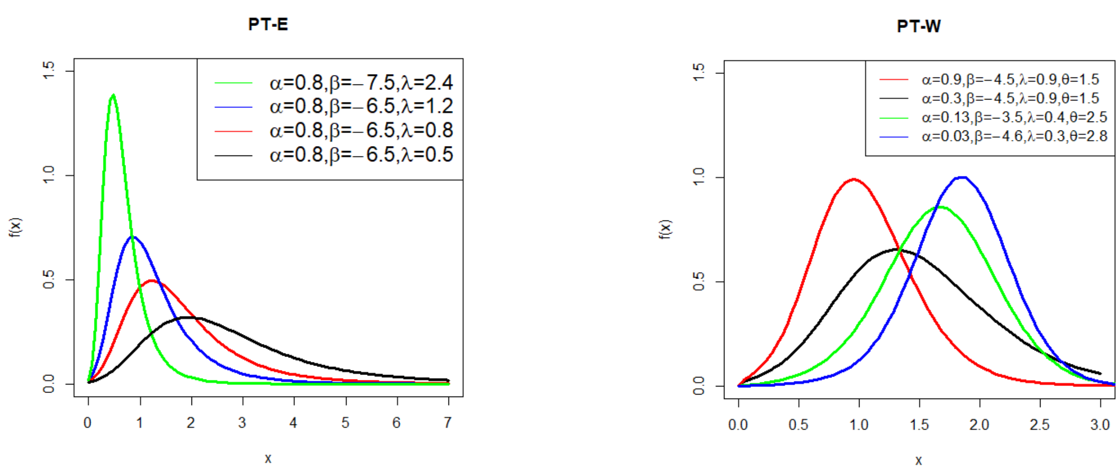

Fig 1: The pdf plots of the PT-E and PT-W distributions.

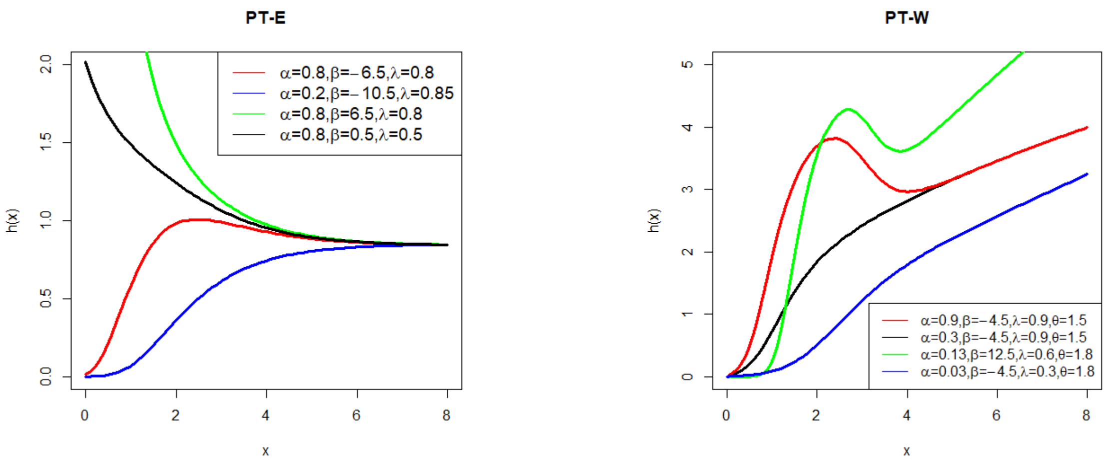

Fig 2: The hrf plots of the PT-E and PT-W distribution.

2. Properties

2.1 Linear Representation

In this Section, equations (5) and (6) can be expressed as infinite series expansion to show that the PT-G can be written as a linear combination of T-G as well as a linear combination of exponentiated-G distributions. These expressions will be helpful to study the mathematical characteristics of the PT-G family.

Using the power series for the exponential function, we can write (5) as

$$f^{\text{PT-G}}(x;\alpha,\beta)=g^{\text{T-G}}(x;\alpha)\sum_{i=0}^{\infty}\delta_i\,[G^{\text{T-G}}(x;\alpha)]^i \tag{7}$$

$$=\sum_{i=0}^{\infty}\delta_i'\,\frac{d}{dx}[G^{\text{T-G}}(x;\alpha)]^{i+1} \tag{8}$$

where $\delta_i'=\dfrac{(-1)^i\,\beta^{i+1}}{(1-e^{-\beta})\,(i+1)i!}$ and $\delta_i=\delta_i'\,(i+1)$.

Using Taylor series expansion of (6) we have

$$F^{\text{PT-G}}(x;\alpha,\beta)=\sum_{j=0}^{\infty}\xi_j\,[G^{\text{T-G}}(x;\alpha))]^j, \tag{9}$$

where $\xi_j=\dfrac{(-1)^{j+1}\,\beta^j}{(1-e^{-\beta})\,j!}$.

2.2 Moment Generating Function

The moment generating function (mgf) of PT-G family can be easily expressed in terms of those of the exponentiated T-G distribution using the results of Section 2.1. For example, using Eq. (8) it can be seen that

$$M_X^{\text{PT-G}}(s)=E[e^{sX}]=\int_{-\infty}^{\infty}e^{sx}f^{\text{PT-G}}(x;\alpha,\beta,)\,dx=\int_{-\infty}^{\infty}e^{sx}\sum_{i=0}^{\infty}\delta_i'\frac{d}{dx}[G^{\text{T-G}}(x;\alpha)]^{i+1}dx,$$

$$= \sum_{i=0}^{\infty} \delta_i' \int_{-\infty}^{\infty} e^{sx} \frac{d}{dx} [G^{\text{T-G}}(x;\alpha)]^{i+1} dx \quad = \sum_{i=0}^{\infty} \delta_i \, M_X^{\text{T-G}}(s) \, .$$

where $M_X^{\text{T-G}}(s)$ is the mgf of a exponentiated T-G distribution.

2.3 Distribution of Order Statistics

Consider a random sample $X_1, X_2, \ldots, X_n$ from any PT-G distribution. Let $X_{r:n}$ denote the r^{th} order statistic. The pdf of $X_{r:n}$ can be expressed as

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} f^{\text{ PT-G}}(x) F^{\text{ PT-G}}(x)^{r-1} \{1 - F^{\text{ PT-G}}(x)\}^{n-r}$$

$$= \frac{n!}{(r-1)!(n-r)!} \sum_{m=0}^{n-r} (-1)^m \binom{n-r}{m} f^{\text{ PT-G}}(x) [F^{\text{ PT-G}}(x)]^{m+r-1} .$$

The pdf of the r^{th} order statistic for of the PT-G can be derived by using the expansion of its pdf and cdf as

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} \sum_{m=0}^{n-r} (-1)^m \binom{n-r}{m} g^{\text{T-G}}(x;\alpha) \sum_{i=0}^{\infty} \delta_i [G^{\text{T-G}}(x;\alpha)]^i \left[\sum_{j=0}^{\infty} \xi_j [G^{\text{T-G}}(x;\alpha))]^j \right]^{m+r-1}$$

where δ_i and ξ_j defined above.

Using power series raised to power for positive integer $n\ (\geq 1)$ (see Gradshteyn and Ryzhik, 2007)

$\left(\sum_{i=0}^{\infty} a_i u^i \right)^n = \sum_{i=0}^{\infty} c_{n,i} u^i$, where the coefficient $c_{n,i}$ for $i = 1, 2, \ldots$ are easily obtained from the recurrence equation $c_{n,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i] a_m c_{n,i-m}$ where $c_{n,0} = a_o^n$.

Now $$\left[\sum_{j=0}^{\infty} \xi_j [G^{\text{TG}}(x;\alpha))]^j \right]^{m+r-1} = \sum_{j=0}^{\infty} d_{m+r-1,\, j} [G^{\text{OMEGP}}(x;\beta,\xi)]^j .$$

Where, $d_{m+r-1,\, j} = (j\, a_0)^{-1} \sum_{k=1}^{j} [k(m+r) - j] a_k \, d_{m+r-1,\, j-k}$.

Therefore the density function of the r^{th} order statistics of PT-G distribution can be expressed as

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} \sum_{m=0}^{n-r} (-1)^m \binom{n-r}{m} \sum_{i,\, j=0}^{\infty} \delta_i \, d_{m+r-1,\, j} [G^{\text{TG}}(x;\alpha)]^{i+j} g^{\text{TG}}(x;\alpha)$$

$$= \sum_{i,\, j=0}^{\infty} \psi_{i,\, j} [G^{\text{TG}}(x;\alpha)]^{i+j} g^{\text{TG}}(x;\alpha) \qquad (10)$$

$$= \sum_{i,j=0}^{\infty} \frac{\psi_{i,j}}{i+j+1} \frac{d}{dx} [G^{\mathrm{TG}}(x;\alpha)]^{i+j+1}$$

$$= \sum_{i,j=0}^{\infty} \psi'_{i,j} \frac{d}{dx} [G^{\mathrm{TG}}(x;\alpha)]^{i+j+1}$$

where $\psi_{i,j} = \frac{n!}{(r-1)!(n-r)!} \sum_{m=0}^{n-r} (-1)^m \binom{n-r}{m} \delta_i \, d_{m+r-1,j}$ and $\psi'_{i,j} = \psi_{i,j}/(i+j+1)$

2.4 Probability Weighted Moments

The probability weighted moments (PWM), first proposed by Greenwood *et al*. (1979), are expectations of certain functions of a random variable whose mean exists. The $(p,q,r)^{th}$ PWM of *T* is defined by

$$\Gamma_{p,q,r} = \int_{-\infty}^{\infty} x^p F(x)^q \, [1-F(x)]^r \, f(x) \, dx.$$

From equation (7) the s^{th} moment of T can be written as

$$E(X^s) = \int_0^{\infty} x^s \, f^{\mathrm{PTG}}(x;\alpha,\beta) \, dx = \sum_{i=0}^{\infty} \delta_i \int_0^{\infty} x^s \, [G^{\mathrm{TG}}(x;\alpha)]^i \, g^{\mathrm{TG}}(x;\alpha) \, dx = \sum_{i=0}^{\infty} \delta_i \, \Gamma_{s,\,i,0}.$$

where δ_i is the defined in section 2.1. Therefore the PWMs of the PT-G can be expressed in terms of the linear combination of the PWMs of the T-G distributions.

Proceeding similarly we can express s^{th} moment of the r^{th} order statistic $X_{i:n}$ a random sample of size *n* from PT-G on using equation (10) as $E(X^s_{i;n}) = \sum_{i,j=0}^{\infty} \psi_{i,j} \, \Gamma_{s,\, i+j,\, 0}$, where $\psi_{i,j}$ defined in above.

2.5 Stress-Strength System Reliability

In stress-strength modeling $R = P(X_1 < X_2)$ is a measure of component reliability of the system with random stress X_1 and strength X_2. It measures the probability that the systems strength X_2 is greater than environmental stress X_1 applied on that system. The probability of failure of a system is based on the probability of stress exceeding strength, whereas, the reliability of the system is the reversed probability. the system reliability is given by

$$R = P(X_1 < X_2) = P(Stress < Strength) = \int_0^{\infty} f_{Stress}(x) \, F_{Strength}(x) \, dx$$

Let X_1 and X_2 be two independent random variables with $\text{PT-G}(x;\alpha_1,\beta_1)$ and $\text{PT-G}(x;\alpha_2,\beta_2)$ distributions respectively. Then we have

$$R = \int_0^{\infty} f^{\text{PT-G}}(x;\alpha_1,\beta_1) \, F^{\text{PT-G}}(x;\alpha_2,\beta_2) \, dx \quad .$$

Note that the pdf and cdf of X_1 and X_2 are given by

$$f^{\text{PT-G}}(x;\alpha_1,\beta_1) = g^{\text{T-G}}(x;\alpha_1) \sum_{i=0}^{\infty} \delta_i^{(1)} [G^{\text{T-G}}(x;\alpha_1)]^i \quad \text{and} \quad F^{\text{PT-G}}(x;\alpha_2,\beta_2) = \sum_{j=0}^{\infty} \xi_j^{(2)} [G^{\text{T-G}}(x;\alpha_2))]^j .$$

Thus $$R=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\delta_i^{(1)}\,\xi_j^{(2)}\int_0^{\infty} g^{\text{T-G}}(x;\alpha_1)[G^{\text{T-G}}(x;\alpha_1)]^i\,[G^{\text{T-G}}(x;\alpha_2))]^j\,dx \quad ,$$

where $\delta_i^{(1)}=\dfrac{(-1)^i\,\beta_1^{\;i+1}}{(1-e^{-\beta_1})\;i!}$ and $\xi_j^{(2)}=\dfrac{(-1)^{j+1}\,\beta_2^{\;j}}{(1-e^{-\beta_2})\;j!}$.

2.6 Residual Life and Reversed Residual Life

Let X be a PT-G random variable with cdf in (6). The n^{th} moment of the residual life, say $m_n(t)=E[(X-t)^n/X>t],\ n=1,2,...$uniquely determines $F(x)$.The n^{th} moment of the residual life of X is given by

$$m_n(t)=\frac{1}{1-F^{\text{PT-G}}(t;\alpha,\beta)}\int_t^{\infty}(x-t)^n\,d\,F^{\text{PT-G}}(x;\alpha,\beta)$$

$$=\frac{1}{1-F^{\text{PT-G}}(t;\alpha,\beta)}\int_t^{\infty}\sum_{r=0}^{n}\binom{n}{r}\,x^r(-t)^{n-r}\;f^{\text{PT-G}}(x;\alpha,\beta)\,dx$$

$$=\frac{1}{1-F^{\text{PT-G}}(t;\alpha,\beta)}\sum_{i=0}^{\infty}\delta_i^{*}\int_t^{\infty}x^r\,[G^{\text{T-G}}(x;\alpha)]^i\,g^{\text{T-G}}(x;\alpha)\,dx\,, \tag{11}$$

where $\delta_i^{*}=\delta_i\sum_{r=0}^{n}\binom{n}{r}(-t)^{n-r}$.

The n^{th} moment of the reverse residual life, say $M_n(t)=E[(t-X)^n/X\le t],\ t>0,\ n=1,2,...$uniquely determines $F^{\text{PT-G}}(x;\alpha,\beta)$. We have

$$M_n(t)=\frac{1}{F^{\text{PT-G}}(t;\alpha,\beta)}\int_0^{t}(t-x)^n\,d\,F^{\text{PT-G}}(x;\alpha,\beta)$$

$$=\frac{1}{F^{\text{PT-G}}(t;\alpha,\beta)}\int_0^{t}\sum_{r=0}^{n}(-1)^r\binom{n}{r}x^r(-t)^{n-r}\;f^{\text{PT-G}}(x;\alpha,\beta)\,dx$$

$$=\frac{1}{F(t;\alpha,\beta)}\sum_{i=0}^{\infty}\delta_i^{**}\int_o^{t}x^r\,[G^{\text{T-G}}(x;\alpha)]^i\,g^{\text{T-G}}(x;\alpha)\,dx\,, \tag{12}$$

where $\delta_i^{**}=\delta_i\,(-1)^n\sum_{r=0}^{n}\binom{n}{r}t^{n-r}$. The mean residual life (MRL) of X can be obtained by setting n = 1 in equation (11) and is defined by $m_1(t)=E[(X-t)/X>t]$ also called the life expectation at age t which represents the expected additional life length for a unit which is alive at age t. The mean inactivity time (MIT) or mean waiting time (MWT), also called the mean reversed residual life function, is given by $M_1(t)=E[(t-X)/X\le t],\ t>0$ and it represents the waiting time elapsed since the failure of an item on the condition that this failure had occurred in $(0,t)$. The MIT of the PT-G family of distributions can be obtained easily by setting $n=1$in equation (12).

2.7 Rényi entropy

The entropy of a random variable is a measure of uncertainty. The Rényi entropy is defined as

$$I_R(\delta)=(1-\delta)^{-1}\log\left(\int_{-\infty}^{\infty} f(t)^{\delta}\,dt\right),$$

where $\delta>0$ and $\delta\neq 1$. Using the power series for the exponential function, we can write (5) as

$$f^{\text{PT-G}}(x;\alpha,\beta)^{\delta}=g^{\text{T-G}}(x;\alpha)^{\delta}\sum_{i=0}^{\infty}\mu_i\,[G^{\text{T-G}}(x;\alpha)]^{i\delta}\ ,\ \text{where}\ \mu_i=\frac{(-1)^i\,\beta^{\delta(i+1)}}{(1-e^{-\beta})^{\delta}\ i!}.$$

Therefore, the Rényi entropy of the PT-G family is given by

$$I_R(\delta)=(1-\delta)^{-1}\log\left(\int_0^{\infty} g^{\text{T-G}}(x;\alpha)^{\delta}\sum_{i=0}^{\infty}\mu_i\,[G^{\text{T-G}}(x;\alpha)]^{i\delta}dx\right)$$

$$=(1-\delta)^{-1}\log\left(\sum_{i=0}^{\infty}\mu_i\int_0^{\infty} g^{\text{T-G}}(x;\alpha)^{\delta}[G^{\text{T-G}}(x;\alpha)]^{i\delta}dx\right).$$

2.8 Mean Deviation

Let X be the PT-G random variable with mean $\mu=E(X)$ and median $M=\text{Median}(X)=Q(0.5)$. The mean deviation from the mean $[\delta_{\mu}(X)=E(|X-\mu|)]$ and the mean deviation from the median $[\delta_M(X)=E(|X-M|)]$ can be expressed as

$$\delta_{\mu}(X)=\int_{-\infty}^{\infty}|X-\mu|\ f(x)\,dx=\int_{-\infty}^{\mu}(\mu-x)\ f(x)\,dx+\int_{\mu}^{\infty}(x-\mu)\ f(x)\,dx=2\,\mu F(\mu)-2\Phi(\mu)$$

and $$\delta_M(X)=\int_{-\infty}^{\infty}|X-M|\ f(x)\,dx=\int_{-\infty}^{M}(M-x)\ f(x)\,dx+\int_{M}^{\infty}(x-M)\ f(x)\,dx=\mu-2\Phi(M)$$

respectively, where $F(\cdot)$ and $f(\cdot)$ are the cdf and pdf of the PT-G distribution, and $\Phi(t)=\int_{-\infty}^{t} x\,f(x)\,dx$.

We compute $\Phi(t)$ as follows:

$$\Phi(t)=\sum_{i=0}^{\infty}\delta_i\int_{-\infty}^{t} x\,g^{\text{T-G}}(x;\alpha)[G^{\text{T-G}}(x;\alpha)]^{i}\,dx\ ,$$ where δ_i defined in section 2.1.

2.9 Numerical computations

Tables 1, 2 and 3 report some numerical results of mean, variance, skewness and kurtosis of the PT-E model using R software.

Table 1: Some descriptive statistics using PT-E model as α grows to 1.

Model Parameters ↓			Mean	Variance	Skewness	Kurtosis
α	β	λ				
- 0.9	0.5	0.5	2.6160	4.4568	1.7635	7.8902
- 0.5			2.2263	4.2132	1.8805	8.4226
0.5			1.3090	2.3618	2.7033	14.4088
0.9			0.9634	1.1972	2.8296	17.3650

Table 2: Some descriptive statistics using PT-E model as β grows.

Model Parameters ↓			Mean	Variance	Skewness	Kurtosis
α	β	λ				
0.8	0.3	1.5	0.4553	0.2881	2.6332	13.5982
2.5			0.2747	0.1448	3.6948	24.9485
5.0			0.1380	0.0416	5.5175	61.1556
10			0.0589	0.0051	4.8573	84.2369

Table 3: Some descriptive statistics using PT-E model as λ grows.

Model Parameters ↓			Mean	Variance	Skewness	Kurtosis
α	β	λ				
0.8	0.7	0.9	0.6327	0.6139	3.125011	19.09734
2.5			0.2025	0.0629	3.125011	19.09734
5.0			0.1012	0.0157	3.125011	19.09734
10			0.0506	0.0039	3.125011	19.09734

From the above Tables 1, 2 and 3, the following observations can be noted:

- The proposed model is suitable of modelling positive skewness data sets.
- The proposed model is suitable of modelling leptokurtic (kurtosis > 3) data sets.
- The mean and variance always decrease with all the model parameters.
- The skewness and kurtosis are constant for fixed values of α and β with $\lambda \to \infty$.

3 Stochastic Characterisation

In this section we establish certain characterizations of the PT-G distribution in three directions: (i) based on two truncated moments; (ii) in terms of the hazard function and (iii) in terms of the reverse hazard function. These characterizations will be presented in three subsections.

3.1 Characterizations based on two truncated moments

This subsection deals with the characterizations of PT-G distribution in terms of a simple relationship between two truncated moments. We will employ a result of Glänzel (1986) given in theorem below.

Theorem 1: Let (Ω, F, P) be a given probability space and let $H=[a,b]$ be an interval for some $d<b$ ($a=-\infty,\ b=\infty$ might as well be allowed). Let $X:\Omega \rightarrow H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real function defined on H such that

$$E[q_2(X)\ /\ X \geq x] = E[q_1(X)\ /\ X \geq t]\ \xi(x)\ , \qquad x \in H,$$

is defined with some real function ξ. Assume that $q_1, q_2 \in C^1(H),\ \xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set *H*. Finally, assume that the equation $\xi q_1 = q_2$ has no real solution in the interior of *H*. Then F is uniquely determined by the functions q_1, q_2 and ξ , particularly

$$F(x) = \int_a^x C \left| \xi'(u) \big/ [\xi(u)\ q_1(u) - q_2(u)] \right| \exp(-s(u))\, du\ ,$$

where the function s is a solution of the differential equation $s' = \xi' q_1 \big/ (\xi q_1 - q_2)$ and *C* is the normalization constant, such that $\int_H d\,F = 1.$

For given a sequence $\{X_n\}$ of random variables with cdfs $\{F_n\}$ such that the functions $q_{1n},\ q_{2n}$ and ξ_n $(n \in N)$ satisfy the conditions of Theorem 1 and let $q_{1n} \rightarrow q_1,\ q_{2n} \rightarrow q_2$ for some continuously differentiable real functions q_1 and q_2 and *X* be a random variable with cdf *F*. Under the condition that $q_{1n}(X)$ and $q_{2n}(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to *X* in distribution if and only if ξ_n converges to

$$\xi(x) = \frac{E[q_2(X)\ /\ X \geq x]}{E[q_1(X)\ /\ X \geq x]}\ .$$

This stability theorem ensures that the convergence of $\{F_n\}$ is reflected by corresponding convergence of the functions $q_1,\ q_2$ and ξ_n respectively. This characterization is stable in the sense of weak convergence.

Proposition 3.1.1: Suppose *X* is a continuous random variable. Let $q_1(x) = \exp[-\beta G(x)\{1+\alpha-\alpha G(x)\}]$ and $q_2(x) = q_1(x)\,[1+\alpha-2\alpha G(x)]^2$ for $x \in \Re$. Then *X* has density (5) if and only if the function ξ defined in Theorem 1 is of the form $\xi(x) = \frac{1}{2}[\{1+\alpha-2\alpha G(x)\}^2 + (1-\alpha)^2],\ x \in \Re$

Proof: If *X* has density (5), then

$$(1-F^{\text{PT-G}}(x;\alpha,\beta))\,E\,[q_1(X)/X \geq x] = \frac{\beta}{4\alpha(1-e^{-\beta})}[\{1+\alpha-2\alpha G(x)\}^2 - (1-\alpha)^2],\ x \in \Re$$

and

$$(1-F^{\text{PTG}}(x;\alpha,\beta))\,E\,[q_2(X)/X \geq x] = \frac{\beta}{8\alpha(1-e^{-\beta})}[\{1+\alpha-2\alpha G(x)\}^4 - (1-\alpha)^4],\ x \in \Re$$

and hence $$\xi(x)=\frac{1}{2}[\{1+\alpha-2\alpha G(x)\}^{2}+(1-\alpha)^{2}],\ x\in\Re.$$

We also have, $\xi(x)q_1(x)-q_2(x)=-\frac{1}{2}q_1(x)[\{1+\alpha-2\alpha G(x)\}^{2}-(1-\alpha)^{2}]<0$ for $x\in\Re$.

Conversely, if ξ is of the above form, then

$$s'(x)=\frac{\xi'(x)q_1(x)}{\xi(x)q_1(x)-q_2(x)}$$
$$=\frac{4\alpha g(x)[1+\alpha-2\alpha G(x)]}{[1+\alpha-2\alpha G(x)]^{2}-(1-\alpha)^{2}},\ x\in\Re$$

and $s(x)=-\log[\{1+\alpha-2\alpha G(x)\}^{2}-(1-\alpha)^{2}]$.

Now, according to Theorem 1, *X* has pdf given in equation (5).

Corollary 3.1.1: Suppose *X* is a continuous random variable. Let $q_1(x)$ be as in Proposition 3.1.1. Then *X* has pdf in equation (5) if and only if there exist functions q_2 and ξ defined in Theorem 1 for which the following first order differential equation holds

$$\frac{\xi'(x)q_1(x)}{\xi(x)q_1(x)-q_2(x)}=\frac{4\alpha g(x)[1+\alpha-2\alpha G(x)]}{[1+\alpha-2\alpha G(x)]^{2}-(1-\alpha)^{2}},\ x\in\Re$$

Corollary 3.1.2: The differential equation in Corollary 3.1.1 has the following general solution

$$\xi(x)=\frac{1}{[\{1+\alpha-2\alpha G(x)\}^{2}-(1-\alpha)^{2}]}\left[-\int 4\alpha g(x)[1+\alpha-2\alpha G(x)]\,(q_1(x))^{-1}q_2(x)dx+D\right].$$

where *D* is a constant. A set of functions satisfying the above differential equation is given in Proposition 3.1.1 with $D=1/2$. Clearly, there are other triplets (q_1,q_2,ξ) satisfying the conditions of Theorem 1.

3.2 Characterization based on hazard function

The hazard function, h_F, of a twice differentiable distribution function, *F*, satisfies the following trivial differential equation

$$f'(x)/f(x)=[h'_F(x)/h_F(x)]-h_F(x).$$

The following proposition establishes a non-trivial characterization of PT-G distribution based on the hazard function.

Proposition 3.2.1: Suppose *X* is a continuous random variable. Then, *X* has density (5) if and only if its hazard function $h^{\text{PT-G}}(x)$ satisfies the following first order differential equation

$$h^{\text{PT-G}}(x)-\beta g(x)[1+\alpha-2\alpha G(x)]h^{\text{PT-G}}(x)=\beta\exp[-\beta G(x)\{1+\alpha-2\alpha G(x)\}]\times\frac{d}{dx}\left\{\frac{g(x)\{1+\alpha-2\alpha G(x)\}}{\exp[-\beta G(x)\{1+\alpha-2\alpha G(x)\}]-e^{-\beta}}\right\}$$

$x \in \Re$ with the boundary condition $\lim_{x\to-\infty} h^{\text{PT-G}}(x) = \frac{\beta(1+\alpha)}{1-e^{-\beta}} \lim_{x\to-\infty} g(x)$.

Proof: Is straightforward and hence omitted.

3.3 Characterization in terms of the reverse hazard function

The reverse hazard function, r_F, of a twice differentiable distribution function, *F*, is defined as

$$r_F = f(x)/F(x), \quad x \in \text{support of } F .$$

In this subsection we present a characterization of the PT-G distribution in terms of the reverse hazard function.

Proposition 3.3.1: Let $X : \Omega \to \Re$ be a continuous random variable. The random variable *X* has density (5) if and only if its reverse hazard function $r^{\text{PT-G}}(x)$ satisfies differential equation

$$r^{\text{PT-G}}(x) - \beta g(x)[1+\alpha-2\alpha G(x)] r^{\text{PT-G}}(x) = \beta \exp[-\beta G(x)\{1+\alpha-2\alpha G(x)\}] \times \frac{d}{dx}\left\{\frac{g(x)\{1+\alpha-2\alpha G(x)\}}{\exp[-\beta G(x)\{1+\alpha-2\alpha G(x)\}]-e^{-\beta}}\right\}$$

$x \in \Re$ with the boundary condition $\lim_{x\to\infty} r^{\text{PT-G}}(x) = \frac{\beta(1+\alpha)}{1-e^{-\beta}} \lim_{x\to\infty} g(x)$.

4. Maximum Likelihood Estimation

Let $x = (x_1, x_2, ..., x_n)$ be a random sample of size n from PT-G with parameter vector $\boldsymbol{\rho} = (\alpha, \beta, \boldsymbol{\eta})$, where $\boldsymbol{\eta} = (\eta_1, \eta_2, ..., \eta_q)$ is the parameter vector of *G*. The log-likelihood function is written as

$$\ell = \ell(\boldsymbol{\rho}) = n\log\beta - n\log(1-e^{-\beta}) + \sum_{i=1}^{n}\log[g(x_i,\boldsymbol{\eta})] + \sum_{i=1}^{n}\log[1+\alpha-2\alpha G(x_i,\boldsymbol{\eta})]$$

$$-\beta\sum_{i=1}^{n} G(x_i,\boldsymbol{\eta})\,[1+\alpha-\alpha G(x_i,\boldsymbol{\eta})] .$$

This log-likelihood function cannot be solved analytically because of its complex form but it can be maximized numerically by employing global optimization methods available with the software's R. By taking the partial derivatives of the log-likelihood function with respect to the parameter α , β and $\boldsymbol{\eta}$, we obtain the components of the score vector $U_{\boldsymbol{\rho}} = (U_\alpha, U_\beta, U_\eta)$.

The asymptotic variance-covariance matrix of the MLEs of parameters can obtained by inverting the Fisher information matrix $\mathrm{I}(\boldsymbol{\rho})$ which in turn can be derived using the second partial derivatives of the log-likelihood function with respect to each parameter. The $i\,j^{th}$ elements of $\mathrm{I}_n(\boldsymbol{\rho})$ are given by

$$\mathrm{I}_{ij} = -E[\partial^2 l(\boldsymbol{\rho}) / \partial\rho_i\,\partial\rho_j] , \quad i, j = 1, 2+q .$$

The exact evaluation of the above expectations may be cumbersome. In practice one can estimate $\mathrm{I}_n(\boldsymbol{\rho})$ by the observed Fisher's information matrix $\hat{\mathrm{I}}_n(\hat{\boldsymbol{\rho}}) = (\hat{\mathrm{I}}_{ij})$ defined as

$$\hat{\mathrm{I}}_{ij} \approx \left(-\partial^2 l(\boldsymbol{\rho}) / \partial\rho_i\,\partial\rho_j\right)_{\boldsymbol{\eta}=\hat{\boldsymbol{\eta}}} , \quad i, j = 1, 2+q .$$

Using the general theory of MLEs under some regularity conditions on the parameters as $n \to \infty$ the asymptotic distribution of $\sqrt{n}\,(\hat{\boldsymbol{\rho}} - \boldsymbol{\rho})$ is $N_k(0, V_n)$ where $V_n = (v_{jj}) = \mathrm{I}_n^{-1}(\boldsymbol{\rho})$. The asymptotic behaviour remains valid if V_n is replaced by $\hat{V}_n = \hat{\mathrm{I}}^{-1}(\hat{\boldsymbol{\rho}})$. Using this result large sample standard errors of j^{th} parameter ρ_j is given by $\sqrt{\hat{v}_{jj}}$.

4.1 Simulation

In order to assess the performance of the MLEs, a simulation study is performed utilizing the statistical software **R** through the package (**stats4**), command ***mle***. Then 1000 replications of samples of size $n = 20, 25, \ldots, 100$ from PT-E (α, β, λ) in section 1.1 model for the following three cases are generated:

(i) $\alpha = 0.5,\ \beta = 0.5,\ \lambda = 0.5$,

(ii) $\alpha = 0.6,\ \beta = 0.7,\ \lambda = 0.8$ and

(iii) $\alpha = 0.6,\ \beta = 0.7,\ \lambda = 0.9$.

To maximize the log likelihood function, the **MaxBFGS** subroutine is used with analytical derivatives. The evaluation of the estimates was performed based on the following quantities for each sample size: the empirical biases, and mean squared errors (MSEs) are calculated utilizing the **R** package from the MC replications, where

$$Bias(\theta) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\theta}_j - \theta)\ \rho_j \quad \text{and} \quad MSE(\theta) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\theta}_j - \theta)^2$$

The simulation results are graphically presented in the Figures 3, 4 and 5. From these figures it is observed that the bias and MSE decrease as the sample size n grows in all cases showing consistency and asymptotic unbiasedness of the mles. It is also noted that the bias for the parameter is always negative unlike for the others. But the bias in all cases converges to zero very quickly as the sample size exceeds 30.

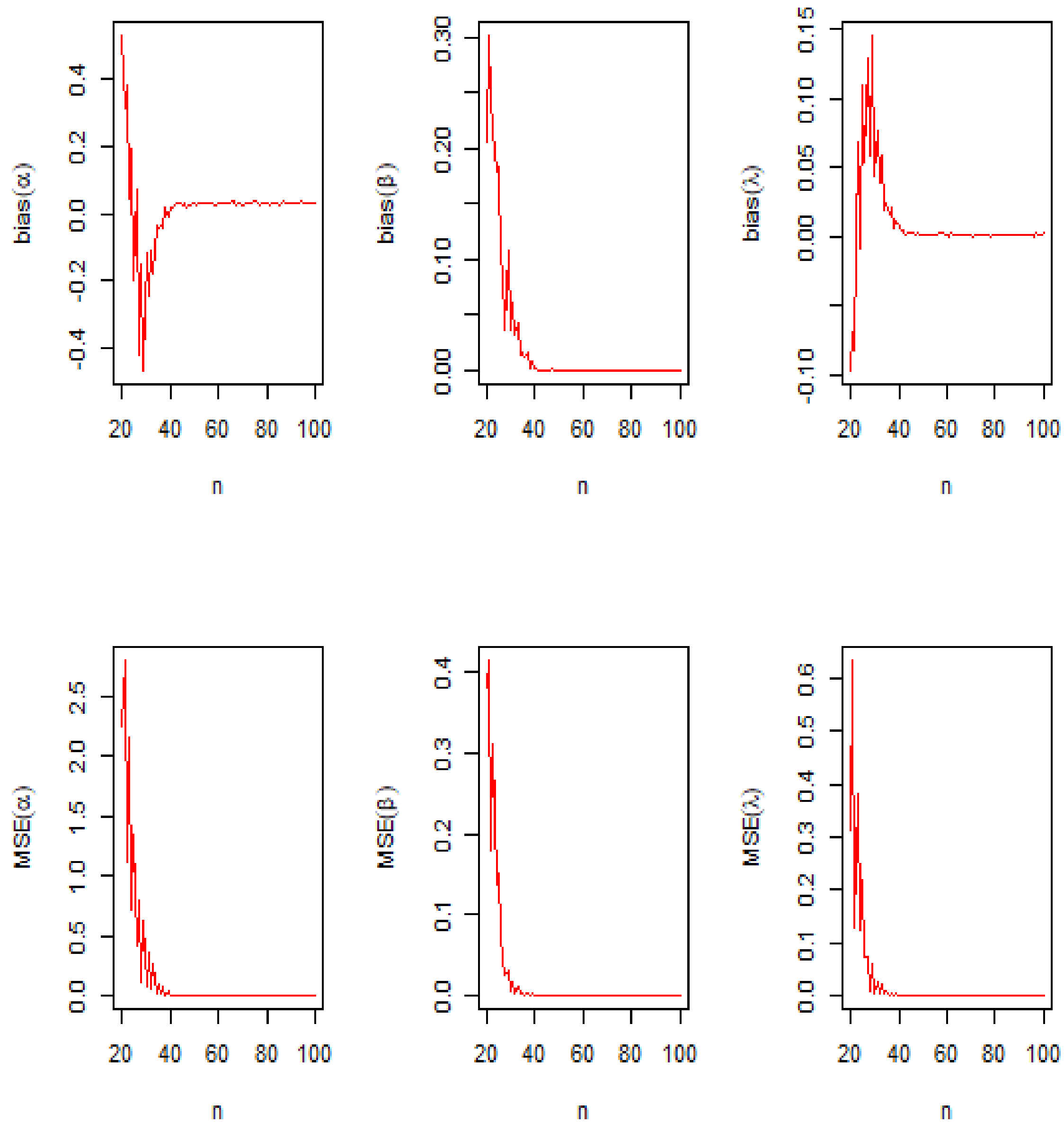

Fig 3: The biases and MSEs of the model parameter estimated versus n based on case I.

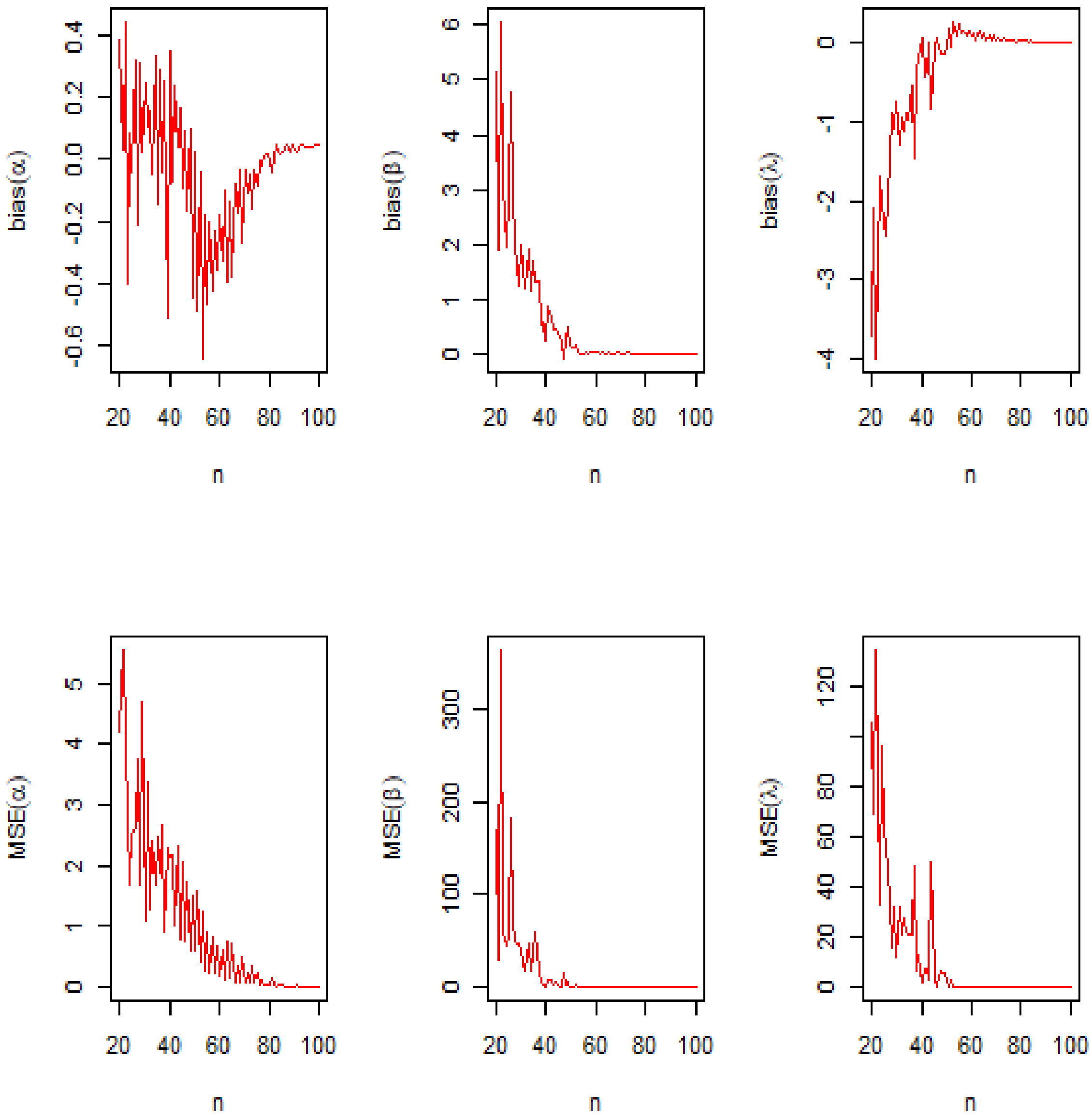

Fig 4: The biases and MSEs of the model parameter estimated versus n based on case II.

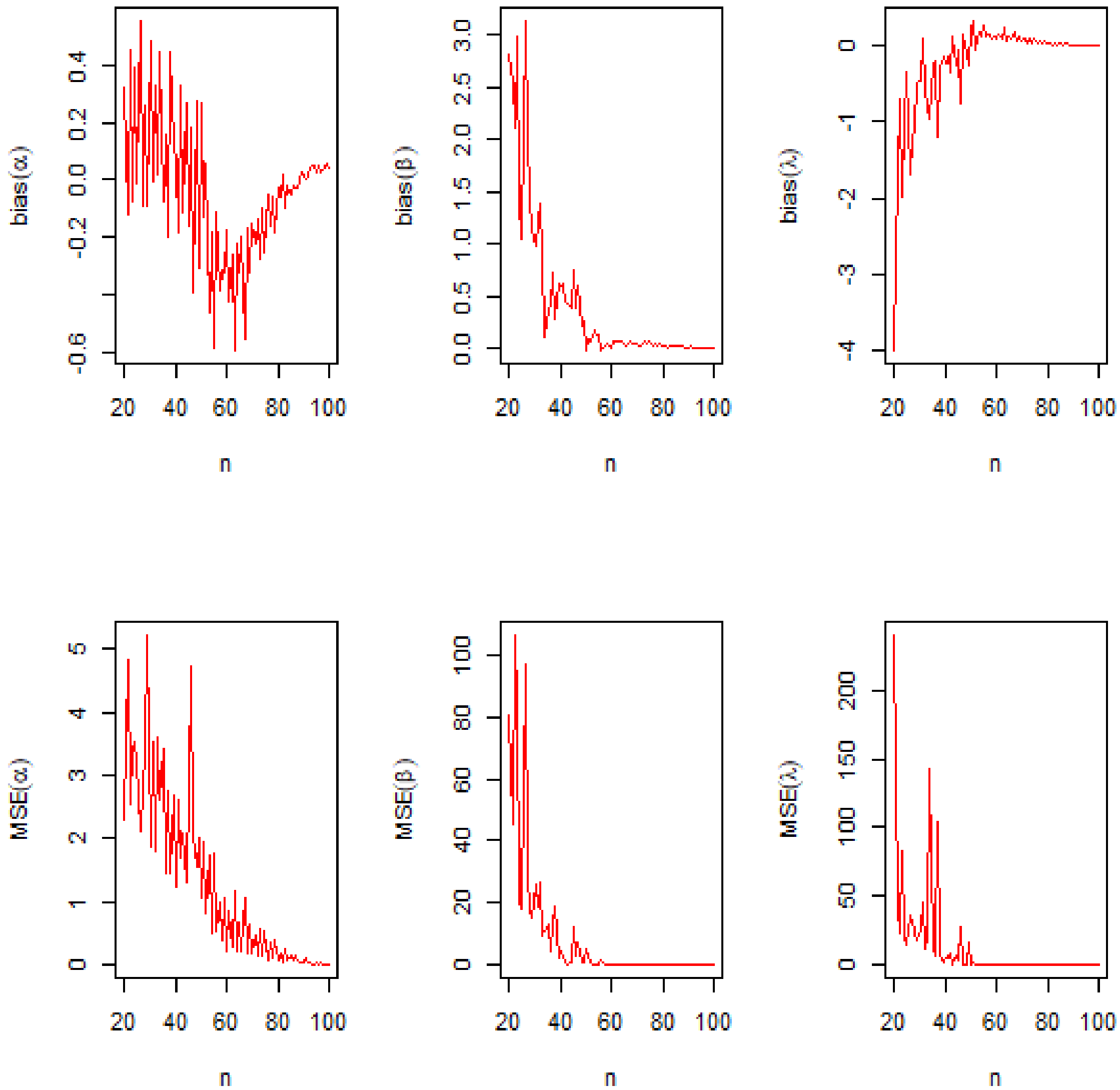

Fig 5: The biases and MSEs of the model parameter estimated versus n based on case III.

4 Real life applications

Here we consider fitting of two failure time data sets to show that the distributions from the proposed $\mathrm{PT-E}(\alpha,\beta,\lambda)$ family can provide better model than the corresponding distributions exponential (Exp), moment exponential (ME), Marshall-Olkin exponential (MO-E) (Marshall and Olkin, 1997), generalized Marshall-Olkin exponential (GMO-E) (Jayakumar and Mathew, 2008), Kumaraswamy exponential (Kw-E) (Cordeiro and de Castro, 2011), Beta exponential (B-E) (Eugene *et al*., 2002), Marshall-Olkin Kumaraswamy exponential (MOKw-E) (Handique *et al*., 2017), Kumaraswamy Marshall-Olkin exponential (KwMO-E) (Alizadeh *et al*., 2015), beta Poisson exponential (Handique *et al*., 2020) and Kumaraswamy Poisson exponential (Chakraborty *et al*., 2020) distribution.

The first data is about survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960), while the second one represents the lifetime data relating to relief times (in minutes) of patients receiving an analgesic. The data was reported by Gross and Clark (1975) and it has twenty (20) observations. The descriptive statistics of the data sets are tabulated in Table 4 and both the data are positively skewed.

Table 4: Descriptive Statistics for the data sets

Data Sets	n	Min.	Mean	Median	s.d.	Skewness	Kurtosis	1^{st} Qu.	3^{rd} Qu.	Max.

I	72	0.100	1.851	1.560	1.200	1.788	4.157	1.080	2.303	7.000
II	20	1.100	1.900	1.700	0.704	1.592	2.346	1.475	2.050	4.100

The total time on test (TTT) plot (see Aarset, 1987) for the data sets shown in the Fig. 6 indicate that the both data sets have increasing hazard rate.

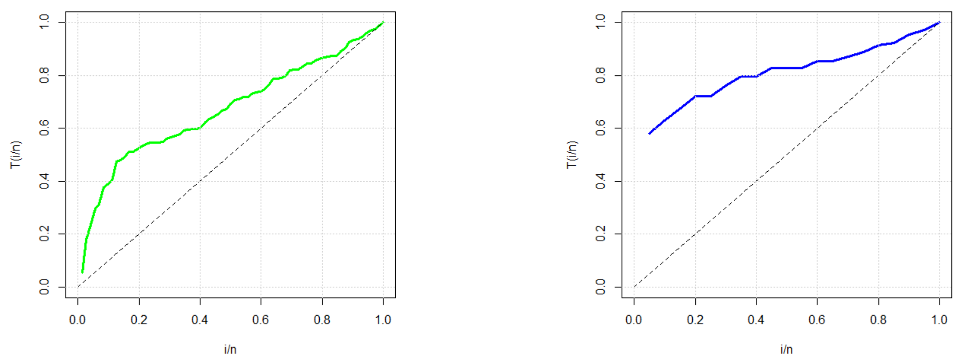

Fig 6: TTT-plots for the Data set I and Data set II

The mle's of the parameters with corresponding standard errors in the parentheses for all the fitted models along are given in Table 5 and Table 7 for data set I and data set II respectively. While the various model selection criteria namely the AIC, BIC, CAIC, HQIC, A, W and KS statistic with p-value for the fitted models of the data sets I and II are presented respectively in Table 6 and Table 8.

We have considered some well known model selection criteria namely the AIC, BIC, CAIC and HQIC and the Kolmogorov-Smirnov (K-S) statistics, Anderson-Darling (A) and Cramer von-mises (W) for goodness of fit to compare the fitted models. We have also provided the asymptotic standard errors and confidence intervals of the mles of the parameters for each competing models. Visual comparison fitted density and the fitted cdf are presented in Figures 7 and 8. These plots reveal that the proposed distributions provide a good fit to these data.

From these findings based on the lowest values different criteria the PT-E is found to be a better model than the models Exp, ME, MO-E, GMO-E, Kw-E, B-E, MOKw-E, KwMO-E, BP-E and KwP-E for both the data sets. More over visual comparison of the closeness of the fitted density with the observed histogram and fitted cdf with the observed ogive of the data sets I and II are presented in the Figures 7 and 8 respectively also indicate that the proposed distributions provide comparatively closer fit to these data sets.

Table 5: MLEs, standard errors (in parentheses) values for the guinea Pigs survival time's data set

Models	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$

Exp (λ)	---	---	---	---	0.540 (0.063)
ME (λ)	---	---	---	---	0.925 (0.077)
MO-E (α, λ)	---	---	8.778 (3.555)	---	1.379 (0.193)
GMO-E (b, α, λ)	---	0.179 (0.070)	47.635 (44.901)	---	4.465 (1.327)
Kw-E (a, b, λ)	3.304 (1.106)	1.100 (0.764)	---	---	1.037 (0.614)
B-E (a, b, λ)	0.807 (0.696)	3.461 (1.003)	---	---	1.331 (0.855)
MOKw-E (α, a, b, λ)	2.716 (1.316)	1.986 (0.784)	0.008 (0.002)	---	0.099 (0.048)
KwMO-E (a, b, α, λ)	3.478 (0.861)	3.306 (0.779)	0.373 (0.136)	---	0.299 (1.112)
BP-E (a, b, β, λ)	3.595 (1.031)	0.724 (1.590)	---	0.014 (0.010)	1.482 (0.516)
KwP-E (a, b, β, λ)	3.265 (0.991)	2.658 (1.984)	---	4.001 (5.670)	0.177 (0.226)
PT-E (α, β, λ)	---	---	0.813 (0.182)	-6.587 (1.448)	0.841 (0.192)

Table 6: Log-likelihood, AIC, BIC, CAIC, HQIC, A, W and KS (p-value) values for the guinea Pigs survival times data set

Models	AIC	BIC	CAIC	HQIC	A	W	KS (p-value)
Exp (λ)	234.63	236.91	234.68	235.54	6.53	1.25	0.27 (0.06)
ME (λ)	210.40	212.68	210.45	211.30	1.52	0.25	0.14 (0.13)
MO-E (α,λ)	210.36	214.92	210.53	212.16	1.18	0.17	0.10 (0.43)
GMO-E (b,α,λ)	210.54	217.38	210.89	213.24	1.02	0.16	0.09 (0.51)
Kw-E (a,b,λ)	209.42	216.24	209.77	212.12	0.74	0.11	0.08 (0.50)
B-E (a,b,λ)	207.38	214.22	207.73	210.08	0.98	0.15	0.11 (0.34)
MOKw-E (α,a,b,λ)	209.44	218.56	210.04	213.04	0.79	0.12	0.10 (0.44)
KwMO-E (a,b,α,λ)	207.82	216.94	208.42	211.42	0.61	0.11	0.08 (0.73)
BP-E (a,b,β,λ)	205.42	214.50	206.02	209.02	0.55	0.08	0.09 (0.81)
KwP-E (a,b,β,λ)	206.63	215.74	207.23	210.26	0.48	0.07	0.09 (0.79)
PT-E (α,β,λ)	202.09	208.92	202.44	204.81	0.36	0.05	0.07 (0.86)

Table 7: MLEs, standard errors (in parentheses) values for the relief times of patients receiving an analgesic failure time data set

Models	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
Exp (λ)	---	---	---	---	0.526 (0.117)
ME (λ)	---	---	---	---	0.950 (0.150)
MO-E (α,λ)	---	---	54.474 (35.582)	---	2.316 (0.374)
GMO-E (b,α,λ)	---	0.519 (0.256)	89.462 (66.278)	---	3.169 (0.772)
Kw-E (a,b,λ)	83.756 (42.361)	0.568 (0.326)	---	---	3.330 (1.188)
B-E (a,b,λ)	81.633 (120.41)	0.542 (0.327)	---	---	3.514 (1.410)
MOKw-E (α,a,b,λ)	33.232 (57.837)	0.571 (0.721)	0.133 (0.332)	---	1.669 (1.814)
KwMO-E (a,b,α,λ)	34.826 (22.312)	0.299 (0.239)	28.868 (9.146)	---	4.899 (3.176)
BP-E (a,b,β,λ)	13.396 (1.494)	9.600 (1.091)	---	1.965 (0.341)	0.244 (0.037)
KwP-E (a,b,β,λ)	11.837 (6.493)	3.596 (2.392)	---	5.983 (1.470)	0.225 (0.098)
PT-E (α,β,λ)	---	---	0.301 (0.037)	-9.997 (3.336)	1.555 (0.241)

Table 8: Log-likelihood, AIC, BIC, CAIC, HQIC, A, W and KS (p-value) values for the relief times of patients receiving an analgesic failure time data set

Models	AIC	BIC	CAIC	HQIC	A	W	KS (*p*-value)
Exp (λ)	67.67	68.67	67.89	67.87	4.60	0.96	0.44 (0.004)
ME (λ)	54.32	55.31	54.54	54.50	2.76	0.53	0.32 (0.07)
MO-E (α,λ)	43.51	45.51	44.22	43.90	0.81	0.14	0.18 (0.55)
GMO-E (b,α,λ)	42.75	45.74	44.25	43.34	0.51	0.08	0.15 (0.78)
Kw-E (a,b,λ)	41.78	44.75	43.28	42.32	0.45	0.07	0.14 (0.86)
B-E (a,b,λ)	43.48	46.45	44.98	44.02	0.70	0.12	0.16 (0.80)
MOKw-E (α,a,b,λ)	41.58	45.54	44.25	42.30	0.60	0.11	0.14 (0.87)
KwMO-E (a,b,α,λ)	42.88	46.84	45.55	43.60	1.08	0.19	0.15 (0.86)
BP-E (a,b,β,λ)	38.07	42.02	40.73	38.78	0.39	0.06	0.14 (0.91)
KwP-E (a,b,β,λ)	38.32	42.28	40.98	39.04	0.41	0.05	0.13 (0.93)
PT-E (α,β,λ)	36.84	39.81	38.34	37.38	0.37	0.04	0.11 (0.95)

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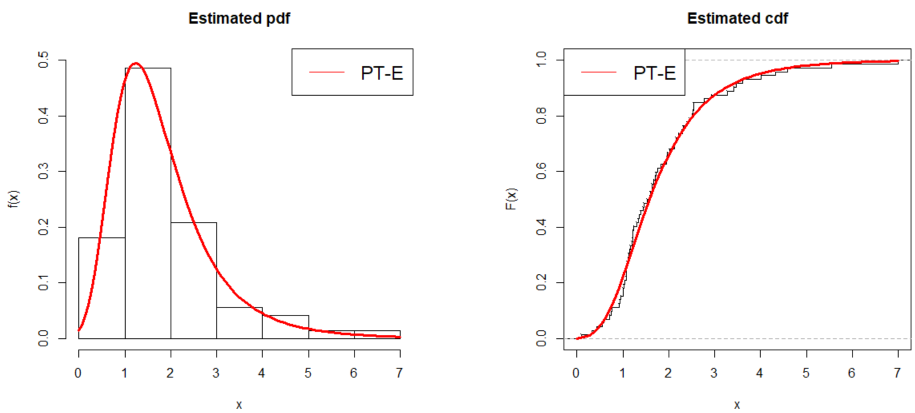

Fig 7: Plots of the observed histogram and estimated pdf on left and observed ogive and estimated cdf for the PT-E model for data set I

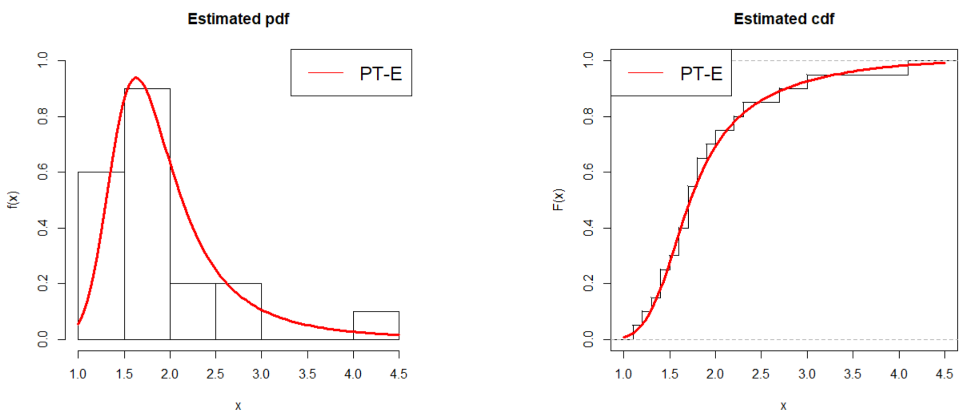

Fig 8: Plots of the observed histogram and estimated pdf on left and observed ogive and estimated cdf for the PT-E model for data set II

Finally, in order to further check how well our model captures the important characteristics of the observed data sets in Table 9 we lists the corresponding estimated values for data sets I and II based on the PT-E distribution.

Table 9: Estimated descriptive statistics using PT-E model for data sets I and II.

Data Set ↓ Measures →	Mean	Std. Dev.	Skewness	Kurtosis
I	1.8317	1.12102	1.7700	5.7168
II	1.6558	0.79997	1.2367	2.7068

From Table 9, it is observed that the PT-E distribution captures them quite well for analyzing data sets I and II because this model gives approximately the same values as compared to the real statistics for both data sets (see Table 4), thus confirming the adequacy of the proposed distribution.

5 Conclusion

We propose a new Poisson transmuted-G (PT-G) family of distributions, which extends the transmuted family by adding two additional parameters. Many well-known distributions emerge as special cases of the PT-G family for particular values for the parameters. The mathematical properties of the new family including explicit expansions for the quantile function, moments generating function, order statistics, Probability weighted moments, stress-strength reliability, residual life, reversed residual life, Rényi entropy and mean deviation are provided. Stochastic characterisations are discussed. Some numerical computations of important characterists are illustrated. The model parameters are estimated by the maximum likelihood method. Simulation study carried out to examine the behaviour of the bias and mean square error of the maximum likelihood estimators returned very good assessment. Two real data sets modelling established that distribution from of the PT-G family can give much better fits than other distributions from some well-known families.

Conflict of interest: All the authors declare that there is no conflict of interest.

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