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K-8 Preservice Teachers' Inductive Reasoning in the Problem-Solving Contexts

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Abstract

This paper reports the results from an exploratory study of K-8 pre-service teachers' inductive reasoning. The analysis of 130 written solutions to seven tasks and 77 reflective journals completed by 20 pre-service teachers lead to descriptions of inductive reasoning processes, i.e. specializing, conjecturing, generalizing, and justifying, in the problem-solving contexts. The uncovered characterizations of the four inductive reasoning processes were further used to describe pathways of successful generalizations. The results highlight the importance of specializing and justifying in constructing powerful generalizations. Implications for teacher education are discussed.

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Introduction

The phrase *mathematical reasoning* is typically associated with formal proof, that is, a process in which one logically deduces, from a set of principles or definitions, conclusions about all classes of particular instances. In their recent publication, the National Council of Teachers of Mathematics, NCTM (2009) defined mathematical reasoning much more broadly, as a process of making inferences on the basis of evidence or stated principles highlighting that "mathematical reasoning can take many forms, ranging from informal explanation and justification to formal deduction, as well as inductive observations," (p.4). Although, without a formal proof, the validity of inductive reasoning, that is generalizing from finite incomplete classes of particular instances cannot be certain, inductive observations assist students in making mathematical discoveries and support their learning of mathematics with understanding. As such, inductive reasoning plays an important role in the K-8 mathematics. Curriculum standards for school mathematics (NCTM, 2000; National Governor's Association (NGA) & Council of Chief State School Officers: Common Core State Mathematics Standards, [CCSSM], 2011) have set expectations that elementary and middle school students develop an understanding of algebraic concepts in a way that supports generalizing and extending the ideas of arithmetic to the concepts of algebra. The standard documents emphasize that K-8 students should learn algebraic concepts as a "set of . . . competencies tied to the representation of quantitative relationships and as a style of thinking for formalizing patterns, functions, and generalizations," (NCTM, 2000; p. 223). Kaput (1999) argued that the process of generalizing supports students' development of more abstract ways of thinking; in the process of generalizing initial objects of one's reasoning (i.e., specific cases and situations) might be replaced with new objects such as patterns, procedures, structures and relationships. In that sense inductive reasoning has the potential to support students' ability to solve problems using abstractions and ultimately operate on mathematical entities logically and independently from the specific context.

Expectations that students reason inductively and make sense of mathematics need to be supported by strong preparation of K-8 pre-service teachers. Little attention, however, has been paid to pre-service teachers' inductive reasoning in contextually-based situations nonetheless that inductive reasoning plays a significant role in the K-8 mathematics; facilitates problem solving, learning and the development of expertise (Haverty, Koedinger, Klahr, & Alibali, 2000). Past studies (e.g., Rivera & Becker, 2007; Hallagan, Rule, & Carlson, 2009; Richardson, Berenson, & Staley, 2009) predominantly focused on pre-service teachers' inductive reasoning in the context of analyzing numerical or geometric (figural) patterns. Given the importance of inductive reasoning in problem solving and the lack of literature that explores pre-service teachers' inductive reasoning in contextually-based situations the goal of this research was to:

(1) Characterize pre-service teachers' inductive reasoning processes in problem-solving contexts, and

(2) Characterize pathways of successful generalizations.

Inductive Reasoning

Inductive reasoning processes that are central to this work represent what Polya (1981) described as the acquisition of new knowledge, namely, discovering properties from specific phenomena and analyzing regularities in a logical way. When describing inductive processes Harverty et al. (2000) specified that when reasoning inductively one engages in (1) data gathering, (2) pattern finding, and (3) hypothesis generation. Cañadas and Castro (2007, 2009) conceptualized inductive reasoning in terms of (1) initial experiences with particular cases, (2) strategies for organizing information about cases, (3) search and prediction of patterns, (4) conjecture formulation, (5) conjecture validation, (6) conjecture generalization, and (7) generalization justification. Burton (1984) identified that inductive reasoning rests on one's ability to: (1) specialize; i.e., engage in activities through which one explores particular cases, (2) conjecture; i.e., engage in activities through which one explores, expresses and substantiates underlying relationships among particular cases, (3) generalize; i.e., follow recognition of patterns with statements of generality, and (4) justify ; i.e., that is convince oneself and others about the robustness of pattern generalization.

The work described in this paper builds on the above interpretations of inductive reasoning, using Burton's classification of the four inductive reasoning processes as a starting point. For this research, the four inductive reasoning processes were operationalized building on Burton (1984), Haverty et al., (2000), and Cañadas and Castro's (2007, 2009) descriptions and the operational definition is presented in Table 1.

Table 1.
Inductive Reasoning[1] Processes

Inductive Reasoning Process	Operational Description
1.Specializing	Recognizable by activities focused on collecting, observing, organizing and representing information about specific cases
2. Conjecturing	Recognizable by one's ability to observe and express information about regularities that characterize explored cases and to identify unknown case
3.Generalizing	Recognizable by statement of generality (a description or a rule) that follows a pattern recognition allowing to make meaning and extrapolate information about cases beyond those directly studied
4. Justifying	Recognizable by the arguments used to validate the truth of the general statement about regularities observed in the analyzed information

[1] based on Burton (1984), Haverty et al., (2000), and Cañadas and Castro (2007, 2009)

The goal of this research was to further capture how these processes demonstrate in solutions to contextually-based tasks.

Method

Participants

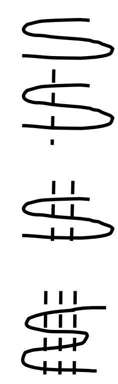
Participants for this study were 20 undergraduate students, nineteen females and one male, all grades 1-8 teaching certification candidates, at a large private Midwestern university. All were sophomores or juniors, enrolled in the Problem Solving and Reasoning course, the first in a 3-course mathematics sequence for 1-8 pre-service teachers. The objective of the course was to deepen pre-service teachers' problem-solving ability, and to strengthen their appreciation and understanding of mathematics. The course capitalized on pre-service teachers' authentic experiences with problem-solving and aimed to provide them with an understanding of the environment that supports students' growth as discoverers and users of mathematics. Throughout the semester, the pre-service teachers solved numerous problems and explicitly discussed a variety of problem-solving strategies. They were encouraged to reflect on their own thinking and progress in developing productive problem-solving dispositions (e.g., persistence, flexibility, ability to build on one another's ideas, and ability to communicate and justify findings). The inquiry-based design of the course supported community building encouraging students' discussions that fostered collective examinations of the merits of proposed ideas, strategies and arguments.

Data Sources

The data consisted of (a) 130 solutions to four investigative tasks (one composed of four parts), and (b) 77 reflective journals. The pre-service teachers had two weeks to complete each investigative task and were explicitly asked to explain and justify mathematical claims they made. They were asked to carefully record their thought processes, questions they asked themselves, conjectures they made and the results of testing them. They were also asked to journal and include reflections on their problem solving processes during each investigation. The goal for the pre-service teachers was to document the progress of their thinking rather than just provide a final, clean, solution to each task. Each of the selected tasks facilitated inductive reasoning and fostered exploration of mathematical ideas that were accessible to the participants. Each task provided an opportunity to generalize from a finite number of specific instances, and none involved rote algebraic manipulations. For example, the Cutting Through the Layers Investigation (Figure 1) encouraged analyzing and generalizing about classes of string-folding situations, as defined by the number of layers, e.g., class of 3-layer strings, class of 4-layer strings, with respect to the number of cuts, and, at the same time, facilitated generalizing across these classes.

Imagine a single piece of string, which can be bent back and forth. In the picture the string is bent so it has 3 "layers." But it is still one piece of string. Imagine now that you take a scissors and cut across the bent string, as indicated by the dotted line. The result will be four separate pieces of string, as shown.
You could have made more than one cut across the bent string, creating more pieces, or, you could start with more layers in the bent string.

In the picture at the right, there are four layers and three cuts. That creates a total of 13 pieces.



Investigate different numbers of layers and different numbers of cuts. Describe how to calculate the number of pieces (P) if somebody gives you the number of layers (L) and the number of cuts (C).

Figure 1. Cutting through the Layers Investigative Task (adapted from Driscoll, 1999).

The remaining investigative tasks are included in Appendix A.

Data Analysis

The analysis of the data was conducted in three phases: (1) In phase one, the data were examined to identify the four processes of interest i.e. specializing, conjecturing, generalizing and justifying. The goal was to identify the four inductive reasoning processes and to sort out the solutions that were based on inductive reasoning from those that were not. (2) In phase two, inductive reasoning-based solutions were examined to identify and uncover their characteristics. In this stage of data analysis, each of the four processes was further delineated. (3) The third phase of data analysis served to examine pathways of successful generalizations.

Two trained research assistants assisted in the first phase of analysis, in which, guided by the operational definition of inductive reasoning (Table 1), all solutions and accompanying reflective journals were analyzed to identify and code the four inductive reasoning processes (i.e., specializing, conjecturing, generalizing, and justifying). Validity and reliability was established in the process of discussing independently coded results. Specific examples were cited to **c**larify the coding schemes and negotiate coding agreement among the three coders to 100%. The author alone carried out the remaining part of data analysis. Using the technique outlined by Miles and Huberman (1994) the data were carefully reexamined to further analyze solutions classified as inductive with a goal of characterizing each of the four processes. Iterative cycles of comparing and contrasting characterizations across all solutions were carried to identify their similarities and differences leading to further refinement of these descriptions. In that process, descriptions of characterizations were collapsed into codes and definitions of codes established. Emergent codes were further examined for overlaps, applied and revised until the coding system stabilized. Furthermore all solutions were analyzed qualitatively according to their correctness and generalization strategy. In addition, successful solutions were further compared with respect to the frequencies of identified characterizations of each of the four processes. More detailed discussion of data analysis accompanied by the examples from the data set is presented in the results section that follows.

Results

Inductive and non-inductive reasoning solutions

As noted earlier, inductive reasoning is conceptualized as a process in which one draws on a collection of observations whose combined strength helps one arrive at a solution. Cañadas and Castro (2005) argued that one's experiences with particular cases provide the starting point of inductive reasoning. Thus, to distinguish inductive reasoning solutions from non-inductive reasoning solutions, the data were examined to identify the overall nature of specializing activities with a specific focus on the *observing* aspect of data collection. Solutions which documented that the pre-service teachers engaged in data gathering activities to generate a final numerical answer, rather than to establish an understanding of the nature of cases they examined for the purpose of analyzing underlying patterns and generalizing about them, were classified as non-inductive and excluded from further analyses. For example, consider excerpts from pre-service teachers' #3 and #5's reflective journals that illustrate the overall nature of data gathering activities in the context of the Eric the Sheep Investigation and the Lots of Squares investigation (Appendix A).

> I started by using blocks to represent the sheep. I tried with 25 right away but I kind of got confused and just ended up picking out 50 blocks and counting out the whole set, 2 at a time, in order to do the problem. I acted out and saw that Eric jumped 17 sheep before he was at the front of the line. Solving the problem with cubes I was able to see right away that 17 sheep will be shorn before Eric. (PST #3, Eric the Sheep Investigation).

> When I tried solving this problem I tried drawing everything out. Drawing diagram always helps me but when the answer could be infinity I could draw only for so long. The second

> part of the question, the largest 'impossible' number, this is where the true difficulty came in. (PST #5, Lots of Squares Investigation).

Both pre-service teachers generated additional data for the problem situation they studied. Their examples illustrate data gathering activities carried for the purpose of finding the specific answer, rather than understanding and uncovering the embedded relationships. Thus, these types of data gathering experiences were coded reenacting and labeled as non-inductive. In contrast, an excerpt from pre-service teacher #1's solution clearly illustrates the sense-seeking purpose of data gathering experiences essential to inductive reasoning:

> I began by drawing a picture of the information given, starting with one sheep in line ahead of Eric. I wanted to see if there was a pattern in the number of sheep shorn and jumped and how many were shorn before Eric. (PST#1, Eric the Sheep Investigation).

Included in Table 2 is a summary of inductive versus non-inductive solutions the pre-service teachers generated for each task.

Table 2.
Distribution of Inductive and non-Inductive Solutions across the Tasks

Task		Number and (%)** of Pre-service Teachers who	
	n*	Reasoned Inductively	Reasoned Non-inductively
Lots of Squares	20	16 (80%)	4 (20%)
Black and White Tiles	19	19 (100%)	0 (0%)
Cutting through the Layers	20	20 (100%)	0 (0%)
Eric the Sheep 1	18	11 (61%)	7 (39%)
Eric the Sheep 2	18	12 (67%)	6 (33%)
Eric the Sheep 3	18	12 (67%)	6 (33%)
Eric the Sheep 4	17	7 (41%)	10 (52%)

*the number reflects missing solutions
** rounded to the nearest whole percent

Only 6 pre-service teachers (29%) consistently engaged in inductive reasoning while solving *all* tasks and three pre-service teachers (20%) solved all tasks but Eric the Sheep, part 4, inductively.

Characterizations of inductive reasoning processes

Specializing. Specializing denoted sense-seeking activities that focused on collecting, representing and organizing information about specific cases. These activities were further described with respect to (a) the use of representations, and (b) organization. The use of representations characterized instances of using different representational systems in the process of examining particular cases and was further characterized as synonymous or syntactic. Organization of data collecting activities was classified with respect to systematicity.

***Synonymous versus Syntactic use of Representations*.** When the pre-service teachers explored and organized information about examined cases using different representational systems in a way that information about examined cases was preserved from one representation to the next, such use of different representations was characterized as synonymous. When the pre-service teachers explored information about examined cases using different representations providing an evidence of augmenting information collected about cases from one representation to the next such use of different representations was characterized as syntactic.

Consider two excerpts from pre-service teachers #9 and #2's solutions to the "Cutting through the Layers" Investigation, respectively illustrated in Figures 2a and 2b. Both examples illustrate how these pre-service teachers constructed a series of diagrams as a mean of exploring the problem situation and further examined and represented information about each examined case numerically.

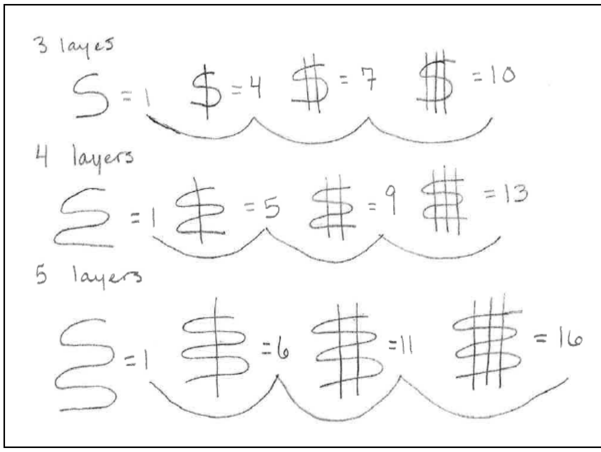


Figure 2a. An example of synonymous use of representations, PST #9

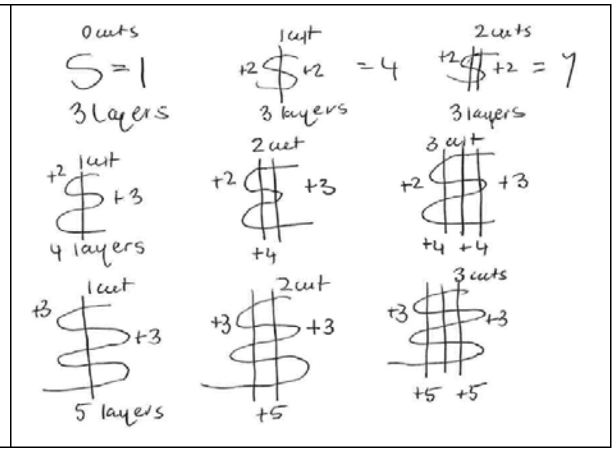


Figure 2b. An example of syntactic use of representations, PST #2

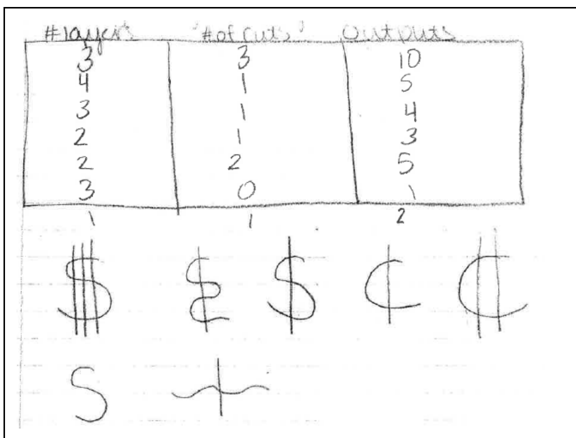
Pre-service teacher #9's graphical and numerical explorations are consistent with the verbal description of the problem. Each representation used (numerical and graphical) appear to directly summarize the information about the total number of pieces, as discussed in the verbal statement of the problem, (thus synonymous use of representations). While pre-service teacher #2 also represented information about each analyzed case graphically and numerically, her numerical representation reveals a shift from thinking about the total number of pieces to thinking about the layer-cut structure. Pre-service teacher #2s' work, included in Figure 2b, illustrates how, in the process of specializing, she explored and augmented information about cases she examined extending the initial information as she moved from one representation to the next (syntactic use of representations).In many ways, syntactic use of representations revealed the pre-service teachers' thinking about the complexity of the cases they explored, rather than the exclusive focus on one specific feature of interest indicated by the statement of the problem. The identified two types of uses of different representations, which characterize the overall nature of data exploration activities, are summarized in Table 3.

Table 3.
Summary of Synonymous and Syntactic use of Representations

Task	Use of representations (%)*			
	Synonymous	Syntactic	Both	None
Lots of Squares	75%	0%	25%	0%
Black and White Tiles	5%	42%	53%	0%
Cutting through the Layers	95%	0%	5%	0%
Eric the Sheep 1	89%	0%	11%	0%
Eric the Sheep 2	100%	0%	0%	0%
Eric the Sheep 3	90%	0%	0%	10%**
Eric the Sheep 4	83%	0%	17%	0%

* based on the number of participants who reasoned inductively about the task; rounded to the nearest %
** no explicit evidence of specializing activities

Systematicity versus Non-systematicity. Systematic organization of data gathering activities, which is also illustrated with the discussed earlier excerpts from pre-service teachers #9 and #2's solutions (Figures 2a and b) supported discovery of classes of problem situations. Non-systematic approach characterized data gathering activity in which a variety of cases from across different classes of situations were explored, appearing to be randomly selected. For example, consider the solution excerpt from the Cutting through the Layers Investigation of the pre-service teacher #15, included in Figure 3.



I drew diagrams of the strings of varying the layers and cuts. I then wrote the information I found in a table. I started to look for patterns and make connections. It was hard to find a pattern between the diagrams. I paid attention to all details and tried to find similarities between diagrams.

Figure 3. Non-systematic data gathering process, PST #15.

PST #15 appeared to select cases randomly without evident focus on folds or cuts. Her table and the accompanying explanation do not suggest that in the process she discovered a possible class of problem situations (as defined by the number of layers or cuts). This is in contrast to pre-service teachers #9 and #2 whose solution excerpts were presented in Figures 2a and 2b. Table 4 gives a summary of data gathering activities analyzed with respect to the systematicity, by task.

Table 4.
Summary of Data Organization

Task	Organization (%)*		
	Systematic	Non-systematic	None
Lots of Squares	94%	6%	0%
Black and White Tiles	100%	0%	0%
Cutting through the Layers	80%	20%	0%
Eric the Sheep 1	100%	0%	0%
Eric the Sheep 2	100%	0%	0%
Eric the Sheep 3	90%	0%	10%**
Eric the Sheep 4	100%	0%	0%

* based on the number of participants who reasoned inductively about the task; rounded to the nearest %
** no explicit evidence of specializing activities

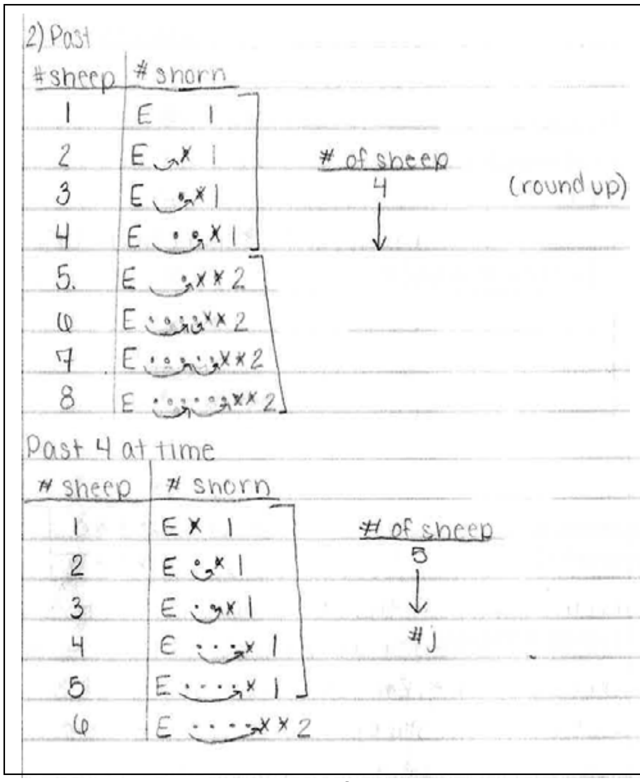
Conjecturing. Consistently with the framework for this study (Table 1) conjecturing was recognizable by one's ability to discover and express information about observed regularities. The expressions of regularity identified in the pre-service teachers' solutions were further classified as (a) local, or (b) global; both formulated in reference to recognition of changing or invariant attributes of examined cases. Local conjectures were defined as statements of recognition of invariant or changing attributes among cases within a specific class of problem situations. Global conjectures were defined as statements of recognition of regularities (changing or invariant attributes) across classes of problem situations.

Local versus Global Conjectures. Consider two excerpts from the Eric the Sheep Investigation part 2 (Appendix A) of pre-service teachers #19 and #18. The first, included in Figure 4, illustrates how the pre-service teacher #19 examined regularity among cases within a specific class of problem situations, as defined by the skip pattern, thus local conjecture.

"... the number of sheep shorn before Eric goes up in groups of three. For every new group of 3's [sheep in front of Eric] there will be one more sheep shorn before Eric."

Figure 4. An Example of Local Conjecture (PST #19)

The second, included in Figure 5, is an example of a global conjecture. In particular, the example shows how the pre-service teacher #18 uncovered and expressed her recognition of regularities across classes of problem situations (as defined by the skip) stating: "I noticed a pattern between Eric skipping 2 sheep every time a sheep is shorn and Eric skipping 3 sheep every time a sheep is shorn." Her work documents that she engaged in broader thinking about analyzed problem situations seeking connections among regularities she identified within specific skip-classes.



I noticed a pattern between Eric skipping 2 sheep every time a sheep is shorn and Eric skipping 3 sheep every time a sheep is shorn. You +1 to the number of sheep skipped, and divide the number of sheep in front of Eric by that number (round up to the nearest whole number.) This is similar to the first problem but the number of sheep in front of Eric changed from 3 to 4 to 10 etc. You still +1 to the number of sheep skipped, and divide by 4, 5, 11 etc.

Figure 5. An Example of Global Conjecture (PST #18)

Table 5 provides a summary of conjecturing activities, by task.

Table 5.
Summary of Conjecturing Activities

Task	Conjectures by Type (%)*			
	Local	Global	Both	None
Lots of Squares	81%	19%	0%	0%
Black and White Tiles	37%	37%	21%	5%
Cutting through the Layers	40%	5%	25%	30%
Eric the Sheep 1	67%	33%	0%	0%
Eric the Sheep 2	91%	0%	9%	0%
Eric the Sheep 3	100%	0%	0%	0%
Eric the Sheep 4	100%	0%	0%	0%

* based on the number of participants who reasoned inductively about the task; rounded to the nearest %

Generalizing

Generalizing was recognizable as a statement of generality (a description or a rule) that followed the recognition of a pattern. Similar to conjectures, general statements were further classified as (a) local or (b) global. Local generalizations were defined as expressions of generality about cases within a given class of problem situations. Global generalizations were defined as expressions of generality about classes of problem situations.

Local versus Global Generalizations. Consider the two excerpts from the pre-service teacher #16's solution to the Cutting Through the Layers Investigation, included in Figures 6a and 6b, as an illustration of local and global generalizations.

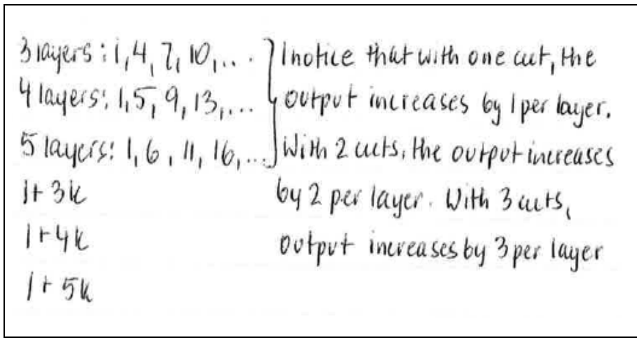


Figure 6a. An Example of Local Generalization, PST #16.

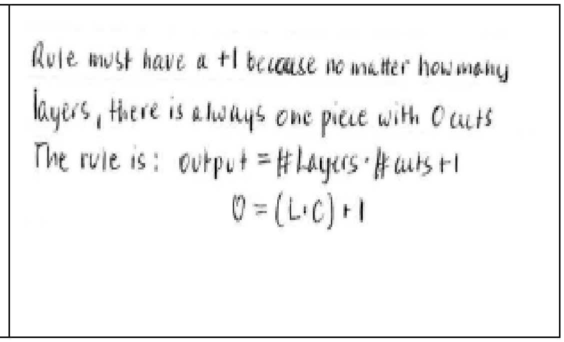


Figure 6b. An Example of Global Generalization, PST #16

The series of rules: 1+3k; 1+4k; 1+5k provide evidence that the pre-service teacher #16 generalized about regularities within each class of problem situations, as defined by the number of layers. Her description: "Output = #Layers *#Cuts+1," which follows her conjecture "Rule must have a +1 because no matter how many layers there are there is always 1 piece with 0 cuts" documents how she globally generalizes across classes of problem situations as defined by the number of layers and cuts, thus global generalization. Table 6 provides a summary of generalization activities across the tasks.

Table 6.
Summary of Generalization Activities

Task	Generalizations (%)*			
	Local	Global	Both	Unable to generalize
Lots of Squares	37.5%	0%	12.5%	50%
Black and White Tiles	11%	47%	37%	5%
Cutting through the Layers	5%	70%	20%	5%
Eric the Sheep 1	67%	33%	0%	0%
Eric the Sheep 2	18%	36%	45%	0%
Eric the Sheep 3	70%	30%	0%	0%
Eric the Sheep 4	17%	66%	0%	17%

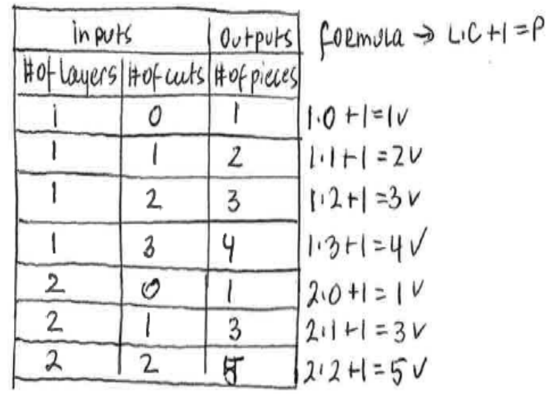
* based on the number of participants who reasoned inductively about the task; rounded to the nearest %

Justifying. The analysis revealed two types of arguments pre-service teachers constructed to validate the truth of formulated generalizations (a) relation-based or (b) empirical. When the pre-service teachers argued the truth of the general statements by providing clear links to the specific relationships uncovered in the context of the problem their justifications were categorized as relation-based. When the pre-service teachers attempted to established the truth of their general statements by testing their validity with selected cases their justifications were categorized as empirical. Frequently, the pre-service teachers considered their general statements obvious and did not engage in validating their truth.

Relation-based versus Empirical Justifications. Consider two explanations of the pre-service teachers #20, and #10, included in Figure 7. Both explanations include arguments pre-service teachers' #20 and #10 formulated to validate generalizations they developed in the context of the Cutting Through the Layers Investigation.

For the same number of layers the number of pieces went up by the number of layers each time a cut was made. There was always one piece to start with because there was one piece even when no cuts were made. This is why the formula is P=CL+1 where P=the number of pieces, C=the number of cuts, and L=the number of layers.

Figure 7a. An Example of Relation-based Justification, PST #20.



Inputs		Outputs	
# of Layers	# of cuts	# of pieces	formula → L·C+1=P
1	0	1	1·0+1=1✓
1	1	2	1·1+1=2✓
1	2	3	1·2+1=3✓
1	3	4	1·3+1=4✓
2	0	1	2·0+1=1✓
2	1	3	2·1+1=3✓
2	2	5	2·2+1=5✓

Figure 7b. An Example of Empirical Justification, PST #10

While the pre-service teacher #20 clearly linked her argument to the context of the problem and the relationships she uncovered the pre-service teacher #10 checked validity of her rule empirically, focusing only on numerical information she collected without attention to any general relationships that fit her examined data. Table 7 includes a summary of identified justification activities, organized by task.

Table 7.
Summary of Justifications

Task	Justifications (%)*		
	Relation-based	Empirical	None
Lots of Squares	19%	19%	62%
Black and White Tiles	16%	79%	5%
Cutting through the Layers	15%	70%	15%
Eric the Sheep 1	0%	89%	11%
Eric the Sheep 2	0%	55%	45%
Eric the Sheep 3	10%	40%	50%
Eric the Sheep 4	83%	0%	17%

* based on the number of participants who reasoned inductively about the task; rounded to the nearest %

Successful Generalizations.

Comparison of generalization strategies across two tasks. The Black and White Tiles Investigation and Cutting through the Layers Investigation were selected to compare and further explore successful generalization strategies identified in the pre-service teachers' solutions. These tasks were selected because 100% of solutions to both of these tasks were identified as inductive. Moreover, all 19 solutions for the Black and White Tiles Investigation were successful and only one of 20 pre-service teachers was not able to develop a correct generalization for the Cutting Thorough the Layers Investigation. For both problems, generalization strategies were further examined and the solutions compared with respect to identified characteristics of specializing, conjecturing and justifying activities. Task-specific global generalization strategies were catalogued and further classified as relation-based or numerical.

Relation-based versus Numerical Generalization Strategies. A task specific generalization was classified as relation-based when the general rule was developed from considering the various relationships uncovered within or among analyzed cases. Generalization strategy was classified as numerical when the general rule was developed by guessing and checking the rule using numerical data collected about analyzed cases. Included in Figure 8 is a summary of global generalization strategies identified in the pre-service teachers' solutions for the two tasks.

Black and White Tiles	Cutting Through the Layers
• Identifying and generalizing about structure (focus on black and white classes): (4n+1)+8(n+1); 4n+1+8n +8; 5+ 4(n-1) +8(n+1); [3n+ (n-1)4]+[(n-1)4]+5; (4n+1)+8(n-1)+16 • Identifying and generalizing about structure (focus on invariant center and groups of tiles in 4 sides): 9+4(3n); • Identifying and generalizing about structure (focus on chunks of three tiles): 3(7+4(n-1)) • Focus on the overall change in the number of tiles; 21+12(n-1); 9+12n	• Identifying and generalizing about structure (invariant fold or cut); lc+1 • Guessing and checking based on the analysis of the numerical relationships between the total number of pieces and the corresponding number of folds and cuts

Figure 8. Task Specific Generalization Strategies

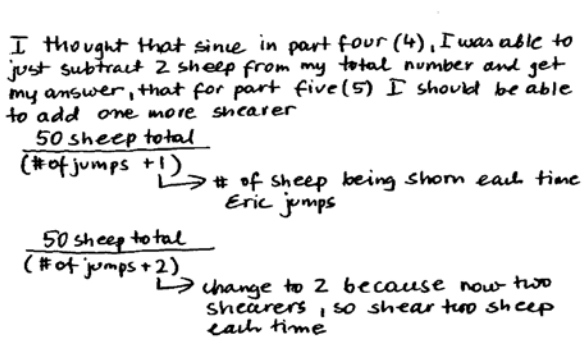
Overall, about half of the pre-service teachers (53 %) generalized about the situation described in the Cutting Through the Layers problem numerically, guessing and checking the relationship between the total number of pieces and the corresponding number of folds and cuts. In contrast, *all* pre-service teachers generalized by considering the various relationships they uncovered within the structure of the figures while solving the Black and White Tiles investigation. The frequency of relation-based generalizations for the Black and White Tiles Investigation was significantly higher than the frequency of relation-based generalizations for the Cutting Through the Layers Investigation, $z=4.10$, $p<0.00$.

A comparison of different characterizations of specializing activities identified in the solutions to these two tasks revealed that pre-service teachers demonstrated significantly more often syntactic use of representations in their solutions to the Black and White Tiles Investigation than they did in their solutions to the Cutting Through the Layers Investigation, $z=2.92$, $p<0.00$. The same was true for synonymous *and* syntactic use of representations; $z=3.72$, $p<0.00$.
The pre-service teachers also demonstrated synonymous use of representations significantly less frequently exploring and collecting information for the Black and White Tiles Investigation than they did for Cutting through the Layers Investigation; $z=12.55$, $p<0.00$. Conjecturing activity (both local or global) was also more frequently observed in solutions to the former than to the latter, $z=2.14$, $p<0.02$. There was no significant difference in frequencies of relation-based and empirical justifications across the solutions to these two investigative tasks.

Qualitative analysis of successful generalizations across the tasks. Further insight into pathways of successful generalizations came from qualitative analysis of solutions that included relation-based generalizations. This sub-group of solutions was characterized by: (1) presence of systematic *and* syntactic use of representations, (2) presence of local *and* global conjectures, and

(3) presence of justifications.

Consider an excerpt from a successful solution of pre-service teacher #1 to the Eric the Sheep Investigation part 4, and the accompanying explanation, included in Figure 8. Initially, the pre-service teacher #1 developed her general rule using analogy to the problem situation described in part 3 of this investigation: "change to 2 because now shearers, so shear two sheep each time."


I thought that since in part four (4), I was able to just subtract 2 sheep from my total number and get my answer, that for part five (5) I should be able to add one more shearer

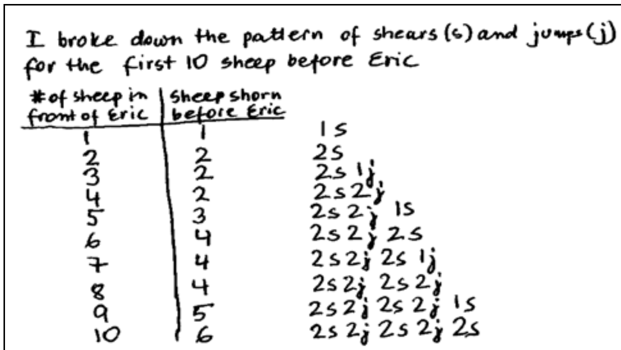
50 sheep total / (# of jumps + 1) → # of sheep being shorn each time Eric jumps

50 sheep total / (# of jumps + 2) → change to 2 because now two shearers, so shear two sheep each time

I started thinking *why* change to 2 and that I could not just assume that two sheep are shorn for every 2 that Eric jumps. Sometimes Eric jumps enough ahead so that he is the 2nd sheep shorn by the 2nd sheerer.

Figure 8. Initial generalization and accompanying reflection, PST # 1.

Her reflection " I started thinking why change to 2 and that I could not just assume that two sheep are shorn for every 2 that Eric jumps," illustrates how, her attempt to justify, fostered her engagement in collecting an additional information and further conjecturing about new patterns she discovered, as illustrated in Figure 9.


I broke down the pattern of shears (s) and jumps (j) for the first 10 sheep before Eric

# of sheep in front of Eric	Sheep shorn before Eric	
1	1	1s
2	2	2s
3	2	2s 1j
4	2	2s 2j
5	3	2s 2j 1s
6	4	2s 2j 2s
7	4	2s 2j 2s 1j
8	4	2s 2j 2s 2j
9	5	2s 2j 2s 2j 1s
10	6	2s 2j 2s 2j 2s

I noticed that each time, as long as there were 2 sheep shorn, the number of sheep shorn before Eric is even. However, if only one sheep is shorn than the number is odd. The number of sheep jumped did not make a difference because this pattern occurred if there were 0j, 1j or 2j.

Figure 9. Data gathering activity stimulated by engagement in justifying, PST# 1.

Consequently, she used the newly discovered patterns to reexamine her initial generalization and develop more powerful one.

Summary and Implication

Despite the agreement about the role of inductive reasoning in mathematics learning and problem solving (e.g., Haverty, et al., 2000, NCTM, 2000), the mathematics education literature lacks a framework for analyzing inductive reasoning processes in contextually-based tasks. Hence the goal of this study was to (1) characterize inductive reasoning processes in problem-solving contexts and (2), to identify and describe pathways of successful generalizations.

Growing out of the analysis of solutions to specific tasks, the results show broad categorizations of the four inductive reasoning processes in contextually-based situations. A general character of identified characterizations makes the presented framework useful for other problems. Each of the inductive reasoning processes (specializing, conjecturing generalizing and justifying) was further characterized in terms of general strategies, rather than task specific ones, making the identified taxonomy applicable to a broader collection of tasks.

The analysis of successful generalizations revealed that specializing activities characterized by syntactic use of representations and systematicity of data collection might support one's ability to develop successful generalizations. While comparing solutions to the Black and White Tiles Investigation and the Cutting Through the Layers Investigation with focus on specializing activities, syntactic use of representations and systematicity of data collection was observed significantly more often in solutions to the former. This result might suggests qualities of specializing activities that might support successful generalizations. This result also indicates that that the role of specializing in successful inductive reasoning might be more important than previously thought. For example, Haverty et al. (2000), based on their study of pre-service teachers inductive reasoning, reported that data gathering activities (specializing) did not differ when comparing successful and unsuccessful generalizations. Haverty et al. examined specializing activities in terms of the amount of data collected (number of cases explored) and the specific types of representations used (tables, graphs, lists, etc.). In contrast, this study focused on qualitative differences in the use of different representations and systematicity of data collection. More research is needed to further explore the role of specializing in successful generalizations.

A finding from this study that deserves special attention of teacher educators is the relative low frequency of justifications, particularly relation-based justifications, the pre-service teachers formulated overall. Frequently, the pre-service teachers in this study considered their general statements obvious and did not engage in validating their truth. At the same time the evidence suggests that justifications support one's ability to construct successful, and more powerful, generalizations. Helping pre-service teachers develop the ability to construct valid arguments and building their awareness of the role justification might play in constructing successful generalizations should be an important goal for teacher education programs. Stylianides (2008), referring to Bills and Rowland (1999), argued that the development of students' abilities to generalize in a way that draws on mathematical structure, rather than by experimenting with limited number of examples, is one of major challenges in mathematics education. Thus a special attention of mathematics teacher educators needs to be placed on helping pre-service teachers to understand the role justifications play in successful generalizations and help them develop the ability to create arguments rooted in relationships that fit the data.

The exploratory study reported in this article has no doubt some limitations. The small sample size, homogeneous group of participants, the selection of problems, and the exclusive use of only written solution protocols might limit the extent to which the results could be generalized. Further studies need to validate, and perhaps further delineate the proposed taxonomy to better inform instruction and assessment in teacher preparation programs. For example, further research can delineate the identified characterizations of inductive processes in problem solving contexts with a focus on mathematical thinking such as thinking about similarity or dissimilarity of problem situations. Another important question to ask is what role sequencing of different inductive reasoning processes plays in the solution to a task. Such research could have the potential to produce more pervasive descriptions of the four activities that underlie inductive reasoning in the problem solving contexts and could provide further insights into an understanding of paths that support successful generalizations.

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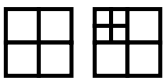
Appendix A

The Investigative Tasks used in the Study
(*adapted from Driscoll, 1999)

"Lots of Squares*" Investigation



The first diagram shows that any square can be divided into 4 smaller squares. The second diagram shows that any square can be divided into 7 smaller squares. Smaller squares don't have to be the same size as each other.

The task of this activity is to investigate what numbers of smaller squares are possible. For example, you can probably see that there is no way to divide a square into 2 smaller squares.

What is the largest "impossible" case? Justify that all cases beyond one that you named are possible.

"Black and White Tiles" Investigation

Mary uses black and white tiles to make figures such as presented below.

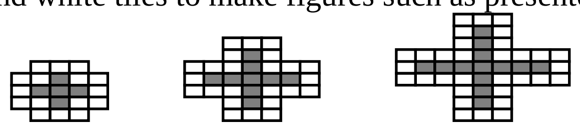


Fig 1 Fig 2 Fig 3

Use the patterns from the shapes to determine the number of tiles needed for Figure 4.
Mary wants to know how many tiles she will need for Figure 10 and for Figure 25 but she does not want to draw all the figures and count the tiles. Write a rule that Mary can use to figure out the number of tiles in any figure. Clearly explain why your rule works.

"Eric the Sheep*" Investigation

1. It's a hot summer day, and Eric the Sheep is at the end of a line of sheep waiting to be shorn. There are 50 sheep in front of him. Being an impatient sort of sheep, though, every time the shearer takes a sheep from the front of the line to be shorn, Eric sneaks up two places in line. How many sheep will be shorn before Eric? Find some way of predicting how many sheep will be shorn before Eric if there are 50 sheep in front of him.
2. Eric gets more and more impatient. Explore how many sheep will be shorn before Eric if Eric sneaks past 3 sheep at a time. How about 4 sheep at a time? 10 sheep at a time? When someone tells you how many sheep there are in front of Eric and how many sheep at a time he can sneak past, describe how you could predict the answer.
3. What if Eric sneaks past 2 sheep first, and then the shearer takes a sheep from the front of the line? Does this change your rule? If so, how? Why?
4. The farmer hires another sheep shearer. There is still one line, but the 1st and 2nd sheep in line get shorn at the same time, then Eric sneaks ahead. Explore what this does to your rule. Explain your answer.