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Abstract
Using algebraic habits of mind as a framework, and focusing on thinking about functions and how they work, we examined the relationship between 18 pre-service middle school teachers' ability to use the features of the algebraic thinking (AT) habit of mind "Building Rules to Represent Functions" and their ability to recognize and interpret the features of the same AT habit of mind in middle school students. We assessed the pre-service teachers' own ability to use the AT habit of mind Building Rules to Represent Functions by examining their

solutions to algebra-based tasks in a semester-long mathematics content course. We assessed the pre-service teachers' ability to recognize and interpret students' facility with the AT habit of mind Building Rules to Represent Functions by analyzing their interpretations of students' written solutions to algebra-based tasks and their ability to plan and analyze AT interviews of middle school students during a concurrent field-based education course. The data revealed that the pre-service teachers had a limited ability to recognize the full richness of algebra-based tasks' potential to elicit the features of Building Rules to Represent Functions in students. The pre-service teachers' own overall AT ability to Build Rules to Represent Functions was related to their ability to recognize the overall ability of students to Build Rules to Represent Functions, as exhibited during one-on-one interviews, but not to their ability to recognize the overall ability to Build Rules to Represent Functions exhibited exclusively in students' written work. Implications for mathematics teacher education are discussed.

Background

Today, most mathematics educators advocate for the inclusion of algebra-based topics in elementary and middle school mathematics classrooms. Early algebra instruction advances students' conceptual knowledge and skills by shifting attention away from symbolic manipulations toward analyzing and generalizing patterns using multiple representations (Kieran, 1996; National Council of Teachers of Mathematics, NCTM, 2000; Silver, 1997). Ideally, focusing on algebraic thinking at the early grades provides students with opportunities to link algebraic ideas to what they know about arithmetic (Kaput, 1998; Kieran, 1996; Silver, 1997).

Algebraic thinking

The phrase algebraic thinking has various connotations that closely relate to what Cuoco, Goldenberg, and Mark (1996) defined as habits of mind: useful ways of thinking about mathematical content. Driscoll (1999, 2001) interpreted algebraic thinking as thinking about quantitative situations that supports making the relationships between variables obvious. He explained that the "facility with algebraic thinking includes being able to think about *functions* and how they work, and to think about the impact that a system's *structure* has on calculations" (Driscoll, 1999, p. 1). Accordingly, he conceptualized these two aspects of algebraic thinking as habits of mind: Building Rules to Represent Functions and Abstracting from Computations situated under the umbrella of a habit of Doing–Undoing. Swafford and Langrall (2000) interpreted algebraic thinking as the ability to think about unknown quantities as known, and Kieran and Chalouh (1993) viewed algebraic thinking as building meaning for the symbols and operations of algebra in terms of arithmetic. Kieran (1996) further specified that algebraic thinking means the ability to use a variety of representations to analyze quantitative situations in a relational way, and she also asserted that algebraic thinking in the early grades can be developed

> …within activities for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without letter-symbolic algebra at all, such as, analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting. (Kieran, 2004, p. 149)

Anchored in Driscoll's (1999, 2001) interpretation of algebraic thinking, this research focuses on the first aspect of algebraic thinking as described by Driscoll (1999), namely, thinking about functions and how they work. Accordingly, we narrowed our work with pre-service teachers to the algebraic thinking (AT) habit of Building Rules to Represent Functions. This mental habit embraces thinking processes that are at the heart of middle school algebra: recognizing and analyzing patterns, investigating and representing relationships, generalizing beyond specific examples, analyzing how processes or relationships change, or seeking arguments for how and why rules and procedures work. Unless otherwise specified, throughout this paper *algebraic thinking* means the kind of thinking that results from exercising the habit of mind Building Rules to Represent Functions

(Driscoll, 1999, 2001). Accordingly, our operational definition of algebraic thinking is based on Driscoll's description of the features that characterize Building Rules to Represent Functions (see Table 1).

Table 1 Features of Building Rules to Represent Functions examined in this study

Features of algebraic habits of mind	**Description of thinking exemplified**
1. Organizing Information	Ability to organize information in ways useful for uncovering patterns, relationships, and the rules that define them
2. Predicting Patterns	Ability to discover and make sense of regularities in a given situation
3. Chunking Information	Ability to look for repeating chunks in information that reveal how a pattern works
4. Different Representations	Ability to think about and try different representations of the problem to uncover different information about the problem
5. Describing a Rule	Ability to describe steps of a procedure or a rule explicitly or recursively without specific inputs
6. Describing Change	Ability to describe change in a process or a relationship explicitly as a functional relationship between variables
7. Justifying a Rule	Ability to justify why a rule works for *any* number

Adapted from Driscoll (2001)

Teacher knowledge and teacher preparation

Teacher knowledge has been identified as an important variable that influences the outcomes of teacher practice (e.g., Borko & Putnam, 1996; Mewborn, 2003). Hill, Rowan, and Ball (2005), among others, documented how students' achievement closely relates to their teachers' mathematical knowledge. At the same time, research shows that teachers often lack a strong foundation for their mathematical knowledge (Ma, 1999), including a lack of flexibility in their understanding of algebraic concepts. Mewborn (2003) and van Dooren, Verschaffel, and Onghema (2002) attributed such difficulties to a fragmented knowledge of a disconnected system of algebraic symbols and procedures. Some of these deficiencies in teacher knowledge might possibly be explained by teachers' own experiences with *traditional* school algebra. Such experiences might not only limit teachers' content knowledge of algebra and algebraic thinking but also counter their efforts to help their students attain algebraic thinking competence.

It is commonly accepted that teachers with a robust knowledge of algebra are better positioned to prepare students for success in algebra. Mathematics teacher educators also agree that teachers need to understand how to help students develop an understanding of algebraic ideas and make connections among them (Algebra Working Group to the National Council of Teachers of Mathematics, 1997; Kieran, 2007). While agreement exists that teachers need a strong knowledge of algebraic thinking to be able to help their students understand algebra-based concepts, there is little agreement about how to strengthen teachers' knowledge.

One suggested way to strengthen teachers' knowledge is through teacher preparation. For example, Philipp et al. (2007) recommended engaging pre-service teachers in learning mathematics content and pedagogy concurrently providing them with opportunities to explore the mathematical thinking of students. The work of Carpenter and colleagues (Carpenter & Fennema, 1992; Carpenter, Fennema, Franke, Levi, & Empson, 1999) underscores the importance of engaging pre-service teachers in the exploration of student thinking by showing that teachers who routinely analyze student thinking position themselves to make better instructional decisions. Hill (2010) reasoned that the design of teacher preparation programs needs to draw on a deep understanding of the specialized content and pedagogical knowledge needed for teaching. She argued for a research agenda that provides a mapping of the specialized knowledge teachers need to be successful in their work. This type of understanding is paramount for the design of strong teacher education programs that successfully prepare teachers to introduce early algebra concepts and foster algebraic thinking in K-8 students. The research reported in this paper responds to Philipp et al. (2007) and Hill's (2010) arguments by seeking an understanding of how teacher preparation programs can foster pre-service teachers' knowledge of algebraic thinking in a way that enables pre-service teachers to use that knowledge effectively to nurture algebraic thinking in students.

A difficult and often misunderstood aspect of algebra is the concept of function (Clement, 2001). We used Driscoll's (2001) descriptions of the seven features of Building Rules to Represent Functions to map specific aspects of content and pedagogical knowledge needed to help students develop the concept of function. We drew on Driscoll's framework, which was developed in collaboration with one of the authors (see Driscoll & Moyer, 2008; Driscoll, Moyer, & Zawojewski, 1998), for two reasons. First, selecting the AT habit Building Rules to Represent Functions focused our work on specific AT features that support thinking about functions and how they work and whose development is essential in middle school algebra. This way we could conduct a fine-grained analysis of the mathematics content and pedagogical knowledge pre-service teachers need to specifically support middle school students in the development of these ways of thinking. Secondly, the framework on which our study builds is widely used in teacher professional development to support middle school teachers' understanding of algebraic thinking (see Driscoll, 2001). In the context of pre-service teacher education, our goals were to (1) scrutinize how teacher educators can assess and strengthen specific aspects of pre-service teachers' (broadly defined) knowledge of algebraic thinking, and (2) determine the relationships that exist between specific features of pre-service teachers' algebraic thinking proficiency and their ability to recognize and interpret the algebraic thinking of students. We define pre-service teachers' knowledge of algebraic thinking as a blend of (a) their ability to use different features of algebraic thinking in their own solutions, (b) their ability to analyze mathematics problems for their potential to elicit students' algebraic thinking, and (c) their ability to recognize, elicit, and interpret students' algebraic thinking in the context of clinical interviews and in samples of student written work.

The following research questions guided this investigation:

1. How does the algebraic thinking of pre-service teachers support their ability to recognize a task's potential to engage middle school students in algebraic thinking?
2. How does the algebraic thinking of pre-service teachers support their ability to recognize and interpret features of algebraic thinking in the work of middle school students?

Derry, Wilsman, and Hackbarth (2007) made the case that complex concepts such as those related to algebraic thinking cannot easily be explained or taught using rule-bound instruction. They believe that teachers develop knowledge of algebraic thinking when they are immersed in situations that elicit different aspects of algebraic thinking. With this idea in mind, we created an instructional approach that immersed pre-service teachers in situations that encouraged them to use features of Building Rules to Represent Functions in their own thinking and to recognize those same features in the thinking of students.

We conducted our study using a multi-tier design (Lesh & Kelly, 2000). For the lower tier, middle school students solved problems during AT interviews conducted by the pre-service teachers. For the middle tier, the pre-service teachers themselves solved AT tasks, analyzed AT tasks, analyzed students' written solutions to AT tasks, and analyzed students' algebraic thinking exhibited during the two AT interviews they conducted for the lower tier. For the upper tier, the authors analyzed the pre-service teachers' algebraic thinking as well as their ability to plan, conduct,[Footnote1] and analyze AT interviews.

Method

Participants

Participants were 18 undergraduate pre-service teachers in their last 2 years of a teacher education program at a large private Midwestern university in the USA and 18 middle school students in a nearby public school. All pre-service teachers were grades 1–8 teaching certification candidates. The pre-service teachers were enrolled concurrently in a mathematics content course taught in the Mathematics Department and a field experience course taught in the College of Education. The content course was the last in a conceptually based three-course sequence in mathematics for elementary education majors. The goal of the content course was to help pre-service teachers develop the ability to interpret, compare, connect, and generalize across multiple algebra topics within the middle school mathematics curriculum. In the content course, the pre-service teachers engaged in activities that solicited multiple solutions and representations of algebra-based tasks. The pre-service teachers were encouraged to share, explain, compare, and interpret various representations and reasoning. The field experience course consisted of 2 weeks of classroom instruction followed by weekly observations of middle school mathematics instruction, and one-on-one sessions conducted by each pre-service teacher with a middle school student. At the heart of this course were activities that involved pre-service teachers in tutoring or conducting one-on-one clinical interviews and analyzing the algebraic thinking of middle school students.

Data sources and data collection

We collected the following data during our semester-long study: (a) solutions to the 125 AT tasks pre-service teachers completed during class, for homework, and on performance assessments, (b) pre-service teachers' analyses of samples of middle school students' written work supplied by the content course instructor, (c) transcripts of two 45-min audio-recorded algebraic-thinking interviews each pre-service teacher conducted with one middle school student, (d) transcripts of two 30-min video-recorded debriefing interviews conducted by trained university researchers following each pre-service teacher's algebraic-thinking interview, (e) ten-page written analysis papers in which pre-service teachers analyzed the algebraic thinking exhibited by their middle school students during their two AT interviews.

Data analysis and results

The three authors independently coded the data. Validity and reliability were established by comparing sets of independent results, citing specific examples, clarifying the coding schemes, and re-coding the data until 100 % agreement was achieved. Once coded, the data were analyzed using a combination of qualitative and quantitative methods. We present the data analysis and results organized by research question.

Research question 1

How does the algebraic thinking of pre-service teachers support their ability to recognize a task's potential to engage middle school students in algebraic thinking?

AT scoring rubric

We rated each pre-service teacher's demonstrated use of an AT feature in his/her written solution to each of the 125 tasks as (3) proficient, (2) emerging, or (1) not evident. If a problem did not encourage the use of a particular feature, we did not use that problem to rate the strength of the pre-service teachers' thinking on that feature.

We rated a pre-service teacher's use of an identified feature as (3) *proficient* if the written solution revealed thinking *characteristic* of that feature, if the feature was carried out correctly, and if the use of the feature was *linked* directly to the context of the problem. We rated a pre-service teacher's use of an identified feature as (2) *emerging* if the written solution articulated thinking *characteristic* of that feature and if the feature was carried out *correctly*, but *without direct links* to the context of the problem. We also rated a pre-service teacher's use of an identified feature of algebraic thinking as (2) *emerging* if the written solution articulated thinking *characteristic* of that feature with *direct links* to the context of the problem, but was carried out *incorrectly*. We rated the strength of a pre-service teacher's thinking as (1) *not evident* on an identified feature if the problem encouraged the use of the feature, but the solution did not show evidence of thinking *characteristic* of that feature.

AT scores

To quantify each pre-service teacher's ability to use each AT feature (AT-feature score), we averaged his/her ratings on each of the seven features across the collection of tasks. This resulted in seven AT-feature scores for each pre-service teacher.[Footnote2] An AT-composite score (average of all seven AT-feature scores) rated a pre-service teacher's overall ability to think algebraically (as defined by our definition of algebraic thinking).

R-feature scores

Prior to conducting their two AT interviews with a middle school student, we asked the pre-service teachers to select two of the seven tasks presented in Appendix 1, one to be used in each interview. All three authors independently determined that each task had the potential to engage middle school students in all seven features of algebraic thinking, and all seven features were observed in the solutions that the pre-service teachers themselves generated for these tasks. Included in Appendix 2 is a sample task solution accompanied by a summary of our analysis that shows how the seven AT features are evident in the pre-service teacher's work.

We followed up each pre-service teacher's AT interview with a debriefing interview during which we asked, "Which features of algebraic thinking did you expect the problem could elicit from your middle school student?" After each response, we followed up with the questions, "Why?" and "Are there any other features of algebraic thinking that you think the task could encourage?" We analyzed the debriefing interviews to identify the features of algebraic thinking our pre-service teachers recognized in their selected tasks.

We quantified each pre-service teacher's ability to recognize each feature of algebraic thinking using a feature recognition score (R-feature score) which we defined as the proportion of the tasks (between 0 and 100 %) that the pre-service teacher recognized as having the potential to engage students in a given feature of algebraic thinking. The means of the resulting seven R-feature scores were compared for differences using repeated-measures ANOVA. We used the R-composite score (the average of the seven R-feature scores) as an overall measure of each pre-service teacher's ability to recognize the features of algebraic thinking that the two interview tasks had the potential to elicit in their middle school student. We examined the correlation between the 18 pairs of R-composite and AT-composite scores, as well as all seven correlations between the R-feature and AT-feature scores.

Recognizing task potential

Despite extensive discussions during the content class of all seven features of Building Rules to Represent Functions, the pre-service teachers demonstrated limited ability to identify them in their interview tasks. The means of the R-feature scores are presented in Table 2.

Table 2 Pre-service teachers' mean R-feature scores

	1. Organizing information (*n* = 18)	**2. Predicting patterns (*n* = 18)**	**3. Chunking information (*n* = 18)**	**4. Different representations (*n* = 18)**	**5. Describing a rule (*n* = 18)**	**6. Describing change (*n* = 18)**	**7. Justifying a rule (*n* = 18)**

Mean	0.72	0.82	0.41	0.38	0.72	0.44	0.70
SD	0.39	0.23	0.35	0.30	0.35	0.42	0.30

Proportion of the tasks recognized as having the potential to foster a given feature of algebraic thinking

The pre-service teachers recognized, in at least 70 % of the tasks, their potential to elicit only four of the seven features: 1, 2, 5, and 7. In fewer than 45 % of the tasks, they recognized the potential to elicit features 3, 4, and 6. There were statistically significant differences among the seven R-feature score means ($F(6,102) = 5.05$; $p < 0.01$). Bonferroni-adjusted pairwise comparisons confirmed that the R-feature mean for Feature 2 was statistically significantly greater than two other R-feature means: Feature 3 ($p < 0.05$) and Feature 4 ($p < 0.01$). The other differences were not statistically significant. The mean of the pre-service teachers' R-composite scores was only 0.60 (SD = 0.38). This indicates that, on average, the pre-service teachers recognized a task's potential to elicit any given feature of algebraic thinking only 60 % of the time.

Task recognition patterns

Two common characteristics underlying the pre-service teachers' perceptions of task potential may explain their poor performance: (a) reliance on previous mathematical experiences with the task and (b) literal use of the task description. When determining a task's potential to elicit various features of algebraic thinking, many of the pre-service teachers simply recalled their own experiences solving the task without considering ways of thinking different from their own. The pre-service teachers' recollection of their own experiences while solving a given task may have hampered their ability to recognize the task's potential to elicit all the features of algebraic thinking, as illustrated by the interview excerpts:

> When I was doing it originally, um, I think immediately you can create a table. (PST #14)
>
> While doing it in class I thought about chunking information by showing that the different, like it starts with bottom one and then you go by two and just keep increasing by two. (PST #3)

While some pre-service teachers judged a task's potential using recollections of their own thinking about the task, others based their recognition on the statement of the task itself:

> I knew that the student would have to justify how she came up with the rule because that was stated in the series of questions. (PST #17)
>
> Well, definitely predicting patterns because pattern is in the title, yeah, so patterns for sure. (PST #4)

Focusing on a task statement without considering the thought processes involved in the solution limited the pre-service teachers' ability to anticipate features of algebraic thinking that the task could potentially foster.

Correlations

We analyzed the relationship between the pre-service teachers' own algebraic thinking ability and their ability to recognize a task's potential to engage students in a given feature of algebraic thinking. The correlation between the pre-service teachers' AT-composite scores (range 1.93–2.82, $\overline{\overline{M}} = 2.46$, max 3, SD = 0.24) and their R-composite scores was not statistically different from zero ($r = 0.159$, p = n.s.). Similarly, none of the pre-service teachers' individual AT-feature scores were correlated with the corresponding individual R-feature scores. None of the possible correlations between pairs of the AT-feature scores and R-feature scores was significantly different from zero. These results suggest that the pre-service teachers' task recognition ability may have developed or been used independently from their own AT abilities. We speculate that the pre-service teachers' recognitions of the features elicited by their interview tasks may be grounded in side effects of the teaching-learning process, rather than in their own AT abilities. We discuss possible side effects that may account for this counterintuitive finding in the Section 4.

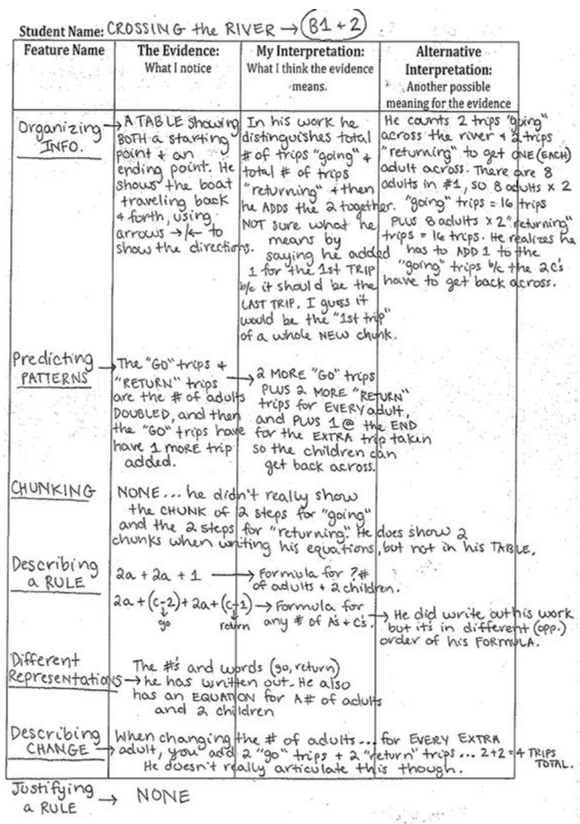
Research question 2

How does the algebraic thinking of pre-service teachers support their ability to recognize and interpret features of algebraic thinking in the work of middle school students?

We analyzed the relationship between the pre-service teachers' AT proficiency and their ability to recognize and interpret (RI) algebraic thinking in two samples of students' written work. We rated the pre-service teachers' abilities to recognize and interpret each of the seven features as (3) *proficient*, (2) *emerging*, or (1) *not evident*, using a recognition and interpretation scoring rubric similar to the AT-scoring rubric described earlier. For each pre-service teacher, we computed recognition-and-interpretation feature scores (RI feature) by averaging his/her ratings on each feature across the analyzed solutions. We quantified the pre-service teachers' overall ability to recognize and interpret algebraic thinking in students' work with an RI-composite score (average of the seven RI-feature scores) and examined eight correlations: those between the pre-service teachers' eight AT scores (one AT-composite score and seven AT-feature scores) and the corresponding eight RI scores (one RI-composite score and seven RI-feature scores), respectively.

Recognizing and interpreting AT features in students' written work

Included in Fig. 1 is PST #9's analysis of Student B's solution to Task 3 (presented in Fig. 2) which we use to illustrate our ratings of the pre-service teachers' ability to interpret algebraic thinking in students' written work.


Student Name: CROSSING the RIVER → (B1 + 2)

Feature Name	The Evidence: What I notice	My Interpretation: What I think the evidence means.	Alternative Interpretation: Another possible meaning for the evidence
Organizing INFO. →	A TABLE showing BOTH a starting point + an ending point. He shows the boat traveling back + forth, using arrows →/← to show the directions.	In his work he distinguishes total # of trips "going" + total # of trips "returning" + then he ADDS the 2 together. NOT sure what he means by saying he added 1 for the 1st TRIP b/c it should be the LAST TRIP. I guess it would be the "1st trip" of a whole NEW chunk.	He counts 2 trips "going" across the river + 2 trips "returning" to get ONE (EACH) adult across. There are 8 adults in #1, so 8 adults × 2 "going" trips = 16 trips PLUS 8 adults × 2 "returning" trips = 16 trips. He realizes he has to ADD 1 to the "going" trips b/c the 2 C's have to get back across.
Predicting PATTERNS →	The "GO" trips + "RETURN" trips are the # of adults DOUBLED, and then the "GO" trips have have 1 MORE trip added. →	2 MORE "GO" trips PLUS 2 MORE "RETURN" trips for EVERY adult, and PLUS 1 @ the END for the EXTRA trip taken so the children can get back across.	
CHUNKING	NONE... he didn't really show the CHUNK of 2 steps for "going" and the 2 steps for "returning." He does show 2 chunks when writing his equations, but not in his TABLE.		
Describing a RULE	2a + 2a + 1 → 2a + (c − 2) + 2a + (c − 1) → (go, return)	Formula for ?# of adults + 2 children. Formula for any # of A's + C's. →	He did write out his work but it's in different (opp.) order of his FORMULA.
Different Representations →	The #'s and words (go, return) he has written out. He also has an EQUATION for A# of adults and 2 children		
Describing CHANGE →	When changing the # of adults... for EVERY EXTRA adult, you add 2 "go" trips + 2 "return" trips ... 2+2 = 4 TRIPS TOTAL. He doesn't really articulate this though.		
Justifying a RULE →	NONE		

Fig. 1 PST #9's analysis of student B's (see Fig. 2) written work

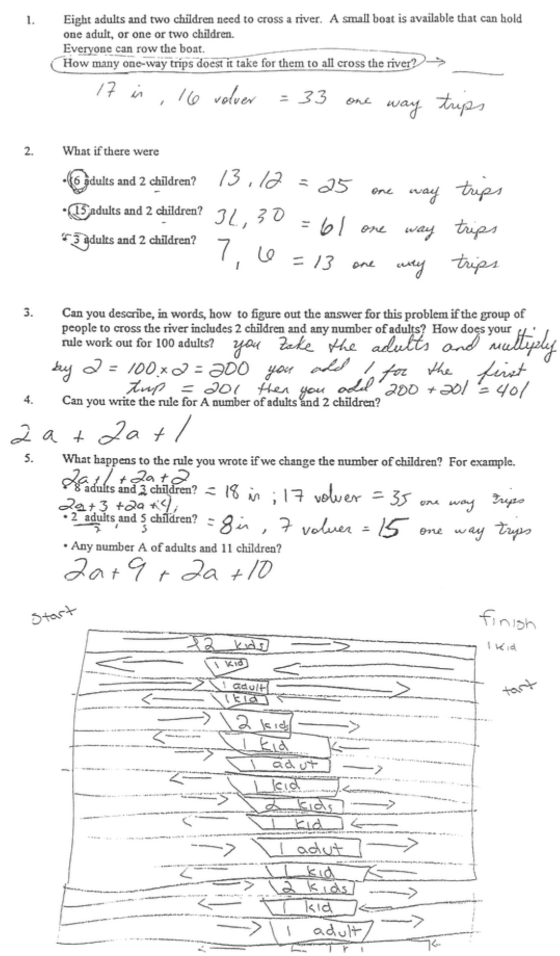


Fig. 2 Student B's work on task 3 (See Appendix 1). From Driscoll (2001)

Proficient ratings

PST #9 recognized and interpreted six out of the seven features of algebraic thinking in the sample of student work. We rated her analysis (Fig. 1) of Student B's work (Fig. 2) as (3) proficient at recognizing and interpreting features 2, 5, and 6. PST #9 cited evidence that Student B found a pattern (Feature 2) in the given situation: "The 'GO' trips & 'RETURN' trips are the # of adults [circled] DOUBLED, and then the 'GO' trips have 1 more trip added." PST #9 correctly cited the rule (stated as 2a + 2a + 1) that Student B wrote for answer 4, and correctly generalized answer 5 as the rule 2a + (c-2) + 2a + (c-1). PST #9 also noted that Student B wrote the number of going and returning trips (answers 2 and 5) "…in different (opp.) order of his formula," (Describing a Rule, Feature 5).

Moreover, she also correctly observed that the sample solution does not provide clear evidence that Student B explicitly considered the change (Feature 6) in the total number of trips that occurs for every additional adult. "When changing the number of adults . . . for every extra adult you add 2 'go' trips and 2 'return' trips . . . 2 + 2 = 4 trips total. He doesn't really articulate this though."

Emerging ratings

We rated PST #9's ability to recognize and interpret features 1, 4, and 3 (Fig. 1) in Student B's work as (2) emerging. First, PST #9 correctly cited (column 2) that Student B (Fig. 2) organized information (Feature 1) in: "[a] table [diagram] showing both a starting point & an ending point. He [the student] shows the boat traveling back & forth, using arrows →/← to show the directions." However, PST #9's interpretation (column 3) did not refer to the evidence cited. Instead, she presented more evidence from answer 1, stating that the student organized the information by distinguishing "…total number of trips 'going' [*ir*] & total # of trips 'returning' [*volver*]… ." failing to cite as evidence the tabular-like way in which Student B organized information in answer 2. This is an important omission since it is likely that Student B used the "table" in answer 2 (including the circles around the numbers of adults) to develop the rule described in answer 3. Taken as a whole, PST #9's analysis indicates some confusion about exactly which aspects of a student's organization are important.

In her analysis of Students B's solution, PST #9 recognized the use of verbal and symbolic representations (Feature 4), but she did not identify the diagram as a form of representation of the problem. She also failed to examine links between the different forms of representation evident in Student B's solution. Specifically, in her analysis PST #9 did not explicitly describe how the diagram supported Student B's development of the pattern or rule for generating the number of trips required for different numbers of adults. PST #9 correctly discussed thinking characteristic of Feature 3: "He [the student] does show 2 chunks when writing his equation…." However, her analysis lacked links between what she observed about the student's thinking and the context of the problem. She did not seem to realize that the left and right arrows in Student B's diagram, together with Student B's rule, provide evidence of Student B's thinking about the problem situation in terms of repeating chunks of one-way trips.

Not evident ratings

We rated PST #9's recognition and interpretation of Feature 7 as (1) *not evident*. She did not recognize the student's statement ". . . you add one for the first trip" as a partial justification for the developed rule explaining that she was "…[n]ot sure what he means by saying he added 1 for the 1st trip b/c it should be the last trip." In particular, she did not realize that the extra trip could be thought of as either the first or the last trip.

Strength of recognition and interpretation

Student B's written work (Fig. 2) was the first of two samples the pre-service teachers were required to analyze. Although we asked the pre-service teachers to analyze Student B's work for all seven features of algebraic thinking, we asked them to analyze the second sample for only features 4, 5, and 6. We used these ratings to generate seven RI-feature scores for each pre-service teacher (averaging the two recognition and interpretation ratings for each of features 4, 5, and 6). Each RI-feature score assesses the pre-service teacher's ability to recognize and interpret one of the features of algebraic thinking in students' written work. The seven means of the 18 pre-service teachers' RI-feature scores are presented in Table 3, where the mean of Feature 7 (1.39) is the lowest (max 3). A repeated-measures ANOVA revealed statistically significant differences among the seven means ($F(6,102) = 9.54$; $p < 0.01$), and Bonferroni-adjusted pairwise comparisons confirmed statistically significant differences ($p < 0.01$) between the mean of Feature 7 and each of the other six means.

Table 3 Means of the pre-service teachers' RI-feature scores

	1. Organizing information (n [a] = 18)	**2. Predicting patterns (n = 18)**	**3. Chunking information (n = 18)**	**4. Different representations (n = 36)**	**5. Describing a rule (n = 36)**	**6. Describing change (n = 36)**	**7. Justifying a rule (n = 18)**
Mean	2.61	2.28	2.56	2.50	2.39	2.28	1.39
SD	0.70	0.57	0.62	0.49	0. 53	0.39	0.70

[a]Number of scores for a given feature across the 2 interpretation tasks and all 18 pre-service teachers

We used the average of all seven RI-feature scores (RI-composite score) to estimate the overall strength of the pre-service teachers' ability to recognize and interpret student work. The RI-composite scores ranged from 1.39 to 2.61. The average of all 18 RI-composite scores was $\overline{\overline{M}} = 2.29$ (max 3); SD = 0.69.

Correlation between AT and RI scores

The correlations between pre-service teachers' AT-composite scores (range, 1.93–2.82, $\overline{\overline{M}} = 2.46$, max 3, SD = 0.24) and their RI-composite scores were not statistically significantly different from zero ($r = 0.121$, p = n.s.). This result suggests that the pre-service teachers' (overall) ability to think algebraically might be independent of their (overall) ability to recognize and interpret algebraic thinking in students' written work. We examined correlations between the pre-service teachers' individual AT-feature scores and their corresponding RI-feature scores. The results revealed statistically significant correlations between three pairs of AT- and RI-feature scores: Feature 1($r = 0.473$, $p < 0.01$), Feature 3 ($r = 0.588$, $p < 0.05$), and Feature 4 ($r = 0.512$, $p < 0.03$). These results suggest that our pre-service teachers' own ability to organize information, chunk information, or use different representations may be good predictors of their ability to recognize and make sense, respectively, of features 1, 3, and 4 in students' written work. Our analysis does not support similar conclusions regarding features 2, 5, 6, and 7. None of the correlations between the AT-feature scores for features 2, 5, 6, and 7 and the corresponding RI feature scores were statistically significantly different from zero.

Recognizing and interpreting AT features in student interviews

We analyzed the pre-service teachers' AT interview transcripts, debriefing interview transcripts, and their written AT analysis papers to determine how well they were able to identify and interpret the features of algebraic thinking elicited by students during one-on-one interviews. To gain insight into how the pre-service teachers' overall AT abilities related to their ability to recognize and interpret the features of algebraic thinking exhibited by students in one-on-one interviews, we qualitatively compared the interpretative analyses conducted by the pre-service teachers who had high (2.58–2.82) and low (1.93–2.34) AT-composite scores.

The analysis revealed that the pre-service teachers with high AT-composite scores not only successfully elicited[Footnote3] evidence of algebraic thinking from their interviewees but also were able to recognize and interpret students' algebraic thinking when it occurred. The pre-service teachers identified as having low AT-composite scores, on the other hand, were much less consistent in eliciting, recognizing, and interpreting situations where students engaged in algebraic thinking. Generally, when attempting to analyze student thinking, the low-AT pre-service teacher group emphasized what the students *did* during their one-on-one interview sessions, rather than analyze *how* they were thinking. The examples that follow demonstrate the qualitative differences between the high- and low-AT pre-service teachers' ability to analyze the algebraic thinking of the students they interviewed.

Excerpts from the high-AT pre-service teachers' group

This first excerpt illustrates how a pre-service teacher (PST #6) in the high-AT group identified and made meaning of her middle school student's attempt to solve Task 3 (Appendix 1). She recognized not only that the student was able to predict a pattern but also that the student exhibited the ability to chunk information to describe how a pattern works:

> She [the middle school student] was able to predict a pattern. She stated "Like two children go over, one comes back, an adult goes over, then a child comes back, wait, so if two children go over and one comes back and then one adult goes over and child comes back, so that's two go over one comes back and adult goes over the child comes back. Wait, it's the same thing over and over again!" . . . At first she was

> counting … then she realized that the pattern repeated itself ever four turns and then "plus one" at the end of the problem was the two children crossing at the end. It was interesting to see her coming up with a rule 4a + 1 because the plus one is for children coming back. She was thinking in chunks CC, C, A, C and CC, C, A, C. (PST #6)

Another pre-service teacher from the high-AT group (PST #17) interpreted how her student was able to describe a rule for the V task (Appendix 1) by consistently thinking about the pattern in terms of two groups of blocks:

> He states "there is three on this side [. . .] if you add three to the four you get seven." This statement, along with his usage of the figure, indicates that he is thinking of the figure in two different sections. The one side that is equal to the figure number and the other side that is equal to one less than the figure number. Later when describing another figure he states: "So, there is fourteen on this side not counting this one, and then there is fifteen." (PST #17)

Excerpts from the low-AT pre-service teachers' group

The pre-service teachers with low AT-composite scores rarely interpreted their student's actions in the context of the features of algebraic thinking. Instead of focusing on students' thinking, pre-service teachers in the low-AT group usually focused on students' actions, simply highlighting what the student did. Consider the following excerpt from PST #18's interview with a student who is attempting to solve Task 1.

1. Student: If the pattern continues, how many of the blocks will be contained in the next letter V? So, there is one in the first, three in the second, five in the third, seven in the sixth, no I mean in the fourth. So… there will be one, two, three, four, five, six, seven, eight, nine blocks.
2. PST #18: How did you solve that?
3. Student: Because I figured out you have two more blocks to every V because one has one, that has to be the tip, and then in the second pattern [second letter V] there are two, and in the third pattern [third letter V] there is two more and so on.
4. PST #18: And what did you mean by tip?
5. Student: Cause, the letter V has to have a point like right there. . .
6. PST #18: So, does the tip ever change as the pattern goes up?
7. Student: No.

In her written analysis, PST #18 described the student actions that accompanied their verbal exchange. It appears that her intent was to explain how the student employed Feature 1 rather than the purpose or usefulness of the student's "interesting organization process," that is whether the process was or was not useful to the student for uncovering patterns, relationships, or the rules that define them. She wrote:

> Within the first problem [Task 1], the letter V, she did begin an interesting organization process: she wrote out the first figure numbers 1 through 15, and then next to it put the number of total blocks in each of these figures. (PST #18)

PST #18's attempt to connect the student's statement in line 1 to Feature 6 revealed her naïve and superficial understanding of this feature:

> She [the student] saw in both problems [Task 1 and Task 2] that the figures changed each time. She used counting to figure out changes that were occurring from one figure to figure. She stated ". . . there is one in the first, three in the second, five in the third, and seven in the fourth" in reference to the change in the number of blocks in the letter V problem. She knew [that] change was occurring and used counting skills to distinguish the differences in figures. (PST #18)

In particular, PST #18 incorrectly interpreted the numbers in the student's statement as specifying "…changes that were occurring from one figure to figure. She [the student] stated '…there is one in the first, three in the second, five in the third, and seven in the fourth…' ." Furthermore, PST #18 failed to recognize that the student's statement (line 3) that "…you have two more blocks to every V" demonstrated the ability to employ Feature 3 (Chunking Information), as well as an emerging ability to employ Feature 6 (Describing Change).

Discussion and implications

This study explored relationships involving pre-service teachers' specialized abilities to: (1) think algebraically, (2) recognize opportunities to engage students in algebraic thinking, and (3) recognize and interpret algebraic thinking in students.

Our first research question provides an important window into pre-service teachers' awareness of the potential of algebra-based tasks to engage students in algebraic thinking. Our pre-service teachers demonstrated a rather limited ability to recognize the full potential of algebra-based tasks to elicit algebraic thinking in students, recognizing only some features in the analyzed tasks. To effectively engage students in algebraic thinking, pre-service teachers need to understand the contexts in which the various features of algebraic thinking might arise. Our analysis revealed that pre-service teachers' ability to recognize a task's potential to engage middle school students in algebraic thinking was not associated with their own overall algebraic thinking (AT) ability. Because this finding seems counterintuitive, we wondered whether some other dynamic might be obscuring this relationship. We have come to believe that the pre-service teachers did not use their AT abilities to fully analyze tasks. For reasons outlined below, we conjecture that the pre-service teachers' recognitions of the features elicited by their interview tasks may be grounded in the teaching-learning process rather than in their own AT abilities.

We believe that it is possible that the pre-service teachers came to expect that all of the algebra-based pattern-finding tasks would require the solver to organize information, identify a pattern, describe a rule, and justify it. In order to prepare the pre-service teachers to conduct their clinical interviews for the subset of algebra-based pattern-finding tasks (43 of the 125 problems), we always required the pre-service teachers to explicitly show evidence that they used these four features. Given that our pre-service teachers were required to discuss these four features in all their solutions, even if the actual statements of the problems did not, it is likely that the pre-service teachers came to expect that pattern-finding algebra-based tasks would always elicit these four features. This expectation may be the main reason that the pre-service teachers so frequently (≥70 %) identified their interview tasks as having the potential to elicit these four features.

On the other hand, we rarely required the pre-service teachers to show evidence that they used Feature 3 (Chunking Information), Feature 4 (Different Representations), or Feature 6 (Describing Change) unless the problem explicitly asked for it. An analysis of the same subset of 43 pre-interview problems revealed that, in this subset of problems, these three features were infrequently prompted: 26, 23, and 21 %, respectively. Furthermore, none of the statements in the interview tasks themselves explicitly asked the solver to use Chunking Information, Different Representations, or Describing Change. We believe it is likely that the pre-service teachers infrequently (≤44 %) identified their interview tasks as eliciting the features Chunking Information, Different Representations, or Describing Change because they based their answers on their prior experiences with these types of problems or on the literal parsing of the task statements rather than careful analysis of the problem solutions. This result implies that to build pre-service teachers' knowledge of algebraic thinking (broadly defined), the various features of algebraic thinking should be equally emphasized. Discussions that explicitly focus on how algebra-based tasks can be implemented to elicit all seven features of algebraic thinking might prove beneficial. Such discussions could be orchestrated in the context of analyzing alternative solutions to algebra-based tasks, with a goal of helping pre-service teachers recognize ways of thinking different from their own that might be embedded in alternate solutions. Explicit consideration of alternative solutions, as well as comparison of the AT features that generate them, might strengthen pre-service teachers' own algebraic thinking and heighten their awareness of how problem situations can provide contexts for engaging students in many different features of algebraic thinking.

Our second research question showed that the pre-service teachers had significantly more difficulty recognizing and interpreting the Justifying a Rule feature in student work than any other AT feature. The pre-service teachers' own overall AT ability was related to their ability to recognize the overall algebraic thinking exhibited by students during one-on-one interviews, but not to their ability to recognize the overall algebraic thinking exhibited exclusively in students' written work. We uncovered strong positive "self" correlations between the pre-service teachers' recognition and interpretation ability and their corresponding algebraic thinking ability relative to the following features: Organizing Information, Chunking Information, and Different Representations. Similar "self" correlations were not apparent for any of the other four features: Predicting Patterns, Describing a Rule, Describing Change, or Justifying a Rule.

A particularly significant implication of these results is that pre-service teachers' ability to recognize and interpret algebraic thinking in clinical settings is highly dependent upon their own AT ability. It reinforces the strength of our appeal to give special emphasis throughout the pre-service mathematics curriculum to all seven features of algebraic thinking. The Justifying the Rule feature of algebraic thinking should be addressed with particular consistency given that pre-service teachers' demonstrated weak ability to justify a rule. This implication also prompts us to reconsider how to implement activities requiring pre-service teachers to interpret samples of students' written work. The pre-service teachers generally did not bring their own AT abilities to bear on the interpretation and analysis of four of the features of algebraic thinking: Predicting Patterns, Describing a Rule, Describing Change, or Justifying a Rule. It may be that they bypassed the use of their own AT abilities in favor of extraneous cues, as they did in their recognition of task potential. If so, special attention needs to be given to ways of inducing the pre-service teachers to bring the full weight of their own algebraic thinking abilities to bear on these tasks. One possible approach would be to coordinate the pre-service teachers' analysis of clinical interviews more closely with their analysis of student written work.

Final remarks

Algebraic thinking is at the heart of teaching and learning algebra at the elementary and middle school levels. Building pre-service teachers' broadly defined knowledge of algebraic thinking should be an important goal for teacher education programs. Pre-service teachers should engage in algebraic thinking, be able to recognize the opportunities for engaging their students in algebraic thinking, and understand the algebraic thinking of their students. Teachers who make sense of students' thinking gain important insights about how students develop mathematical ideas (Carpenter & Fennema, 1992; Tirosh, 2000; Vacc & Bright, 1999). Paying attention to students' thinking positions teachers to determine what their students already know or do not know, supporting their instructional decisions. Our window into the complexity of the relationship between pre-service teachers' knowledge of algebraic thinking and their ability to help students develop AT abilities helps mathematics teacher educators and researchers design programs sensitive to important issues related to early algebra instruction.

Given the exploratory nature of our research we recognize that our study has limitations. A small number of participants, a lack of comparison groups, and a lack of consideration given to other types of courses or settings dictate the caution with which these results should be interpreted. Caution is also dictated because we used a small range of problems and because we limited our assessment of the pre-service teachers' algebraic thinking to the analysis of only their written solutions. Despite these limitations, we believe that our results identify promising avenues for mathematics teacher educators to pursue and underscore the importance of clinical work in teacher preparation programs.

Notes

1. We report our analysis of the pre-service teachers' ability to conduct AT interviews in a separate paper (see van den Kieboom, Magiera, & Moyer, 2010)
2. For a complete description of this analysis and descriptions of pre-service teachers' AT proficiency see Magiera, van den Kieboom, and Moyer (2011).
3. See van den Kieboom et al. (2010)

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Appendices

Appendix 1

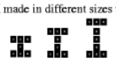
Tasks pre-service teachers analyzed for their potential to engage students in algebraic thinking

Task 1

1. If the pattern continues, how many blocks will be contained in the next letter V?
2. How many blocks would be in the 15th figure in the sequence? How did you figure out your answer?
3. How could you figure out the number of blocks in any letter V in this pattern?
4. Can you build a letter V that follows that pattern and uses 36 blocks?
5. Would any of the letter V's in this pattern have an even number of blocks? Why or why not?

Task 2

Here is a letter I made in different sizes using small tiles.



Size 1 Size 2 Size 3

1. Describe how the letter grows from one size to the next.
2. How many tiles would you need to make a letter I of: a) Size 6? b) Size 10? c) Size 38? d) Size 100?
3. Write a rule that helps to predict the number of tiles for any size letter I? You may write a rule either in words or using variables.
4. Suppose you had 39 tiles. What is largest size of I that you could make?

Task 3

Eight adults and two children need to cross a river. They have a small boat available that can hold one adult or one or two children. Everyone can row the boat.

1. How many one way trips does it take for them all to cross the river?
2. What if there were 6 adults and 2 children? 15 adults and 2 children? 3 adults and 2 children?
3. Can you describe in words how to figure out the answer for this problem if the group of people to crosses the river includes 2 children and any number of adults? How does your rule work out for 100 adults?
4. Write the rule for "A" number of adults and 2 children.
5. What happens to the rule you wrote if we change the number of children? For example 8 adults and 3 children? 2 adults and 5 children? Any number of adults and 11 children?
6. One group of adults and children took 27 trips to cross the river. How many adults and how many children were in the group? Is there more than one solution?

Task 4

The shapes shown below are made with toothpicks. Look for patterns in the number of toothpicks in the perimeter of each shape.

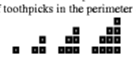


Fig.1 Fig.2 Fig.3 Fig.4

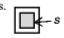
1. Use the pattern from the shapes to determine the perimeter of the fifth figure in the sequence. Clearly explain how you arrived at the answer.
2. Write a formula that you could use to find the perimeter of any figure *n*. Explain how you found your formula.

Task 5

Sally is having a party. The first time the doorbell rings, one guest enters. If on each successive ring a group enters that has 2 more persons than the group that entered on the previous ring how many guests will have arrived after 20th ring?

Task 6

Below is a picture of an in-ground swimming pool surrounded by a border of square tiles.



1. How many 1-foot square tiles will be needed for the border of a square-shaped pool that has edges length *s* feet?
2. In as many ways as you can express the total number of tiles needed.
3. How do you know that your expressions are equivalent? Provide convincing arguments that your expressions are equivalent.

Task 7

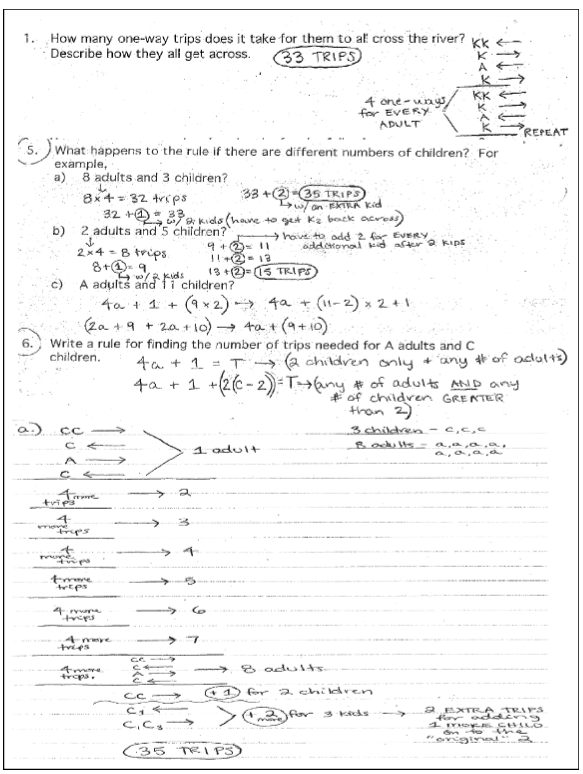
Each house below was built using pattern-block tiles: triangles and squares.

House 1 House 2 House 3 House 4

1. Determine the total number of tiles needed for each house.
2. Draw a sketch of house 5and describe what house 5 would look like.
3. Predict the total number of tiles you will need to build house 15. Explain your thinking.
   Write a rule that gives the total number of pieces to build any house in this sequence.

Appendix 2

An excerpt of a pre-service teacher's solution to Task 3 (see Appendix 1) accompanied by our analysis of the task's potential to elicit the seven features of AT



Feature name	Feature description	Evidence in pre-service teacher's work
Organizing Information	Ability to organize information in ways useful	Lists and clearly labeled diagrams provide evidence of PST's organizing problem information in a useful way.

	for uncovering patterns, relationships, and the rules that define them	
Predicting Patterns	Ability to discover and make sense of regularities in a given situation	The list in the upper right shows PST's understanding of how the pattern of trips (kk, k, A, k, repeat) gives rise to the pattern for the number of trips (4 times the number of adults + 1).
Chunking Information	Ability to look for repeating chunks in information that reveal how a pattern works	Both diagrams show the PST's understanding that a "chunk" of 4 trips is needed to move each adult.
Describing a Rule (either recursively or explicitly)	Ability to describe steps of a procedure or a rule explicitly or recursively without specific inputs	The PST gives explicit rules that generate correct predictions of the number of trips regardless of the input.
Different Representations	Ability to think about and try different representations of the problem to uncover different information about the problem	The PST examines and describes the problem information through the use of diagrams, words, symbols. The information presented works together to support the development of a rule.
Describing Change	Ability to describe change in a process or a relationship explicitly as a functional relationship between variables	The PST shows thinking about the change (+4) in the total number of trips that corresponds to each change (+1) in the total number of adults (writing on the side of a top diagram).
Justifying a Rule	Ability to justify why a rule works for *any* number	Assuming the rule for finding the number of trips required for 2 children (which she previously justified, but which is not shown here), the PST informally justifies the rule for 2 children and any number of adults by explaining and showing: "+1 for 2 children"; "+2 more for 3 kids→2 extra trips for adding 1 more child on to the 'original' two." (bottom diagram) and "+1 w/2 kids" (have to get k_2 back across), in her comment to #5a. She not only justifies the rule by appealing to the numerical pattern of two additional 2 trips for each extra child, but she also connects the context of the problem to the numerical pattern in a diagram showing the extra trips needed.