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On Order Statistics for GS-Distributions

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Abstract

In this article, a class of distributions is used to establish several recurrence relations satisfied by single and product moments of order statistics and progressive Type-II right censoring. The recurrence relations for moments of some specific distributions including uniform (a, b), exponential (λ), generalized exponential (α, λ, ν), beta $(1, b)$, beta $(b, 1)$, logistic (α, β) and other distributions from order statistics and progressive Type-II right censoring can be obtained as special cases. A short explanation of GS-distribution can be found in reference [27]. As an example, means, variances and covariances for standard exponential distribution of progressive Type-II right censored order statistics are computed. Various characterizations of the recently introduced GS-distributions are presented. These characterizations are based on a simple relationship between two truncated moments ; on hazard function ; and on functions of order statistics. A characterization of the GS-distributions based on conditional moment of order statistics is extended to truncated moment of order statistics.

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1 Introduction

Recently, Muiñoa et al. [27] introduced a class of distributions consisting of unimodal distributions called GS-distributions. This new class contains certain well-known distributions for which cumulative distribution functions have closed forms. GS-distributions has several interesting properties. First, it includes as special cases, various statistical distributions for which the cumulative distribution functions have a closed form (see Table 1). Second, the GS-distributions can be used to model observed data, when the true underlying distribution is not known. For further details and the domain of applicability of this class, we refer the interested reader to Muiñoa et al. [27] . The GS-distributions are a 4-parameter family of distributions.

In this paper, we obtained some recurrence relations between moments of order statistics as well as those of single and product moments of progressive Type II right censoring for GS-distributions when all the parameters are assumed to be positive integers. Various characterizations of the recently introduced GS-distributions are presented. These characterizations are based on a simple relationship between two truncated moments ; on hazard function ; and on functions of order statistics. A characterization of the GS-distributions based on conditional moment of order statistics is extended to truncated moment of order statistics. In the applications where the underlying distribution is assumed to be GS-distribution, the investigator needs to verify that this distribution is in fact the GS-distribution. To this end the investigator has to rely on the characterizations of the underlying distribution and determine if the corresponding conditions are satisfied. So, the problem of characterizing the GS-distribution becomes essential. One of our objectives here, is to present characterizations of GS-distributions. We shall do this in three different directions as discussed in Section 5.

The *pdf* (probability density function) of a GS- distribution with *cdf* (cumulative distribution function), F , is given by

$$f(x) = \alpha \ (F(x))^{\beta} (1 - (F(x))^{\rho})^{\gamma}, a < x < b, \tag{1}$$

where $\alpha, \rho > 0$ and $\beta, \gamma \geq 0$ are parameters.

Table 1

Distribution	*GS − distributions*			
	α	β	ρ	γ
$Uniform(a,b)$	$\frac{1}{b-a}$	0	1	0
$Exponential(\lambda)$	λ	0	1	1
$Generalized\ Exponential\ (\alpha,\lambda,\nu)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha-1}{\alpha}$	$\frac{1}{\alpha}$	1
$Logistic(\alpha,\beta)$	$\frac{1}{\beta}$	1	1	1
$Beta(b,1)$	b	$\frac{b-1}{b}$	1	0
$Beta(1,b)$	b	0	1	$\frac{b-1}{b}$
$F(2,m)$	1	0	1	$\frac{2+m}{m}$
$F(n,2)$	$\frac{n^2}{4}$	$\frac{n-2}{n}$	$\frac{2}{n}$	2

Note: $F(2,m)$ (and $F(n,2)$) is the F distribution with parameters $(2,m)$ (and $(n,2)$).

Let $X_1, X_2, \cdots, X_n$ be independent random variables having *pdf* $f(x)$ and *cdf* $F(x)$. Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote their corresponding the order statistics. The *pdf* of $X_{r:n}$ and the *pdf* of $(X_{r:n}, X_{s:n})$ are respectively:

$$f_{r:n}(x) = \left[\frac{(n)!}{(r-1)!(n-r)!}\right](F(x))^{r-1}f(x)(1-F(x))^{n-r}, \qquad 1 \leq r \leq n \tag{2}$$

$$\begin{aligned} f_{r,s:n}(x,y) = &\left[\frac{(n)!}{(r-1)!(s-r-1)!(n-r)!}\right](F(x))^{r-1}f(x)(F(y))-F(x))^{s-r-1} \\ &\times f(y)(1-F(y))^{n-s}, \qquad 1 \leq r \leq s \leq n. \end{aligned} \tag{3}$$

Progressive Type-II censoring schemes are the most popular censoring schemes which are used in practice. It can be briefly described as follows: Suppose n independent items are put on life test with continuous identically distributed failure times $X_1, X_2, \cdots, X_n$. Suppose a censoring scheme $(R_1, R_2, \cdots, R_m)$ is prefixed such that at the first failure, R_1 surviving items are removed from the experiment at random; at the second observed failure, R_2 surviving items are removed from the experiment at random; this process continues until at the m^{th} observed value, R_m items are removed from the test at random, $n = m + \sum_{i=1}^{m} R_i$. We will denote the m order observed failure times by $X_{1:m:n}^{(R_1,R_2,\cdots,R_m)}, X_{2:m:n}^{(R_1,R_2,\cdots,R_m)}, \cdots, X_{m:m:n}^{(R_1,R_2,\cdots,R_m)}$ and call them the progressive Type-II right censored order statistics from a sample of size n with progressive censoring scheme $(R_1, R_2, \cdots, R_m)$. Many authors have studied progressive censoring. Among them are Cohen ([7], [8]), Aggarwala, and Balakrishnan [2], Balakrishnan and Sandhu [5] and Balakrishnan [7]. For an extensive survey of progressive censored see Balakrishnan and Aggarwala [4].

The probability density function of $X_{1:m:n}^{(R_1,R_2,\cdots,R_m)}, X_{2:m:n}^{(R_1,R_2,\cdots,R_m)}$, $\cdots, X_{m:m:n}^{(R_1,R_2,\cdots,R_m)}$ is given by (Balakrishnan and Sandhu [5]):-

$$f_{1,2,\cdots,m:m:n}(x_1,x_2,\cdots,x_m) = A(n,m-1)\prod_{i=1}^{m} f(x_i)(1-F(x_i))^{R_i},$$

$$a < x_1 < x_2 < \cdots < x_m < b,$$

where

$$A(n,m-1) = n(n-R_1-1)(n-R_1-R_2-2)\cdots(n-R_1-R_2-\cdots-R_{m-1}-m+1).$$

2 Recurrence Relations between Moments of Order Statistics for GS-Distributions

This section deals with obtaining several recurrence relations satisfied by single and product moments for order statistics of GS-distributions as follow:

Note: Throughout Sections 2 and 3 we assume that $\alpha, \rho > 0, \beta,\ \gamma \geq 0$ are positive integers.

Relation 1 For all $1 \leq r \leq n, k = 0, 1, 2, \cdots$.

$$\alpha \sum_{i=0}^{\gamma} \binom{\gamma}{i} (-1)^i \frac{(\beta+\rho i + r - 1)!(n)!}{(n+\beta+\rho i-1)!(r-1)!}[\mu_{\beta+\rho i+r:n+\beta+\rho i-1}^{(k+1)}] = (k+1)\mu_{r:n}^{(k)}$$
$$+\,\alpha \sum_{i=0}^{\gamma} \binom{\gamma}{i} (-1)^i \frac{(\beta+\rho i + r - 1)!(n)!}{(n+\beta+\rho i-1)!(r-1)!}[\mu_{\beta+\rho i+r-1:n+\beta+\rho i-1}^{(k+1)}].$$

Proof. We have

$$f(x) = \alpha(F(x))^{\beta}(1-(F(x))^{\rho})^{\gamma}, a \leq x \leq b.$$

Let

$$\mu_{r:n}^{(k)} = \left[\frac{(n)!}{(r-1)!(n-r)!}\right] \int_a^b x^k (F(x))^{r-1} f(x)(1-F(x))^{n-r} dx \ .$$

Using binomial expansion to expand $(1-(F(x))^{\rho})^{\gamma}$ in (1), we obtain

$$f(x) = \alpha \sum_{i\,=\,0}^{\gamma} \binom{\gamma}{i} (-1)^i (F(x))^{\beta+\rho i}, \tag{4}$$

and hence

$$\mu_{r:n}^{(k)} = \alpha \textstyle\sum_{i=0}^{\gamma} \binom{\gamma}{i} (-1)^i \left[\frac{(n)!}{(r-1)!(n-r)!}\right] \int_a^b x^k (F(x))^{\beta+\rho i+r-1} (1-F(x))^{n-r} dx.$$

Integrating by parts yields

$$\begin{aligned} (\frac{k+1}{\alpha})\mu_{r:n}^{(k)} = & \sum_{i=0}^{\gamma} \binom{\gamma}{i} (-1)^i [-(\beta+\rho i+r-1) \left[\frac{(n)!}{(r-1)!(n-r)!}\right] \\ & \times \int_a^b x^k (F(x))^{\beta+\rho i+r-2} f(x)(1-F(x))^{n-r} dx \\ & + (n-r) \left[\frac{(n)!}{(r-1)!(n-r)!}\right] \int_a^b x^k (F(x))^{\beta+\rho i+r-1} f(x) \\ & \times (1-F(x))^{n-r-1} dx] \ . \end{aligned} \tag{5}$$

The proof is completed by rewriting (5).

Relation 2 For all $2 \leq n$

$$\begin{aligned} \alpha \sum_{i\,=\,0}^{\gamma} \binom{\gamma}{i} (-1)^i \frac{(\beta+\rho i)!(n)!}{(n+\beta+\rho i-1)!} [\mu_{\beta+\rho i+1,\beta+\rho i+1:n+\beta+\rho i-1}] = \mu_{2:n} \\ + \alpha \sum_{i\,=\,0}^{\gamma} \binom{\gamma}{i} (-1)^i \frac{(\beta+\rho i)!(n)!}{(n+\beta+\rho i-1)!} [\mu_{\beta+\rho i,\beta+\rho i+1:n+\beta+\rho i-1}]. \end{aligned}$$

Relation 3 For all $1 \leq r < n$

$$\begin{aligned} \alpha \sum_{i\,=\,0}^{\gamma} \binom{\gamma}{i} (-1)^i \frac{(\beta+\rho i+r-1)!(n)!}{(n+\beta+\rho i-1)!(r-1)!} [\mu_{\beta+\rho i+r,n+\beta+\rho i-1:n+\beta+\rho i-1}] = \mu_{n:n} \\ + \alpha \sum_{i\,=\,0}^{\gamma} \binom{\gamma}{i} (-1)^i \frac{(\beta+\rho i+r-1)!(n)!}{(n+\beta+\rho i-1)!(r-1)!} [\mu_{\beta+\rho i+r-1,n+\beta+\rho i-1:n+\beta+\rho i-1}], \end{aligned}$$

Relation 4 For all $1 \leq r < s < n$

$$\alpha \sum_{i\,=\,0}^{\gamma} \binom{\gamma}{i} (-1)^i \frac{(\beta{+}\rho i + r - 1)!(n)!}{(n + \beta{+}\rho i - 1)!(r-1)!} [\mu_{\beta+\rho i+r, s+\beta+\rho i-1:n+\beta+\rho i-1}] = \mu_{s:n}$$
$$+ \alpha \sum_{i=0}^{\gamma} \binom{\gamma}{i} (-1)^i \frac{(\beta{+}\rho i + r - 1)!(n)!}{(n + \beta{+}\rho i - 1)!(r-1)!} [\mu_{\beta+\rho i+r-1, s+\beta+\rho i-1:n+\beta+\rho i-1}].$$

Proof. Let

$$\begin{aligned}\mu_{s:n} &= E(X^0_{r:n} X_{s:n}) \\ &= \left[\frac{(n)!}{(s-r-1)!(r-1)!(n-s)!}\right] \int_a^b \int_a^y y(F(x))^{r-1} f(x) \\ &\times (F(y) - F(x))^{s-r-1} f(y)(1 - F(y))^{n-s} dxdy \\ &= \left[\frac{(n)!}{(s-r-1)!(r-1)!(n-s)!}\right] \int_a^b y\xi(y) f(y)(1-F(y))^{n-s} dx,\end{aligned} \tag{6}$$

$$\xi(y) = \textstyle\int_a^y (F(x))^{r-1} f(x)(F(y) - F(x))^{s-r-1} dx.$$

From (1), we have

$$\xi(x) = \alpha \sum_{i=0}^{\gamma} \binom{\gamma}{i} (-1)^i \int_a^y (F(x))^{\beta+\rho i+r-1} (F(y) - F(x))^{s-r-1} dx. \tag{7}$$

Integrating by parts yields

$$\begin{aligned}\xi(x) = \alpha \sum_{i=0}^{\gamma} \binom{\gamma}{i} &(-1)^i [-(\beta + \rho i + r - 1) \int_a^y x(F(x))^{\beta+\rho i+r-2} f(x) \\ &\times (F(y) - F(x))^{s-r-1} dx + (s-r-1) \int_a^y x(F(x))^{\beta+\rho i+r-1} f(x) \\ &\times (F(y) - F(x))^{s-r-2} dx\ ].\end{aligned}$$

Substituting this in (6), we obtain

$$\mu_{s:n} = \alpha \sum_{i=0}^{\gamma} \binom{\gamma}{i} (-1)^i [-(\beta+\rho i+r-1) \left[\frac{(n)!}{(s-r-1)!(r-1)!(n-s)!}\right]$$
$$\times \int_a^b \int_a^y xy(F(x))^{\beta+\rho i+r-2} f(x)(F(y)-F(x))^{s-r-1}$$
$$\times f(y)(1-F(y))^{n-s} dxdy + (s-r-1) \left[\frac{(n)!}{(s-r-1)!(r-1)!(n-s)!}\right] \quad (8)$$
$$\times \int_a^b \int_a^y xy(F(x))^{\beta+\rho i+r-1} (F(y)-F(x))^{s-r-2} f(y)(1-F(y))^{n-s} dxdy] \ .$$

The proof is completed by rewriting (8).

3 Recurrence Relations for Single and Product Moments of progressive Type-II right censoring

In this section, some recurrence relations for moments for progressive Type-II right censoring of GS-distributions are established as follow:

Relation 5 For all $1 \leq m \leq n, k = 0, 1, 2, \cdots$.

$$\alpha \sum_{\xi=0}^{\gamma} \sum_{\tau=0}^{\beta+\rho\xi} \binom{\gamma}{\xi} \binom{\beta+\rho\xi}{\tau} (-1)^{\xi}(-1)^{\tau}(R_1+\tau)\left(\frac{n}{n+\tau-1}\right)[\mu_{1:m:n+\tau-1}^{(R_1+\tau-1,R_2,\cdots,R_m)}]^{(k+1)}$$
$$= [(k+1)[\mu_{1:m:n}^{(R_1,R_2,\cdots,R_m)}]^{(k)} - \alpha \sum_{\xi=0}^{\gamma} \sum_{\tau=0}^{\beta+\rho\xi} \binom{\gamma}{\xi} \binom{\beta+\rho\xi}{\tau} (-1)^{\xi}(-1)^{\tau}$$
$$\times \{\frac{n(n-R_1-1)}{n+\tau-1}[\mu_{1:m-1:n+\tau-1}^{(R_1+R_2+\tau,R_3,\cdots,R_m)}]^{(k+1)}\}].$$

Relation 6 For all $1 \leq m \leq n, k = 0, 1, 2, \cdots$.

$$\alpha \sum_{\xi=0}^{\gamma} \sum_{\tau=0}^{\beta+\rho\xi} \binom{\gamma}{\xi} \binom{\beta+\rho\xi}{\tau} (-1)^{\xi}(-1)^{\tau}(R_m+\tau)\left(\frac{A(n,m-1)}{A(n+\tau-1,m-1)}\right)$$
$$\times [\mu_{m:m:n+\tau-1}^{(R_1,R_2,\cdots,R_{m-1},R_m+\tau-1)}]^{(k+1)} = [(k+1)[\mu_{m:m:n}^{(R_1,R_2,\cdots,R_m)}]^{(k)}$$
$$-\alpha \sum_{\xi=0}^{\gamma} \sum_{\tau=0}^{\beta+\rho\xi} \binom{\gamma}{\xi} \binom{\beta+\rho\xi}{\tau} (-1)^{\xi}(-1)^{\tau}\{-\frac{A(n,m-1)}{A(n+\tau-1,m-2)}$$
$$\times [\mu_{m-1:m-1:n+\tau-1}^{(R_1,R_2,\cdots,R_{m-1}+R_m+\tau)}]^{(k+1)}\}].$$

Relation 7 For all $1 \leq r \leq m \leq n, k = 0, 1, 2, \cdots$.

$$\alpha \sum_{\xi=0}^{\gamma} \sum_{\tau=0}^{\beta+\rho\xi} \binom{\gamma}{\xi} \binom{\beta+\rho\xi}{\tau} (-1)^{\xi} (-1)^{\tau} (R_r + \tau) \left(\frac{A(n, r-1)}{A(n+\tau-1, r-1)} \right)$$
$$\times [\mu_{r:m:n+\tau-1}^{(R_1, R_2, \cdots, R_{r-1}, R_r+\tau-1, R_{r+2}, \cdots, R_m)}]^{(k+1)} = [(k+1)[\mu_{r:m:n}^{(R_1, R_2, \cdots, R_m)}]^{(k)}$$
$$- \alpha \sum_{\xi=0}^{\gamma} \sum_{\tau=0}^{\beta+\rho\xi} \binom{\gamma}{\xi} \binom{\beta+\rho\xi}{\tau} (-1)^{\xi} (-1)^{\tau} \{ \left(\frac{A(n, r)}{A(n+\tau-1, r-1)} \right)$$
$$\times \mu_{r:m-1:n+\tau-1}^{(R_1, R_2, \cdots, R_{r-1}, R_r+R_{r+1}+\tau, R_{r+2}, \cdots, R_m)}]^{(k+1)}$$
$$- \frac{A(n, r-1)}{A(n+\tau-1, r-2)} [\mu_{r-1:m-1:n+\tau-1}^{(R_1, R_2, \cdots, R_{r-1}+R_r+\tau, \cdots, R_m)}]^{(k+1)} \}].$$

Proof. We have

$$[\mu_{r:m:n}^{(R_1, R_2, \cdots, R_m)}]^{(k)} = E(X_{r:m:n}^{(R_1, R_2, \cdots, R_m)(k)}) = A(n, m-1)$$
$$\times \int \cdots \int_{0<x_1<x_2<\cdots\ <\ x_m\ <\ \infty} \left\{ \int_{x_{r-1}}^{x_{r+1}} x_r^k \left[1 - F(x_r)\right]^{R_r} f(x_r) dx_r \right\}$$
$$\times (1 - F(x_1))^{R_1} f(x_1) (1 - F(x_2))^{R_2} f(x_2) \cdots (1 - F(x_m))^{R_m}$$
$$\times f(x_m) dx_1 dx_2 \cdots dx_{r-1} dx_{r+1} \cdots dx_m,$$

$$0 < x_1 < x_2 < \cdots < x_r < \cdots < x_m < \infty.$$

Let

$$\Omega = \int_{x_{r-1}}^{x_{r+1}} x_r^k \left[1 - F(x_r)\right]^{R_r} f(x_r) dx_r.$$

We have from (1)

$$f(x) = \alpha \sum_{\xi=0}^{\gamma} \sum_{\tau\ =\ 0}^{\beta+\rho\xi} \binom{\gamma}{\xi} \binom{\beta+\rho\xi}{\tau} (-1)^{\xi} (-1)^{\tau} (1 - F(x))^{\tau}.$$

We have

$$\Omega = \alpha \sum_{\xi=0}^{\gamma} \sum_{\tau\ =\ 0}^{\beta+\rho\xi} \binom{\gamma}{\xi} \binom{\beta+\rho\xi}{\tau} (-1)^{\xi} (-1)^{\tau} \int_{x_{r-1}}^{x_{r+1}} x_r^k \left[1 - F(x_r)\right]^{R_r+\tau} dx_r.$$

Integrating by parts yields

$$\Omega = \frac{\alpha}{k+1}\sum_{\xi=0}^{\gamma}\sum_{\tau=0}^{\beta+\rho\xi}\binom{\gamma}{\xi}\binom{\beta+\rho\xi}{\tau}(-1)^{\xi}(-1)^{\tau}\{x_{r+1}^{k+1}\left[1-F(x_{r+1})\right]^{R_r+\tau}$$
$$-x_{r-1}^{k+1}\left[1-F(x_{r-1})\right]^{R_r+\tau}+(R_r+\tau)\int_{x_{r-1}}^{x_{r+1}} x_r^{k+1}\left[1-F(x_r)\right]^{R_r+\tau-1} f(x_r)dx_r\}.$$

Then

$$[\mu_{r:m:n}^{(R_1,R_2,\cdots,R_m)}]^{(k)} = (\frac{\alpha}{k+1})A(n,m-1)\sum_{\xi\,=\,0}^{\gamma}\sum_{\tau\,=\,0}^{\beta+\rho\xi}\binom{\gamma}{\xi}\binom{\beta+\rho\xi}{\tau}(-1)^{\xi}(-1)^{\tau}$$
$$\times\{\iint_{0<x_1<x_2<\cdots<x_m\,<\,\infty}\cdots\int_{x_{r-1}}^{x_{r+1}} x_{r+1}^{k+1}(1-F(x_1))^{R_1}f(x_1)(1-F(x_2))^{R_2}$$
$$\times f(x_2)\cdots\left[1-F(x_{r-1})\right]^{R_{r-1}} f(x_{r-1})\left[1-F(x_{r+1})\right]^{R_r+R_{r+1}+\tau} f(x_{r+1})\cdots$$
$$\times(1-F(x_m))^{R_m}f(x_m)dx_1dx_2\cdots dx_{r-1}dx_{r+1}\cdots dx_m$$
$$-\iint_{0<x_1<x_2\,<\,\cdots\,<\,x_m\,<\,\infty}\cdots\int_{x_{r-1}}^{x_{r+1}} x_{r-1}^{k+1}(1-F(x_1))^{R_1}$$
$$\times f(x_1)(1-F(x_2))^{R_2}f(x_2)\cdots\left[1-F(x_{r-1})\right]^{R_{r-1}+R_r+\tau}\ f(x_{r-1})\left[1-F(x_{r+1})\right]^{R_{r+1}}$$
$$\times f(x_{r+1})\cdots(1-F(x_m))^{R_m}f(x_m)\ dx_1dx_2\cdots dx_{r-1}dx_{r+1}\cdots dx_m+(R_r+\tau)$$
$$\times\iint_{0<x_1\,<\,x_2<\cdots\,<\,x_m\,<\,\infty}\cdots\int_{x_{r-1}}^{x_{r+1}} x_r^{k+1}(1-F(x_1))^{R_1}f(x_1)(1-F(x_2))^{R_2}f(x_2)\cdots$$
$$\times\left[1-F(x_r)\right]^{R_r+\tau-1} f(x_r)\cdots(1-F(x_m))^{R_m}f(x_m)dx_1dx_2\cdots dx_r\cdots dx_m.$$

We obtain

$$[\mu_{r:m:n}^{(R_1,R_2,\cdots,R_m)}]^{(k)} = (\frac{\alpha}{k+1})\sum_{\xi=0}^{\gamma}\sum_{\tau=0}^{\beta+\rho\xi}\binom{\gamma}{\xi}\binom{\beta+\rho\xi}{\tau}(-1)^{\xi}$$
$$\times(-1)^{\tau}\{\left(\frac{A(n,r)}{A(n+\tau-1,r-1)}\right)$$
$$\times\mu_{r:m-1:n+\tau-1}^{(R_1,R_2,\cdots,R_r+R_{r+1}+\tau,R_{r+2}\cdots,R_m)(k+1)}$$
$$-\left(\frac{A(n,r-1)}{A(n+\tau-1,r-2)}\right)\mu_{r-1:m-1:n+\tau-1}^{(R_1,R_2,\cdots,R_{r-1}+R_r+\tau,R_{r+1}\cdots,R_m)(k+1)}$$
$$+(R_r+\tau)\left(\frac{A(n,r-1)}{A(n+\tau-1,r-1)}\right)\mu_{r:m:n+\tau-1}^{(R_1,R_2,\cdots,R_r+\tau-1,R_{r+1},\cdots,R_m)(k+1)}\}.$$

The proof is completed.

Relation 8 For all $1 < s \leq m \leq n$.

$$\alpha\sum_{\xi=0}^{\gamma}\sum_{\tau=0}^{\beta+\rho\xi}\binom{\gamma}{\xi}\binom{\beta+\rho\xi}{\tau}(-1)^{\xi}(-1)^{\tau}(R_s+\tau)\left(\frac{A(n,s-1)}{A(n+\tau-1,s-1)}\right)$$
$$\times\,\mu_{1,s:m:n+\tau-1}^{(R_1,R_2,\cdots,R_s+\tau-1,R_{s+1},\cdots,R_m)}=\mu_{1:m:n}^{(R_1,R_2,\cdots,R_m)}$$
$$-\,\alpha\sum_{\xi=0}^{\gamma}\sum_{\tau=0}^{\beta+\rho\xi}\binom{\gamma}{\xi}\binom{\beta+\rho\xi}{\tau}(-1)^{\xi}(-1)^{\tau}\{\left(\frac{A(n,s)}{A(n+\tau-1,s-1)}\right)$$
$$\times\,\mu_{1,s:m-1:n+\tau-1}^{(R_1,R_2,\cdots,R_s+R_{s+1}+\tau,\cdots,R_m)}-\left(\frac{A(n,s-1)}{A(n+\tau-1,s-2)}\right)$$
$$\times\,\mu_{1,s-1:m-1:n+\tau-1}^{(R_1,R_2,\cdots,R_{s-1}+R_s+\tau,\cdots,R_m)}\}].$$

Relation 9 For all $1\le r<m\le n$.

$$\alpha\sum_{\xi=0}^{\gamma}\sum_{\tau=0}^{\beta+\rho\xi}\binom{\gamma}{\xi}\binom{\beta+\rho\xi}{\tau}(-1)^{\xi}(-1)^{\tau}(R_m+\tau)\left(\frac{A(n,m-1)}{A(n+\tau-1,m-1)}\right)$$
$$\times\,\mu_{r,m:m:n+\tau-1}^{(R_1,R_2,\cdots,R_m+\tau-1,R_{m+1},\cdots,R_m)}=\mu_{r:m:n}^{(R_1,R_2,\cdots,R_m)}$$
$$-\,\alpha\sum_{\xi=0}^{\gamma}\sum_{\tau=0}^{\beta+\rho\xi}\binom{\gamma}{\xi}\binom{\beta+\rho\xi}{\tau}(-1)^{\xi}(-1)^{\tau}\{-\left(\frac{A(n,m-1)}{A(n+\tau-1,m-2)}\right)$$
$$\times\,\mu_{r,m-1:m-1:n+\tau-1}^{(R_1,R_2,\cdots,R_r,R_{r+1},\cdots,R_{m-1}+R_m+\tau,\cdots,R_m)}\}].$$

Relation 10 For all $1\le r<s\le m\le n$.

$$\alpha\sum_{\xi=0}^{\gamma}\sum_{\tau=0}^{\beta+\rho\xi}\binom{\gamma}{\xi}\binom{\beta+\rho\xi}{\tau}(-1)^{\xi}(-1)^{\tau}(R_s+\tau)\left(\frac{A(n,s-1)}{A(n+\tau-1,s-1)}\right)$$
$$\times\,\mu_{r,s:m:n+\tau-1}^{(R_1,R_2,\cdots,R_s+\tau-1,R_{s+1},\cdots,R_m)}=\mu_{r:m:n}^{(R_1,R_2,\cdots,R_m)}$$
$$-\,\alpha\sum_{\xi=0}^{\gamma}\sum_{\tau=0}^{\beta+\rho\xi}\binom{\gamma}{\xi}\binom{\beta+\rho\xi}{\tau}(-1)^{\xi}(-1)^{\tau}\{\left(\frac{A(n,s)}{A(n+\tau-1,s-1)}\right)$$
$$\times\,\mu_{r,s:m-1:n+\tau-1}^{(R_1,R_2,\cdots,R_r,R_{r+1},\cdots,R_s+R_{s+1}+\tau,\cdots,R_m)}-\left(\frac{A(n,s-1)}{A(n+\tau-1,s-2)}\right)$$
$$\times\,\mu_{r,s-1:m-1:n+\tau-1}^{(R_1,R_2,\cdots,R_r,R_{r+1},\cdots,R_{s-1}+R_s+\tau,\cdots,R_m)}\}].$$

Proof. We have

$$\mu_{r,s:m:n}^{(R_1,R_2,\cdots,R_m)} = E(X_{r:m:n}^{(R_1,R_2,\cdots,R_m)} X_{s:m:n}^{(R_1,R_2,\cdots,R_m)(0)}) = A(n,m-1)$$
$$\times \iint_{0<x_1\ <\ x_2<\cdots\ <\ x_m\ <\ \infty} \cdots \int_{x_{r-1}}^{x_{r+1}} x_r(1-F(x_1))^{R_1} f(x_1)(1-F(x_2))^{R_2}$$
$$\times f(x_2)\cdots[1-F(x_r)]^{R_r} f(x_r)\cdots(1-F(x_m))^{R_m} f(x_m)dx_1dx_2\cdots dx_r\cdots dx_m,$$

$$0 < x_1 < x_2 < \cdots < x_r < \cdots x_s < \cdots < x_m < \infty.$$

Let

$$\Delta = \int_{x_{s-1}}^{x_{s+1}} [1{-}F(x_s)]^{R_s} f(x_s)dx_s.$$

We have from (1)

$$f(x) = \alpha \sum_{\xi=0}^{\gamma} \sum_{\tau=0}^{\beta+\rho\xi} \binom{\gamma}{\xi} \binom{\beta+\rho\xi}{\tau} (-1)^{\xi}(-1)^{\tau}(1-F(x))^{\tau}.$$

Then

$$\Delta = \alpha \sum_{\xi=0}^{\gamma} \sum_{\tau\ =\ 0}^{\beta+\rho\xi} \binom{\gamma}{\xi} \binom{\beta+\rho\xi}{\tau} (-1)^{\xi}(-1)^{\tau} \int_{x_{s-1}}^{x_{s+1}} [1-F(x_s)]^{R_s+\tau} dx_s.$$

Integrating by parts yields

$$\Delta = \alpha \sum_{\xi=0}^{\gamma} \sum_{\tau\ =\ 0}^{\beta+\rho\xi} \binom{\gamma}{\xi} \binom{\beta+\rho\xi}{\tau} (-1)^{\xi}(-1)^{\tau}\{x_{s+1}[1-F(x_{s+1})]^{R_s+\tau}$$
$$- x_{s-1}[1-F(x_{s-1})]^{Rs+\tau} + (R_s+\tau)\int_{x_{s-1}}^{x_{s+1}} x_s [1-F(x_s)]^{R_s+\tau-1} f(x_s)dx_s\}.$$

Then

$$\mu_{r:m:n}^{(R_1,R_2,\cdots,R_m)} = (\frac{\alpha}{k+1})A(n,m-1)\sum_{\xi\,=\,0}^{\gamma}\sum_{\tau=0}^{\beta+\rho\xi}\binom{\gamma}{\xi}\binom{\beta+\rho\xi}{\tau}(-1)^{\xi}(-1)^{\tau}$$

$$\times\{\iint_{0<x_1\,<\,x_2<\cdots<x_m\,<\,\infty}\cdots\int_{x_{r-1}}^{x_{r+1}} x_r x_s(1-F(x_1))^{R_1}f(x_1)(1-F(x_2))^{R_2}$$

$$\times f(x_2)\cdots[1-F(x_r)]^{R_r}f(x_r)\cdots[1-F(x_{s-1})]^{R_{s-1}}f(x_{s-1})\,[1-F(x_{s+1})]^{R_s+R_{s+1}+\tau}$$

$$\times f(x_{s+1})\cdots(1-F(x_m))^{R_m}f(x_m)dx_1dx_2\cdots dx_r dx_{r+1}\cdots dx_{s-1}dx_{s+1}\cdots dx_m$$

$$-\iint_{0<x_1<x_2\,<\,\cdots\,<\,x_m\,<\,\infty}\cdots\int_{x_{r-1}}^{x_{r+1}} x_r x_s(1-F(x_1))^{R_1}f(x_1)(1-F(x_2))^{R_2}f(x_2)\cdots$$

$$\times[1-F(x_r)]^{R_r}f(x_r)\cdots[1-F(x_{s-1})]^{R_{s-1}+R_s+\tau}f(x_{s-1})\,[1-F(x_{s+1})]^{R_{r+1}}f(x_{s+1})$$

$$\times\cdots(1-F(x_m))^{R_m}f(x_m)dx_1dx_2\cdots dx_r dx_{r+1}\cdots dx_{s-1}dx_{s+1}\cdots dx_m$$

$$+(R_s+\tau)\iint_{0<x_1\,<\,x_2<\cdots<x_m\,<\,\infty}\cdots\int_{x_{r-1}}^{x_{r+1}} x_r x_s(1-F(x_1))^{R_1}f(x_1)(1-F(x_2))^{R_2}f(x_2)$$

$$\times\cdots[1-F(x_r)]^{R_r}f(x_r)\cdots[1-F(x_s)]^{R_s+\tau-1}f(x_s)\cdots(1-F(x_m))^{R_m}$$

$$\times f(x_m)dx_1dx_2\cdots dx_r\cdots dx_s\cdots dx_m.$$

We have,

$$\mu_{r:m:n}^{(R_1,R_2,\cdots,R_m)} = \alpha\sum_{\xi=0}^{\gamma}\sum_{\tau=0}^{\beta+\rho\xi}\binom{\gamma}{\xi}\binom{\beta+\rho\xi}{\tau}(-1)^{\xi}(-1)^{\tau}$$

$$\times\{(n-s-\sum_{i\,=\,1}^{s}R_i)\mu_{r,s:m-1:n+\tau-1}^{(R_1,R_2,\cdots,R_r,R_{r+1},\cdots,R_s+R_{s+1}+\tau,\cdots,R_m)}$$

$$-(n-s+1-\sum_{i\,=\,1}^{s-1}R_i)\mu_{r,s-1:m-1:n+\tau-1}^{(R_1,R_2,\cdots,R_r,R_{r+1},\cdots,R_{s-1}+R_s+\tau,\cdots,R_m)}$$

$$+(R_s+\tau)\;\mu_{r,s:m:n+\tau-1}^{(R_1,R_2,\cdots,R_s+\tau-1,\cdots,R_m)}\}.$$

The proof is completed.

Special Cases

1. Recurrence relations for moments from progressive Type-II right censoring for some specific distributions (see Table 1) can be obtained as special cases from Relations $[5\;-10]$.

2. Recurrence relations for moments from Type-II censored can be obtained by letting $R_1 = R_2 = \cdots = R_{m-1} = 0, R_m \neq 0$.

3. The Relations for logistic distribution of progressive Type-II right censored are obtained as a special case from Relations $[5 \ -10]$, $\alpha = 1, \beta = 1, \rho = 1$ and $\gamma = 1$,(see Balakrishnan et al. [6]).

4. The Relations for standard exponential distribution of progressive Type-II right censored are obtained as a special case from Relations $[5 \ -10]$, $\alpha = 1, \beta = 0, \rho = 1$ and $\gamma = 1$, we have

Relation 11 For all $1 \leq r \leq n$

$$\begin{aligned}
&(R_r+1)[\mu_{r:m:n}^{(R_1,R_2,\cdots,R_m)}]^{(k+1)} = (k+1)[\mu_{r:m:n}^{(R_1,R_2,\cdots,R_m)}]^{(k)} \\
&+ (n-r+1-\sum_{i=1}^{r-1} R_i)[\mu_{r-1:m-1:n}^{(R_1,R_2,\cdots,R_{r-1}+R_r+1,\cdots,R_m)}]^{(k+1)} \\
&- (n-r-\sum_{i=1}^{r} R_i)[\mu_{r:m-1:n}^{(R_1,R_2,\cdots,R_r+R_{r+1}+1,\cdots,R_m)}]^{(k+1)}.
\end{aligned}$$

Relation 12 For all $1 \leq r < s \leq n$

$$\begin{aligned}
&(R_s+1)\mu_{r,s:m:n}^{(R_1,R_2,\cdots,R_m)} = \mu_{r:m:n}^{(R_1,R_2,\cdots,R_m)} \\
&- (n-s-\sum_{i=1}^{s} R_i)\mu_{r,s:m-1:n}^{(R_1,R_2,\cdots,R_r,R_{r+1},\cdots,R_s+R_{s+1}+1,\cdots,R_m)} \\
&+ (n-s+1-\sum_{i\,=\,1}^{s-1} R_i)\mu_{r,s-1:m-1:n}^{(R_1,R_2,\cdots,R_r,R_{r+1},\cdots,R_{s-1}+R_s+1,\cdots,R_m)}.
\end{aligned}$$

Remark 1

1. Recurrence relations for moments of progressively censored order statistics from logistic distribution are obtained by Balakrishnan et al. [6].

2. Aggarwala and Balakrishnan [2] have established recurrence relations for moments like the Relations (11 and 12).

3. Depending on Relations [11 and 12], aggarwala and Balakrishnan [2] have obtained recursive algorithm for exponential distribution.

4. The recurrence relations for moments of usual order statistics for standard exponential distribution are obtained as a special case from Relations [11 and 12] as follows:

Relation 13 For all $1 \leq r \leq n$

$$(n-r+1)\mu_{r:n}^{(k+1)} = (k+1)\mu_{r:n}^{(k)} + (n-r+1)\mu_{r-1:n}^{(k+1)}.$$

Relation 14 For all $1 \leq r < s \leq n$

$$\mu_{r,s:n} = \mu_{r:n} + (n-s+1)\mu_{r,s-1:n} - (n-s)\mu_{r,s:n}.$$

Remark 2

Relations (13 and 14) have been established by Joshi [24] and Joshi and Balakrishnan [25] and presented by Aggarwala and Balakrishnan [2] as special cases.

4 Numerical Results

In this section, using Relations [11 and 12] , means, variances and covariances for standard exponential distribution of progressive Type-II right censored order statistics are computed. The computations are presented in Tables (2 - 5) below :

Table 2

$m \diagdown r$	1	2	3	4
1	0.1			
2	0.1	0.2428572		
3	0.1	0.2428572	0.5761905	
4	0.1	0.2428572	0.5761905	1.076190

Means for standard exponential distribution of progressive Type $-$ II right censored, $m = 1(1)4, n = 10, R = (2,3,0,1)$ and $R_m = n - m - \sum_{i=1}^{m-1} R_i$

Table 3

m	$r\diagdown s$	1	2	3	4
1	1	0.01			
2	1	0.01	0.01		
	2		0.0304082		
3	1	0.01	0.01	0.01	
	2		0.0304082	0.0304082	
	3			0.1415193	
4	1	0.01	0.01	0.01	0.01
	2		0.0304082	0.0304082	0.0304082
	3			0.1415193	0.1415193
	4				0.3915193

Variances and covariances for standard exponential distribution of progressive Type $-$ II right censored, $m = 1(1)4, n = 10, R = (2, 3, 0, 1)$ and $R_m = n - m - \sum_{i=1}^{m-1} R_i$

Table 4

$m\diagdown r$	1	2	3	4	5	6	7	8	9	10
1	0.05									
2	0.05	0.1088								
3	0.05	0.1088	0.1713							
4	0.05	0.1088	0.1713	0.2379						
5	0.05	0.1088	0.1713	0.2379	0.3289					
6	0.05	0.1088	0.1713	0.2379	0.3289	0.4539				
7	0.05	0.1088	0.1713	0.2379	0.3289	0.4539	0.6206			
8	0.05	0.1088	0.1713	0.2379	0.3289	0.4539	0.6206	0.8206		
9	0.05	0.1088	0.1713	0.2379	0.3289	0.4539	0.6206	0.8206	1.0706	
10	0.05	0.1088	0.1713	0.2379	0.3289	0.4539	0.6206	0.8206	1.0706	1.4039

Means for standard exponential distribution of progressive Type $-$ II right censored, $m = 1(1)10, n = 20, R = (2, 0, 0, 3, 2, 1, 0, 0, 0, 2)$ and $R_m = n - m - \sum_{i=1}^{m-1} R_i$

Table 5

m	$r\diagdown s$	1	2	3	4	5	6	7
1	1	0.0025						
2	1	0.0025	0.0025					
	2		0.006					
3	1	0.0025	0.0025	0.0025				
	2		0.006	0.006				
	3			0.0099				
4	1	0.0025	0.0025	0.0025	0.0025			
	2		0.006	0.006	0.006			
	3			0.0099	0.0099			
	4				0.01431			
5	1	0.0025	0.0025	0.0025	0.0025	0.0025		
	2		0.006	0.006	0.006	0.006		
	3			0.0099	0.0099	0.0099		
	4				0.0143	0.0143		
	5					0.02258		
6	1	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025	
	2		0.006	0.006	0.006	0.006	0.006	
	3			0.0099	0.0099	0.0099	0.0099	
	4				0.0143	0.0143	0.0143	
	5					0.02258	0.02258	
	6						0.0382	
7	1	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025
	2		0.006	0.006	0.006	0.006	0.006	0.006
	3			0.0099	0.0099	0.0099	0.0099	0.0099
	4				0.0143	0.0143	0.0143	0.0143
	5					0.02258	0.02258	0.02258
	6						0.0382	0.0382
	7							0.065978

Table 5 (continued)

m	$r \diagdown s$	1	2	3	4	5	6	7	8	9	10
8	1	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025		
	2		0.006	0.006	0.006	0.006	0.006	0.006	0.006		
	3			0.0099	0.0099	0.0099	0.0099	0.0098	0.0099		
	4				0.0143	0.0143	0.0143	0.0143	0.0143		
	5						0.02258	0.02258	0.0226		
	6						0.0382	0.0382	0.0382		
	7							0.066	0.066		
	8								0.0106		
9	1	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025	
	2		0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	
	3			0.0099	0.0099	0.0099	0.0099	0.0099	0.0099	0.0099	
	4					0.0143	0.0143	0.0143	0.0143	0.0143	
	5						0.02258	0.02258	0.0226	0.0226	
	6						0.0382	0.0382	0.0382	0.0382	
	7							0.066	0.066	0.066	
	8								0.0106	0.0106	
	9									0.1685	
10	1	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025
	2		0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006
	3			0.0099	0.0099	0.0099	0.0099	0.0099	0.0099	0.0099	0.0099
	4				0.0143	0.0143	0.0143	0.0143	0.0143	0.0143	0.0143
	5					0.0226	0.0226	0.0226	0.0226	0.0226	0.0226
	6						0.0382	0.0382	0.0382	0.0382	0.0382
	7							0.066	0.066	0.066	0.066
	8								0.0106	0.0106	0.0106
	9									0.1685	0.1685
	10										0.2796

Variances and covariances for standard exponential distribution of progressive Type − II right censored, $m = 1(1)10, n = 20,\ \ R = (2, 0, 0, 3, 2, 1, 0, 0, 0, 2)$ and $R_m = n - m - \sum_{i\,=\,1}^{m-1} R_i$

Remark 3

If r is increasing, the single and product moments are increasing too.

5 Characterization Results

As we mentioned in the introduction section, the GS- distributions have applications in many fields of study which have been mentioned in [27] and the references therein. So, an investigator will be vitally interested to know if their model fits the requirements of GS- distribution. To this end, the investigator relies on characterizations of this distribution, which provide conditions under which the underlying distribution is indeed a GS- distribution. In this section we will present various characterizations of the GS- distributions.

Throughout this section we assume, when needed, that the distribution function F is twice differentiable on its support.

5.1. Characterization based on two truncated moments

In this subsection we present characterizations of the GS- distributions in terms of truncated moments. We like to mention here the works of Galambos and Kotz [9], Kotz and Shanbhag [27], Glä nzel [10 − 12], Glänzel [6], Glänzel et al. [13] , Glänzel and Hamedani [15] and Hamedani [17 − 19] in this direction. Our characterization results presented here will employ an interesting result due to Glänzel [11] (Theorem G below).

Theorem 1. Let (Ω, F, P) be a given probability space and let $H = [a, b]$ be an interval for some $a < b\,(a = -\infty, b = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let g and h be two real functions defined on H such that

$$E\left[g\left(X\right)|X \geq x\right] = E\left[h\left(X\right)|X \geq x\right]\eta\left(x\right), \qquad x \in H,$$

is defined with some real function η. Assume that $g, h \in C^1(H), \eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $h\eta = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g, h and η, particularly

$$F\left(x\right) = \int_a^x C\left|\frac{\eta'\left(u\right)}{\eta\left(u\right)h\left(u\right) - g\left(u\right)}\right|\exp\left(-s\left(u\right)\right)\,du,$$

where the function s is a solution of the differential equation $s' = \frac{\eta'\,h}{\eta h - g}$ and C is a constant, chosen to make $\int_H dF = 1$.

Remarks 4. (a) In Theorem 1, the interval H need not be closed. (b) The goal is to have the function η as simple as possible. For a more detailed discussion on the choice of η, we refer the reader to Glänzel and Hamedani [15] and Hamedani [17 – 19].

Proposition 1. Let $X : \Omega \to (a, b)$ be a continuous random variable with *cdf* F and let $h(x) = 1$ and $g(x) = \int_a^x (F(u))^\beta (1 - (F(u))^\rho)^\gamma du$ for $x \in (a, b)$. The *pdf* of X is (1) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2\alpha} + \frac{1}{2}\int_a^x (F(u))^\beta (1 - (F(u))^\rho)^\gamma du, \qquad a < x < b.$$

Proof. Let X have *pdf* (1), then

$$(1 - F(x)) E[h(X) \ |X \geq x] = (1 - F(x)), \quad a < x < b,$$

and

$$(1 - F(x)) E[g(X) |X \geq x] = \int_x^b \int_a^t (F(u))^\beta (1 - (F(u))^\rho)^\gamma du f(t) dt$$
$$= \int_x^b \frac{1}{\alpha} F(t) f(t) dt = \frac{1}{2\alpha}\left(1 - (F(x))^2\right), \qquad a < x < b,$$

and finally

$$\eta(x) h(x) - g(x) = \frac{1}{2}\left(\frac{1}{\alpha} - \int_a^x (F(u))^\beta (1 - (F(u))^\rho)^\gamma du\right) \neq 0 \qquad for \qquad a < x < b.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x) h(x)}{\eta(x) h(x) - g(x)} = \frac{(F(x))^\beta (1 - (F(x))^\rho)^\gamma}{\left(\frac{1}{\alpha} - \int_a^x (F(u))^\beta (1 - (F(u))^\rho)^\gamma du\right)},$$

and hence

$$s(x) = -\ln\left(\frac{1}{\alpha} - \int_a^x (F(u))^\beta (1 - (F(u))^\rho)^\gamma du\right), \ a < x < b.$$

Now, in view of Theorem 1 (with $C = \alpha$), X has *pdf* (1).

Corollary 1. Let $X : \Omega \to (a,b)$ be a continuous random variable and let $h(x) = 1$ for $x \in (a,b)$. The *pdf* of X is (1) if and only if there exist functions g and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)}{\eta(x) - g(x)} = \frac{(F(x))^{\beta}(1-(F(x))^{\rho})^{\gamma}}{\left(\frac{1}{\alpha} - \int_a^x (F(u))^{\beta}(1-(F(u))^{\rho})^{\gamma}\, du\right)}, \ a < x < b.$$

Remark 5. The general solution of the differential equation given in Corollary 1 is

$$\eta(x) = \left(\frac{1}{\alpha} - \int_a^x (F(u))^{\beta}(1-(F(u))^{\rho})^{\gamma}\, du\right)^{-1} \times \left[-\int g(x)\left((F(x))^{\beta}(1-(F(x))^{\rho})^{\gamma}\right) dx + D\right],$$

for $a < x < b$, where D is a constant. One set of appropriate functions is given in Proposition 1 with $D = \frac{1}{2\alpha^2}$.

5.2. Characterization based on the hazard function

For the sake of completeness, we state the following definition.

Definition 1. Let F be an absolutely continuous distribution with the corresponding *pdf* f. The hazard function corresponding to F is denoted by λ_F and is defined by

$$\lambda_F(x) = \frac{f(x)}{1 - F(x)}, x \in SuppF, \tag{9}$$

where $SuppF$ is the support of F.

It is obvious that the hazard function of a twice differentiable distribution function satisfies the first order differential equation

$$\frac{\lambda_F'(x)}{\lambda_F(x)} - \lambda_F(x) = k(x), \tag{10}$$

where $k(x)$ is an appropriate integrable function. Although this differential equation has an obvious form since

$$\frac{f'(x)}{f(x)} = \frac{\lambda_F'(x)}{\lambda_F(x)} - \lambda_F(x),$$

for many univariate continuous distributions (10) seems to be the only differential equation in terms of the hazard function. The goal here is to establish a differential equation which has as simple form as possible and is not of the trivial form (10). For some general families of distributions this may not be possible. Here is our characterization result for the GS-distributions.

Proposition 2. Let $X : \Omega \rightarrow (a, b)$ be a continuous random variable. The *pdf* of X is (1) with $\rho = 1$ if and only if its hazard function λ_F satisfies the differential equation

$$\lambda_F' (x) - q (x) \lambda_F^2 (x) = 0, a < x < b, \tag{11}$$

with boundary condition $\lambda_F (x_m) = \alpha 2^{-(\beta+\gamma-1)}$, where $F (x_m) = \frac{1}{2}$ and

$$q (x) = \frac{(\beta + \gamma - 1) F (x) - \beta}{F (x)}, \qquad a < x < b.$$

Proof. If X has *pdf* (1) , then obviously (11) holds. If λ_F satisfies (11) , then

$$\frac{\lambda_F' (x)}{\lambda_F (x)} = \left(\frac{\beta}{F (x)} - \frac{\gamma - 1}{1 - F (x)} \right) f (x) .$$

Integrating both sides of the above equation with respect to x from x_m to x , we arrive at

$$\ln \left(\frac{\lambda_F (x)}{\lambda_F (x_m)} \right) = \ln \left(2^{\beta+\gamma-1} (F (x))^{\beta} (1 - F (x))^{\gamma} \right),$$

from which , after some computations and using the boundary condition, (1) is obtained.

Remark 6. For characterizations of other well-known continuous distributions based on the hazard function, we refer the reader to Hamedani [20] and Hamedani and Ahsanullah [22] .

5.3. Characterization based on truncated moments of certain functions of order statistics

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be n order statistics from a continuous *cdf* F.

Proposition 3. Let $X : \Omega \rightarrow (a, b)$ be a continuous random variable with *cdf* F and *pdf* f then X has pdf (1) iff

$$E \left[X_{r+1:n}^k | X_{r:n} = x \right] = x^k + \frac{k \; \Lambda (x)}{(1 - F (x))^{n-r}}, \tag{12}$$

where $k \in \mathbb{N}$ and

$\Lambda(x) = \int_x^b y^{k-1} \left[1 - \alpha \int_a^y (F(u))^\beta (1 - (F(u))^\rho)^\gamma du\right]^{n-r} dy$.

Proof. If X has pdf (1) , then

$$E(X_{r+1:n}^k \mid X_{r:n} = x) = \frac{c_{r,r+1:n}}{c_{r:n}} \left(\frac{\int_x^b y^k F(x)^{r-1} f(x) f(y) (1 - F(y))^{n-r-1} dy}{F(x)^{r-1} f(x) (1 - F(x))^{n-r}} \right)$$

$$= \frac{(n-r)}{(1 - F(x))^{n-r}} \int_x^b y^k (1 - F(y))^{n-r-1} f(y) dy.$$

Integrating by parts yields

$$E(X_{r+1:n}^k \mid X_{r:n} = x) = \frac{(n-r)}{(1 - F(x))^{n-r}} \left(\frac{-y^k (1 - F(y))^{n-r}}{n-r} \mid bx + k \int_x^b \frac{y^{k-1} (1-F(y))^{n-r}}{n-r} dy \right)$$

$$= x^k + \frac{k}{(1 - F(x))^{n-r}} \int_x^b y^{k-1} \left[1 - \alpha \int_a^y (F(t))^\beta (1 - (F(t))^\rho)^\gamma dt\right]^{n-r} dy.$$

$$= x^k + \frac{k\Lambda(x)}{(1 - F(x))^{n-r}}.$$

If (12) holds, then

Let $E(X_{r+1:n}^k \mid X_{r:n} = x) = x^k + \frac{k\Lambda(x)}{(1 - F(x))^{n-r}}$, then

$$\frac{(n-r)}{(1 - F(x))^{n-r}} \int_x^b y^k (1 - F(y))^{n-r-1} f(y) dy = x^k + \frac{k\Lambda(x)}{(1 - F(x))^{n-r}}.$$

Multiplying both sides by $(1 - F(x))^{(n-r)}$ we obtain

$$(n-r) \int_x^b y^k (1 - F(y))^{n-r-1} f(y) dy = x^k (1 - F(x))^{n-r} + k\Lambda(x).$$

Differentiating both sides with respect to x yields

$$- (n-r) x^k (1 - F(x))^{n-r-1} f(x) = k x^{k-1} (1 - F(x))^{n-r}$$

$$- (n-r) x^k (1 - F(x))^{n-r-1} f(x) - k x^{k-1}$$

$$\times \left[1 - \alpha \int_a^x (F(y))^\beta (1 - (F(y))^\rho)^\gamma dy\right]^{n-r}.$$

Simplifying, we get

$F(x) = \alpha \int_a^x (F(y))^\beta (1 - (F(y))^\rho)^\gamma dy.$

We present here characterization results similar to Proposition 3 but based on truncated moments of the order statistics. We refer the reader to Ahsanullah and Hamedani [1] , Hamedani et al. [23] and Hamedani [21] , among others , for characterizations of other well-known continuous distributions in this direction.

Proposition 4. Let $X : \Omega \to (a, b)$ be a continuous random variable with *cdf* F such that $\lim_{x\to b} x^{\delta} (1 - F(x))^{n} = 0$, for some $\delta \geq 0$. Then

$$E\left[X_{1:n}^{\delta} | X_{1:n} > t\right] = t^{\delta} + \frac{\delta \Phi(t)}{(1 - F(t))^{n}}, a < t < b, \tag{13}$$

where

$$\Phi(t) = \int_t^b x^{\delta - 1} \left[1 - \alpha \int_a^x (F(u))^{\beta} (1 - (F(u))^{\rho})^{\gamma} du\right]^n dx,$$

for some $\alpha, \rho > 0$ and $\beta, \gamma \geq 0$ if and only if X has *pdf* (1) .

Proof. If X has *pdf* (1) , then clearly (13) is satisfied. Now, if (13) holds, then using integration by parts on the left hand side of (13) , in view of the assumption $\lim_{x\to b} x^{\delta} (1 - F(x))^{n} = 0$, we have (after some simplifications)

$$\int_t^b x^{\delta - 1} (1 - F(x))^{n} dx = \Phi(t), a < t < b. \tag{14}$$

Differentiating both sides of (14) with respect to t , we arrive at

$$(1 - F(t))^{n} = \left[1 - \alpha \int_a^t (F(u))^{\beta} (1 - (F(u))^{\rho})^{\gamma} du\right]^n, a < t < b, \tag{15}$$

from which we obtain

$$F(t) = \int_a^t \alpha (F(u))^{\beta} (1 - (F(u))^{\rho})^{\gamma} du, a \leq t \leq b.$$

Proposition 5. Let $X : \Omega \to (a, b)$ be a continuous random variable with *cdf* F and let δ be a positive number. Then

$$E\left[X_{n:n}^{\delta} | X_{n:n} < t\right] = t^{\delta} - \frac{\delta \Psi(t)}{(F(t))^{n}}, a < t < b, \tag{16}$$

where

$$\Psi(t) = \int_a^t x^{\delta-1} \left[\alpha \int_a^x (F(u))^{\beta} (1 - (F(u))^{\rho})^{\gamma} du \right]^n dx,$$

for some $\alpha, \rho > 0$ and $\beta, \gamma \geq 0$ if and only if X has *pdf* (1).

Proof. Is similar to that of Proposition 4.

Let $X_j, j = 1, 2, \cdots, n$ be n *i.i.d.* (independent and identically distributed) random variables with *cdf* F and corresponding *pdf* f and let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be their corresponding order statistics. Let $X^*_{1:n-r}$ be the $1st$ order statistic from a sample of size $n - r$ of random variables with *cdf* $F_t(x) = \frac{F(x)-F(t)}{1-F(t)}, x \leq t < b$ (t is fixed) and corresponding *pdf* $f_t(x) = \frac{f(x)}{1-F(t)}, x \leq t < b$. Then

$$(X_{r+1:n} | X_{r:n} = t) \overset{d}{=} X^*_{1:n-r} \; (\overset{d}{=} \textit{meansequalin distribution}),$$

that is

$$f_{X_{r+1:n}|X_{r:n}}(x|t) = f_{X^*_{1:n-r}}(x) = (n-r)(1 - F_t(x))^{n-r-1} \frac{f(x)}{1 - F(t)}, x \leq t < b.$$

Remarks 7. (i) In view of the explanation in the above paragraph, we can obtain Proposition 3 directly via the proof of Proposition 4. (ii) Propositions 4 and 5 can be extended in a straightforward manner to include possibly other distributions by replacing the given functions of the order statistics with more general functions and of course under appropriate conditions. (iii) A more general form of (1) can be considered via

$$f(x) = \Pi_{i=1}^k H_i(x), \qquad a < x < b,$$

where

$$H_i(x) = (F(x))^{\beta_i} \; or \; (1 - (F(x))^{\rho_i})^{\gamma_i}.$$

(iv) In fact $f(x)$ can be any appropriate function of $F(x)$.

6 Conclusion

In this article,

1. Several recurrence relations satisfied by single and product moments for GS-distributions of order statistics and progressive Type-II right censoring are established.

2. Recurrence relations for moments of progressive Type-II right censoring and order statistics for some specific distributions including (Uniform (a, b), exponential (λ), generalized exponential (α, λ, ν), beta$(1, b)$, beta $(b, 1)$, logistic $(\alpha, \beta), F(2, m), F(n, 2)$) can be obtained as special cases.

3. The recurrence relations for moments of usual order statistics from GS-distributions are also obtained as special cases.

4. Some computations including means, variances and covariances for standard exponential distribution of progressive Type-II right censored order statistics are computed.

5. Some characterizations of GS-distributions based on moments of order statistics; on truncated moments; on hazard function; on truncated moments of functions of order statistics are given.

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