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Abstract

Certain characterizations of the discrete Lindley and discrete Poisson-Lindley distributions, originally introduced by Bakouch, Jazi and Nadarjah (2014) and Sankaran (1970), respectively, are presented. Al-Babtain, Gemeay and Afify (2020) revisited these distributions and provided estimation methods and actuarial measures as well as their applications in medicine. This short note is intended to complete, in some way, Al-Babtain, Gemeay and Afify (2020)'s work. It should be mentioned that the probability mass functions reported in the two papers mentioned above are not correct. In this note, it will be explained why they are not correct.

Keywords: discrete Lindley distribution, discrete Poisson-Lindey distribution, characterizations of distributions, discrete random variables

1. Introduction

To understand the behavior of the data obtained through a given process we need to be able to describe this behavior via its approximate probability law. This, however, requires to establish conditions which govern the required probability law. In other words, we need to have certain conditions under which we may be able to recover the probability law of the data. So, the problem of characterizing a distribution is an important problem in applied sciences, where an investigator is vitally interested to know if their model follows the right distribution. To this end, the investigator relies on conditions under which their model would follow specifically the chosen distribution. Al-Babtain, Gemeay and Afify (2020) revisited two discrete distributions introduced by Sankaran (1970), which is called Discrete Lindley (*DL*) distribution and by Bakouch, Jazi and Nadarjah (2014), which is called Discrete Poisson-Lindley (*DPL*) distribution. Al-Babtain, Gemeay and Afify (2020) argued that such a distribution is needed in several applied fields such as medicine, public health, epidemiology, agriculture, sociology, to name a few. Al-Babtain, Gemeay and Afify (2020) use three real-life data sets from medical science to show the superiority of the DL and DPL distributions by comparing them with some well-known discrete distriutions such as discrete Pareto and discrete Burr distributions (Krishna and Pundir, 2009), and discrete Burr-Hatke distribution (El-Morshedy, Eliwa and Altun, 2020). In this very short note, we present two characterizations of the DL and DPL distributions based on: (*i*) the conditional expectation of certain function of the random variable and (*ii*) the reverse hazard function. We will also point out the probability mass functions given for these two distributions are not correct. The correct versions will be given here.

The cumulative distribution functions (cdfs), $F(x)$, corresponding probability mass functions (pmfs), $f(x)$ and hazard rate functions, $h_F(x)$ of the DL and DPL are given, respectively, by

$$F(x;\beta) = 1 - \frac{1+\beta+\beta x}{1+\beta}\rho^{x}, \qquad x \in \mathbb{N}, \tag{1}$$

$$f(x;\beta) = \frac{(1+\beta x)(1-\rho)-\beta\rho}{1+\beta}\rho^{x-1}, \qquad x \in \mathbb{N}, \tag{2}$$

$$h_F(x) = \frac{1-\rho}{\rho} - \frac{\beta}{\rho(1+\beta+\beta x)}, \qquad x \in \mathbb{N}, \tag{3}$$

where $\beta > 0$ is a parameter and $\rho = e^{-\beta}$,

and

$$F(x;\alpha) = 1 - \frac{1+\alpha(\alpha+x+2)}{(\alpha+1)^{x+2}}, \qquad x \in \mathbb{N}, \tag{4}$$

$$f(x;\alpha) = \frac{\alpha^2(\alpha+x+1)}{(\alpha+1)^{x+2}}, \qquad x \in \mathbb{R}, \tag{5}$$

$$h_F(x) = \frac{\alpha^2(\alpha+x+1)}{1+\alpha(\alpha+x+2)}, \qquad x \in \mathbb{N}, \tag{6}$$

where $\alpha > 0$ is a parameter.

Remark 1. In defining the pmfs of the DL and DPL distributions, the original authors did not have the correct expressions of the pmfs. They assumed $x \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$. The correct versions of their pmfs are given in (2) and (5) above and for $x \in \mathbb{N}$.(Al-Babtain, Gemeay and Afify, 2020) gave the following cdfs and pmfs for the DL and DPL, which are slightly different but for $x \in \mathbb{N}^*$,

$$\begin{aligned}
F(x;\beta) &= 1 - \frac{\rho^{x+1}}{1+\beta}\left[1+\beta(2+x)\right], && x \in \mathbb{N}^*,\\
f(x;\beta) &= \frac{\rho^{x}}{1+\beta}\left[\beta(1-2\rho)+(1-\rho)(1+\beta x)\right], && x \in \mathbb{N}^*,\\
F(x;\alpha) &= 1 - \frac{1+\alpha(\alpha+x+3)}{(\alpha+1)^{x+3}}, && x \in \mathbb{N}^*,\\
f(x;\alpha) &= \frac{\alpha^2(\alpha+x+2)}{(\alpha+1)^{x+3}}, && x \in \mathbb{N}^*.
\end{aligned}$$

2. Characterization Results

We present our characterizations (i) *and* (ii) via two subsections 2.1 and 2.2.

2.1 Characterizations of the DL and DPL Distributions in Terms of the Conditional Expectation of Certain Function of the Random Variable

Proposition 2.1.1. Let $X : \Omega \to \mathbb{N}$ be a random variable and let E be the conditional expectation of a function of the random variable. The pmf of X is (2) if and only if

$$\begin{aligned}
&E\left\{\left[\left(\frac{1}{(1+\beta X)(1-\rho)-\beta\rho}\right)\right] \mid X > k\right\}\\
&= \frac{1}{(1-\rho)(1+\beta+\beta k)}, \qquad k \in \mathbb{N}.
\end{aligned} \tag{7}$$

Proof. If X has pmf (2), then the left-hand side of (7) ,will be

$$\begin{aligned}
&\frac{1}{1+\beta}(1-F(k))^{-1}\sum_{x=k+1}^{\infty}\left\{\rho^{x-1}\right\} = \frac{1}{(1+\beta+\beta k)\rho^{k}}\left(\frac{\rho^{k}}{1-\rho}\right)\\
&= \frac{1}{(1-\rho)(1+\beta+\beta k)}.
\end{aligned}$$

Conversely, if (7) holds, then

$$\begin{aligned}
&\sum_{x=k+1}^{\infty}\left\{\left(\frac{1}{(1+\beta x)(1-\rho)-\beta\rho}\right)f(x)\right\}\\
&=(1-F(k))\left(\frac{1}{(1-\rho)(1+\beta+\beta k)}\right)\\
&=[1-F(k+1)+f(k+1)]\left(\frac{1}{(1-\rho)(1+\beta+\beta k)}\right),
\end{aligned}\tag{8}$$

where we have used $F(k) = F(k+1) - f(k+1)$.

From (7), we also have

$$\begin{aligned}
&\sum_{x=k+2}^{\infty}\left\{\left(\frac{1}{(1+\beta x)(1-\rho)-\beta\rho}\right)f(x)\right\}\\
&=(1-F(k+1))\left(\frac{1}{(1-\rho)(1+\beta+\beta(k+1))}\right).
\end{aligned}\tag{9}$$

Now, subtracting (9) from (8), we arrive at

$$\begin{aligned}
&(1-F(k+1))\left[\frac{1}{(1-\rho)(1+\beta+\beta k)}-\frac{1}{(1-\rho)(1+\beta+\beta(k+1))}\right]\\
&=\left[\frac{1}{(1+\beta(k+1))(1-\rho)-\beta\rho}-\frac{1}{(1-\rho)(1+\beta+\beta k)}\right]f(k+1),
\end{aligned}$$

or

$$\begin{aligned}
&\frac{\beta}{1-\rho}(1-F(k+1))\left[\frac{1}{[(1+\beta+\beta k)][(1+\beta+\beta(k+1))]}\right]\\
&=\left[\frac{\beta\rho}{[(1+\beta(k+1))(1-\rho)-\beta\rho][(1-\rho)(1+\beta+\beta k)]}\right]f(k+1).
\end{aligned}$$

From the last equality, we have

$$\begin{aligned}
h_F(k+1)=\frac{f(k+1)}{1-F(k+1)}&=\frac{1+\beta(k+1)(1-\rho)-\beta\rho}{\rho(1+\beta+\beta(k+1))}\\
&=\frac{1}{\rho}\left[(1-\rho)-\frac{\beta}{(1+\beta+\beta(k+1))}\right],
\end{aligned}$$

which, in view of (3), implies that X has pmf (2).

Proposition 2.1.2. Let $X:\Omega\rightarrow\mathbb{N}$ be a random variable and let E be the conditional expectation of a function of the random variable. The pmf of X is (5) if and only if

$$E\left\{\left[\left(\frac{1}{\alpha+X+1}\right)\right]\mid X>k\right\}=\frac{\alpha}{1+\alpha(\alpha+k+2)},\quad k\in\mathbb{N}.\tag{10}$$

Proof. If X has pmf (5), then the left-hand side of (10) ,will be

$$\alpha^2 (1 - F(k))^{-1} \sum_{x=k+1}^{\infty} \left\{ \frac{1}{(\alpha+1)^{x+2}} \right\}$$
$$= \alpha^2 \left(\frac{(\alpha+1)^{k+2}}{1+\alpha(\alpha+k+2)} \right) \left(\frac{(\alpha+1)^{-(k+3)}}{1-(\alpha+1)^{-1}} \right) = \frac{\alpha}{1+\alpha(\alpha+k+2)}.$$

Conversely, if (10) holds, then

$$\sum_{x=k+1}^{\infty} \left\{ \left(\frac{1}{\alpha+k+1} \right) f(x) \right\} = (1 - F(k)) \left(\frac{\alpha}{1+\alpha(\alpha+k+2)} \right)$$
$$= [1 - F(k+1) + f(k+1)] \left(\frac{\alpha}{1+\alpha(\alpha+k+2)} \right). \tag{11}$$

From (10), we also have

$$\sum_{x=k+2}^{\infty} \left\{ \left(\frac{1}{\alpha+k+1} \right) f(x) \right\}$$
$$= (1 - F(k+1)) \left(\frac{\alpha}{1+\alpha(\alpha+(k+1)+2)} \right). \tag{12}$$

Now, subtracting (12) from (11), we arrive at

$$(1 - F(k+1)) \left[\frac{\alpha}{1+\alpha(\alpha+k+2)} - \frac{\alpha}{1+\alpha(\alpha+k+3)} \right]$$
$$= \left[\frac{1}{\alpha+k+2} - \frac{\alpha}{1+\alpha(\alpha+k+2)} \right] f(k+1),$$

or

$$(1 - F(k+1)) \left[\frac{\alpha}{1+\alpha(\alpha+k+3)} \right] = \left[\frac{1}{\alpha+k+2} \right] f(k+1).$$

From the last equality, we have

$$h_F(k+1) = \frac{f(k+1)}{1 - F(k+1)} = \frac{\alpha^2 (\alpha+k+2)}{1+\alpha(\alpha+k+3)},$$

which, in view of (6), implies that X has pmf (5).

2.2 Characterization of the DL and DPL Distributions Based on the Hazard Function

Proposition 2.2.1. Let $X : \Omega \to \mathbb{N}$ be a random variable. The pmf of X is (2) if and only if its hazard rate function satisfies the difference equation

$$h_F(k+1) - h_F(k) = -\frac{\beta\rho^{-1}}{(1+\beta+\beta k)(1+\beta+\beta(k+1))}, \tag{13}$$

$k \in \mathbb{N}$, with the initial condition $h_F(1) = \frac{1}{\rho}\left[(1-\rho) - \frac{\beta}{1+2\beta}\right]$.

Proof. If X has pmf (2), then clearly (13) holds. Now, if (13) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=1}^{x-1} \{h_F(k+1) - h_F(k)\}$$
$$= \frac{\beta}{\rho} \sum_{k=1}^{x-1} \left\{ \frac{1}{1+\beta+\beta k} - \frac{1}{(1+\beta+\beta(k+1))} \right\},$$

or

$$h_F(x) - h_F(1) = \frac{\beta}{\rho}\left(\frac{1}{1+\beta+\beta} - \frac{1}{1+\beta+\beta x}\right).$$

In view of the fact that $h_F(1) = \frac{1}{\rho}\left[(1-\rho) - \frac{\beta}{1+2\beta}\right]$, from the last equation we have

$$h_F(x) = \frac{1-\rho}{\rho} - \frac{\beta}{\rho(1+\beta+\beta x)},$$

which, in view of (3), implies that X has pmf (2).

Proposition 2.2.2. Let $X : \Omega \to \mathbb{N}$ be a random variable. The pmf of X is (5) if and only if its hazard rate function satisfies the difference equation

$$h_F(k+1) - h_F(k) = -\frac{\alpha^3}{[1+\alpha(\alpha+k+3)][1+\alpha(\alpha+k+2)]}, \tag{14}$$

$k \in \mathbb{N}$, with the initial condition $h_F(1) = \frac{\alpha^2(\alpha+2)}{1+\alpha(\alpha+3)}$.

*I*f X has pmf (5), then clearly (14) holds. Now, if (14) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=1}^{x-1} \{h_F(k+1) - h_F(k)\}$$
$$= \sum_{k=1}^{x-1} \left\{ \frac{\alpha^2(\alpha+k+2)}{1+\alpha(\alpha+k+3)} - \frac{\alpha^2(\alpha+k+1)}{1+\alpha(\alpha+k+2)} \right\},$$

or

$$h_F(x) - h_F(1) = \frac{\alpha^2(\alpha+x+1)}{1+\alpha(\alpha+x+2)} - \frac{\alpha^2(\alpha+2)}{1+\alpha(\alpha+3)}.$$

In view of the fact that $h_F(1) = \frac{\alpha^2(\alpha+2)}{1+\alpha(\alpha+3)}$, from the last equation we have

$$h_F(x) = \frac{\alpha^2(\alpha+x+1)}{1+\alpha(\alpha+x+2)},$$

which, in view of (6), implies that X has pmf (5).

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