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END MASS EFFECTS ON THE FREQUENCY RESPONSE OF CANTILEVERS: ANALYTICAL RESULTS

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INTRODUCTION

Dynamic-mode cantilever-based structures supporting end masses are frequently used as MEMS/ NEMS devices in application areas as diverse as chemical/ biosensing, atomic force microscopy, and energy harvesting [1-3]. However, due to the wide variety of end mass geometries, dimensions, and material composition, the vast majority of theoretical work targeted at understanding the behavior of such structures is performed via numerical (e.g., finite element) analysis on a case-by-case basis. Thus, any simple relationships existing between the frequency response and the end mass characteristics, as governed by the underlying mechanics, may be hidden. This serves as the motivation for the present study in which a simple analytical formula is derived for replacing an arbitrary end mass with an "effective point mass" at the beam tip which incorporates the effects of rotational inertia and eccentricity of the end mass in addition to its translational mass. The utility of the result lies not only in its generality but also in that it may permit one to convert known dynamic solutions for a cantilever with a point mass (e.g., [4]) into approximate solutions applicable to more realistic end masses.

PROBLEM STATEMENT

The problem of interest is depicted in Fig. 1. Our objective is to replace the end mass on the cantilever with an effective point mass M_{eff} at the beam tip in such a way that the rotational inertia J and the eccentricity e are accounted for in addition to the translational inertia (M). In doing so, we assume that (a) the beam is elastic, of uniform cross section, and monolithic with the end mass; (b) the end mass is rigid; and (c) only horizontal eccentricity (Fig. 1) is considered. We also restrict our attention to the first flexural mode, whose shape we assume is dominated by the inertial force at the beam tip, i.e., the vibrational shape is approximated by the static shape due to an end force. In addition to the dimensions defined in Fig. 1, the following symbols are employed: I=second moment of area of beam cross section; $\bar{m}$=mass per unit length of beam; E=Young's modulus of the beam material; J=mass moment of inertia of the end mass about the axis through its center of mass G (for rotation in the plane of Fig. 1). The dynamic deflection is denoted by $w(\xi,t)$, where $\xi=x/L$ and t represents time.

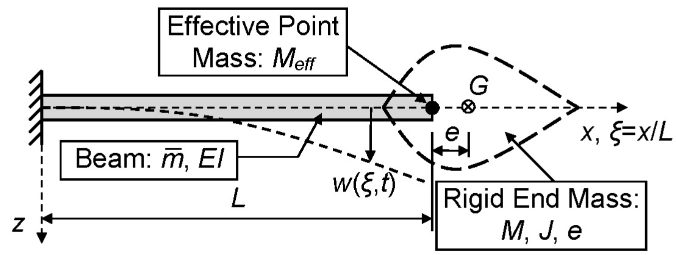

Fig. 1. Replacement of rigid end mass on a cantilever with an effective point mass M_{eff} at the beam tip.

DERIVATION OF EFFECTIVE POINT MASS

Assuming that no external loads act on the end mass, an equilibrium analysis of the end mass results in the following boundary conditions (BCs) at the end of the beam for the cases of the original end mass (1a,b) and its effective point mass counterpart (2a,b):

$$w''(1,t) + \frac{JL}{EI}\ddot{w}'(1,t) + \frac{ML^2 e}{EI}\left[\ddot{w}(1,t) + \frac{e}{L}\ddot{w}'(1,t)\right] = 0 \ , \quad (1a)$$

$$w'''(1,t) - \frac{ML^3}{EI}\left[\ddot{w}(1,t) + \frac{e}{L}\ddot{w}'(1,t)\right] = 0 \ , \quad (1b)$$

$$w''(1,t) = 0 \ , \qquad w'''(1,t) - \frac{M_{eff}L^3}{EI}\ddot{w}(1,t) = 0 \ , \quad (2a,b)$$

where primes and dots denote differentiation with respect to ξ and t, respectively. These two sets of BCs may be interpreted as two sets of force and moment end loads on the beam. By requiring that the work done by the end loads of (2a,b) is equal to that done by those of (1a,b), the effective point mass becomes

$$M_{eff} = M\left[\left(1+3\bar{e}\right) + \frac{9}{4}\left(\bar{e}^2 + \bar{J}\right)\right] \ , \quad (3)$$

in which $\bar{e} \equiv e/L$ and $\bar{J} \equiv J/ML^2$ are the normalized eccentricity and mass moment of inertia of the original end mass, respectively. Equation (3) is general in the sense that no specific end mass geometry has been assumed.

SPECIAL CASE: T-SHAPED CANTILEVERS

In many cases of practical interest the device is fabricated with a rectilinear geometry such as that of a T-beam (Fig. 2). In this case (3) reduces to

$$M_{eff} = M\left[1 + \frac{3}{2}\left(\frac{L_0}{L}\right) + \frac{3}{4}\left(\frac{L_0}{L}\right)^2 + \frac{3}{16}\left(\frac{h_0}{L}\right)^2\right] \ , \quad (4)$$

in which L_0 and h_0 are the length and thickness of the head. This expression may be used in place of M in

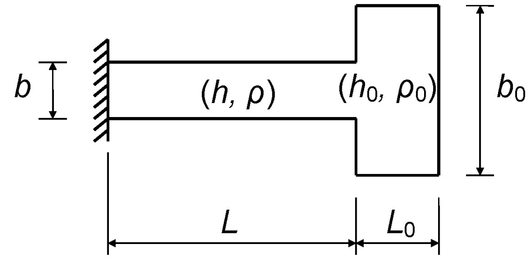

Fig. 2. Geometry (plan view) and notation for a T-shaped cantilever. (Thicknesses and mass densities in parentheses.)

existing solutions for a cantilever having a concentrated tip mass M, thereby accounting for the effects of eccentricity and rotational inertia of the head mass in an approximate manner.

ILLUSTRATIVE EXAMPLE

Here we use the result (4) to estimate the fundamental frequency f of a T-beam having stem parameters (h, b, L) = (10, 300, 600) μm, ρ=1000 kg/m^3, E=4 GPa; and head characteristics (h_0, b_0, L_0) = (50, 1000, 500) μm, ρ_0=4000 kg/m^3, E_0=40 GPa. A benchmark value of f=350.7 Hz was obtained from a finite element analysis using a mesh of higher-order 3D brick elements and assuming Poisson's ratio values of ¼ for the stem and head. The point mass solution (e.g., [4]) with I=bh^3/12 (and M=100 μg for the head mass) gives

$$f = \frac{1}{2\pi}\sqrt{\frac{3EI}{\bar{m}L^4\left(\frac{M}{\bar{m}L} + \frac{33}{140}\right)}} = 591.9\ Hz \ , \quad (5)$$

which, as expected, is a very poor estimate due to the large head size. However, if we replace M in (5) with M_{eff} = 2.772M = 277.2 μg as given by (4) to account for eccentricity and rotational inertia of the head, we achieve an excellent estimate of f = 356.0 Hz, which is only 1.5% larger than the 3D FEA result.

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