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# Geometric Construction-Based Realization of Spatial Elastic Behaviors in Parallel and Serial Manipulators

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## Abstract:

This paper addresses the realization of spatial elastic behavior with a parallel or a serial manipulator. Necessary and sufficient conditions for a manipulator (either parallel or serial) to realize a specific elastic behavior are presented and interpreted in terms of the manipulator geometry. These conditions completely decouple the requirements on component elastic properties from the requirements on mechanism kinematics. New construction-based synthesis procedures for spatial elastic behaviors are developed. With these synthesis procedures, one can select each elastic component of a parallel (or serial) mechanism based on the geometry of a restricted space of allowable candidates. With each elastic component selected, the space of allowable candidates is further restricted. For each stage of the selection process, the geometry of the remaining allowable space is described.

# SECTION I.

## Introduction

Spatial compliant behavior is an important topic in robotics, particularly for collaborative robots. General compliant behavior can be modeled as a rigid body suspended by elastic components. If only small displacements from equilibrium are considered, the deflection can be represented by a 6-vector twist $\mathbf{t}$ and the force can be represented by a 6-vector wrench $\mathbf{w}$. The elastic behavior can then be characterized by a $6 \times 6$ symmetric positive semidefinite (PSD) matrix, the stiffness matrix $\mathbf{K}$, which maps $\mathbf{t}$ to $\mathbf{w}$. If the suspension is fully elastic, the behavior can be equivalently represented by the compliance matrix $\mathbf{C}$, the inverse of $\mathbf{K}$.

A common means to realize a desired compliance is to use a parallel or serial mechanism consisting of many elastic components. Realization of a specified elastic behavior involves the identification of the geometric structure of the mechanism and the spring rate of each component. When used in robotic tasks, the size and configuration of the manipulator may be constrained, and therefore, must be considered in the realization of a desired compliance.

In some manipulators, variable stiffness actuators (VSAs) [1] are used to provide a time-varying compliant behavior. Since VSAs allow joint compliance to be changed in real time, the use of VSAs significantly enlarges the space of realizable compliant behaviors. Varying the joint stiffness values alone, however, may not be adequate to achieve a desired behavior. The identification of the mechanism geometry required to realize a given compliance (as well as the joint compliances) is the primary motivation for this work.

This work is also motivated by the desire to design spatial manipulators that compactly realize specified elastic behavior. The manipulators considered in this paper are either parallel or serial mechanisms having simple connection topology. Understanding the abilities and limitations of these manipulators to realize a general spatial behavior provides a foundation for design of manipulators having more complicated topologies (i.e., hybrid mechanisms composed of parallel and serial elastic components).

In this paper, elastic behavior realization using a geometric construction-based approach is addressed. A synthesis procedure in which each elastic component is selected based on its geometric characteristics is developed. In each step of the synthesis procedure for a parallel manipulator, a spring is selected and a corresponding reduction in the space of available wrenches for subsequent spring selection is identified. Each spring can be selected from the available space by selecting its pitch, direction, and/or location. With the selection of these geometric parameters, the spring rate is uniquely determined. An equivalent dual procedure is provided for serial manipulators.

### A. Related Work

Ball [2] introduced the theory of screws and used it to describe the motion of a rigid body in a general spatial potential field. Dimentberg [3] used screw theory to analyze the static and dynamic behavior of an elastically suspended body. More recently, Griffis and Duffy [4] and Patterson and Lipkin [5], [6] have also studied spatial compliance behavior using screw theory. Chen and Kao [7]derived the conservation congruence transformation for stiffness mapping between joint and Cartesian spaces for finite deflection from equilibrium.

In elastic behavior realization work, the bounds of elastic behaviors achieved with *simple elastic mechanisms* (i.e., simply connected parallel and serial mechanisms without helical joints) have been identified [8], [9]. It was shown [8] that a stiffness matrix can be realized with simple springs connected in parallel if and only if the trace condition $\mathrm{tr}(\mathbf{K\Delta}) = 0$ is satisfied, where $\mathbf{\Delta}$ is the $6 \times 6$ matrix that exchanges the first three columns with the last three columns of a matrix. A similar restriction, $\mathrm{tr}(\mathbf{C\Delta}) = 0$, holds for compliance matrices achieved with simple elastic serial mechanisms [9]. Synthesis procedures for spatial stiffnesses and compliances satisfying the appropriate "simple elastic mechanism" restrictions have been identified [8], [9] and refined [10], [11]. Each of these procedures [8]–[11] are based on a single-step matrix decomposition that does not consider implications on mechanism geometry.

The synthesis of *general* spatial elastic behavior has also been addressed. It was shown that any stiffness matrix (including those that do not satisfy the trace condition) can be synthesized with a parallel mechanism if screw springs [12]are included and that any compliance matrix can be synthesized with a serial mechanism if screw joints [13] are included. The minimum number of screw springs (joints) required for realization of a stiffness (compliance) is defined as the "degree of translational–rotational coupling" of the stiffness (compliance) matrix [14]. The synthesis of a compliance with the minimum number of screw springs was presented in [15].

Choi *et al.* [16] investigated the synthesis of spatial stiffnesses including some geometric considerations. The investigation was restricted to those stiffness matrices satisfying the trace condition $\mathrm{tr}(\mathbf{K\Delta}) = 0$ using only line spring elastic components. It was shown that a stiffness matrix of this type can be realized with six line springs, among which three intersects at an arbitrary point and the remaining three lie on a quadric surface. Hong and Choi [17] also considered geometry in the realization of stiffness matrices satisfying the trace condition using simple line and torsional springs. It was shown that in the realization of a stiffness with rank $n \geq 4$, a line spring or torsional spring having an arbitrary axis can be selected. The approach, however, does not allow every spring to be selected based on its geometry. In each of these approaches [16], [17], although geometric parameters are considered, the geometry of the allowable space for each spring selection was not described.

Our previous work has also addressed geometrical considerations in compliance realizations. In [18], realization of compliance with a compact parallel (or serial) mechanism that has concurrent spring axes (or concurrent joint axes) was presented; and in [19], synthesis of a compliance that has one or more compliant-axes using compact mechanisms was also addressed.

In recent work, compliance realization conditions for a serial mechanism having specified geometry were identified and interpreted in terms of geometric relationships among the joints for point compliance in $E(2)$ [20], [21].

In the most closely related work [22], construction-based synthesis procedures for *planar* elastic behaviors using parallel and serial mechanisms were developed. Using these procedures, one can select the geometry of each elastic component from a restricted space of geometrically allowable candidates. As a result, a physical appreciation of the elastic behavior realized with parallel and serial mechanisms is provided. However, due to the significant difference between *planar* and *spatial* compliances, the approach used in [22] cannot be directly applied to the spatial case. New theory and methods are needed to address the realization of spatial elastic behaviors. The geometry of the space of allowable elastic behaviors is much more complicated for the spatial case. Planar screws have either zero or infinite pitch and, in general, intersect or are parallel; whereas, spatial screws have arbitrary pitch and, in general, do not intersect and are not parallel.

## B. Technical Background

It is known that any rank-$m$ $6 \times 6$ PSD matrix $\mathbf{K}$ can be decomposed into a sum of m rank-1 PSD matrices, i.e.,

$$\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2 + \cdots + \mathbf{K}_m \qquad (1)$$

where each rank-1 component $\mathbf{K}_i$ is symmetric and has the form

$$\mathbf{K}_i = k_i \mathbf{w}_i \mathbf{w}_i^T \qquad (2)$$

where $k_i > 0$ is a constant and $\mathbf{w}_i \in \mathbb{R}^6$ is a unit wrench defined as the *spring wrench* [8]. Each rank-1 PSD stiffness $\mathbf{K}_i$ can be uniquely realized with a simple spring [8] or a screw spring [12] (depending on the pitch of the spring wrench) having a line of action along $\mathbf{w}_i$ and spring constant $k_i$.

If $\mathbf{W}$ is denoted as the $6 \times m$ matrix composed of the $m$ unit spring wrenches: $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m] \in \mathbb{R}^{6\times m}$, then decomposition (1) can be equivalently expressed as follows:

$$\mathbf{K} = \mathbf{W}\mathbf{K}_d\mathbf{W}^T \qquad (3)$$

where $\mathbf{K}_d = \text{diag}(k_1, k_2, \ldots, k_m)$ and each $k_i$ is the spring constant of spring $\mathbf{w}_i$.

Thus, if a stiffness is decomposed into the form of (1) or (3), the elastic behavior is realized with a set of springs connected in parallel. For a stiffness matrix $\mathbf{K}$ with $\text{rank}(\mathbf{K}) \geq 2$, the rank-1 decomposition is not unique. There are infinitely many sets of springs that can realize a given elastic behavior.

By duality [13], a decomposition of a compliance $\mathbf{C}$ (the inverse of stiffness matrix $\mathbf{K}$) yields a set of compliant joint twists associated with a serial mechanism. Thus, using a similar process, a compliance matrix $\mathbf{C}$ can be realized with a serial mechanism in which each joint twist provides a rank-1 PSD component.

There are different ways to decompose a stiffness or compliance matrix into rank-1 components, including Cholosky decomposition [8], eigenvalue decomposition [12], eigenscrew decomposition [23], and eigen-compliant-axis decomposition [24]. These decompositions do not consider the physical layout or the geometry of the mechanism that realizes the desired elastic behavior. As stated previously, very little work has considered the geometry of the elastic mechanism.

Important remaining compliance realization needs addressed in the paper include:
1. means for assessing whether an elastic behavior can be realized at a given mechanism configuration;
2. procedures to describe the entire space of spring wrenches (or joint twists) that will help in realizing a given elastic behavior when some spring wrenches (or joint twists) have already been selected;
3. synthesis procedures for elastic behavior realization using a construction-based approach in which each elastic component is selected based on its geometry.

## C. Overview

This paper addresses the realization of elastic behaviors with parallel and serial mechanisms. Geometric construction-based synthesis procedures to realize a general stiffness matrix are developed. The paper is outlined as follows. Section II presents necessary and sufficient conditions for an elastic behavior to be realized with a parallel mechanism or a serial mechanism. The physical significance of these conditions is identified. In Section III, a new elastic parallel mechanism synthesis procedure based on the realization conditions is developed. The restrictions on the wrench space for each sequential spring selection are identified. In Section IV, geometric properties of the space of acceptable spring wrench locations are presented. For each space dimension, the acceptable subspaces of spring wrenches are described using the screw pitch, direction, or point along the screw axis. Observations related to the pitch and geometry of the screws selected in the synthesis procedure for parallel and serial manipulators are presented in Section V. In Section VI, a numerical example is provided to illustrate the synthesis procedure. A brief summary is presented in Section VII.

# SECTION II.

## Realization Conditions for Elastic Behaviors

In this section, the realizability of an elastic behavior with a given parallel or serial mechanism is addressed. Necessary and sufficient conditions for a parallel mechanism (or serial mechanism) to realize a specified elastic behavior are identified. A physical appreciation of these conditions is then presented.

### A. Realization Conditions

We consider a parallel mechanism having six springs, each with an associated spring wrench $\mathbf{w}_i$. In order to realize a full-rank stiffness, the six 6-vectors $\mathbf{w}_i$s must be linearly independent, and the spring wrench matrix

$$\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_6] \in \mathbb{R}^{6\times 6}$$

must be full rank. This feature can be used to readily assess whether a given positive definite (PD) stiffness $\mathbf{K}$ can be realized with the mechanism described by $\mathbf{W}$.

By (3), a stiffness $\mathbf{K}$ can be realized with the mechanism, if and only if $\mathbf{K}$ can be expressed as follows:

$$\mathbf{K} = \mathbf{W}\mathbf{K}_d\mathbf{W}^T \qquad (4)$$

where $\mathbf{K}_d = \mathrm{diag}(k_1, k_2, \dots, k_6)$ is the joint-space stiffness matrix (a diagonal matrix with positive entries). Since $\mathbf{W}$ is full-rank, (4) can be equivalently expressed as follows:

$$\mathbf{W}^{-1}\mathbf{K}\mathbf{W}^{-T} = \mathbf{K}_d. \qquad (5)$$

The specified behavior can be realized with the mechanism described by $\mathbf{W}$ if the diagonal elements of $\mathbf{K}_d$ are positive. Since (5) involves the inverse of $\mathbf{W}$, the physical significance of the equation is not evident. Taking the inverse of both sides of (5) yields,

$$\mathbf{W}^T\mathbf{C}\mathbf{W} = \mathbf{K}_d^{-1} \qquad (6)$$

where $\mathbf{C} = \mathbf{K}^{-1}$ is the compliance matrix. Because the relationship is now expressed in terms of $\mathbf{W}$, the result can be physically interpreted. Equation (6)can be equivalently expressed as follows:

$$\begin{aligned} \mathbf{w}_i^T\mathbf{C}\mathbf{w}_j &= 0 \forall i \neq j, \\ \mathbf{w}_i^T\mathbf{C}\mathbf{w}_i &= k_i^{-1} > 0, i = 1, \dots, 6. \end{aligned} \qquad (7)(8)$$

Since $\mathbf{C} = \mathbf{K}^{-1}$ is also a PD matrix, $\mathbf{w}_i^T\mathbf{C}\mathbf{w}_i$ in (8) is always positive for any 6-vector $\mathbf{w}_i$. Therefore, condition (7) is a necessary and sufficient condition for the mechanism to realize the behavior. Thus, we have the following proposition.

### Proposition 1:

An elastic behavior $\mathbf{K}$ ($\mathbf{C}$) can be realized with a *parallel* mechanism of six springs $\mathbf{w}_i$ if and only if:

$$\mathbf{w}_i^T\mathbf{C}\mathbf{w}_j = 0 \forall i \neq j. \qquad (9)$$

The spring constant associated with $\mathbf{w}_i$ is determined using:

$$k_i = \frac{1}{\mathbf{w}_i^T \mathbf{C} \mathbf{w}_i}, i = 1,2, \ldots, 6. \qquad (10)$$

Note the result obtained for stiffness matrix $\mathbf{K}$ for a parallel mechanism applies to its elastic dual involving the compliance matrix $\mathbf{C}$ for a serial mechanism. One can simply replace the stiffness matrix $\mathbf{K}$ with the compliance matrix $\mathbf{C}$ and replace the spring wrenches $\mathbf{w}_i$ with the joint twists $\mathbf{t}_i$ to assess the realizability of a given elastic behavior with a serial mechanism at a given configuration. Thus we have the following proposition.

### Proposition 2:

An elastic behavior $\mathbf{C}$ ($\mathbf{K}$) can be realized with a *serial* mechanism of six joint twists $\mathbf{t}_i$ if and only if:

$$\mathbf{t}_i^T \mathbf{K} \mathbf{t}_j = 0 \forall i \neq j. \qquad (11)$$

The joint compliance associated with $\mathbf{t}_i$ is determined using:

$$c_i = \frac{1}{\mathbf{t}_i^T \mathbf{K} \mathbf{t}_i}, i = 1,2, \ldots, 6. \qquad (12)$$

These propositions are identical to the necessary and sufficient conditions for realizing specified elastic behavior in planar systems [22]. The proof presented here, however, is more concise and generalized to the spatial case.

### B. Physical Significance of the Realization Conditions

In the realization of a full-rank stiffness $\mathbf{K}$ with a mechanism having six elastic components, if we denote:

$$\mathbf{t}_i = \mathbf{C} \mathbf{w}_i, i = 1,2, \ldots, 6 \qquad (13)$$

then, $\mathbf{t}_i$ is the twist caused by wrench $\mathbf{w}_i$ acting on the body.

Since wrenches $\mathbf{w}_i$ and twists $\mathbf{t}_i$ are expressed in Plücker's ray and axis coordinates, respectively, condition (9) indicates that twist $\mathbf{t}_i$ must be reciprocal to all the other spring wrenches in the mechanism. Thus, physically the twist caused by wrench $\mathbf{w}_i$ performs no work along any of the mechanism's other spring wrenches. For an elastic behavior, every pair of spring wrenches $\mathbf{w}_i$s must satisfy the conditions of Ball's *conjugate screws of inertia* [2] to realize the behavior. This property for parallel mechanisms is illustrated in Fig. 1(a).

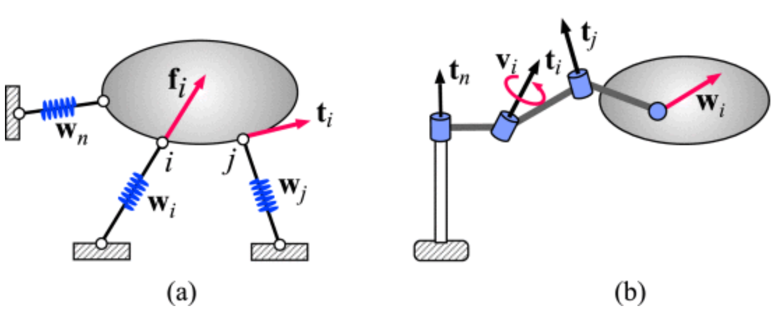


**Fig. 1.** Physical significance of the realization conditions. (a) For a parallel mechanism, a wrench $\mathbf{f}_i$ along the spring wrench $\mathbf{w}_i$ yields a twist $\mathbf{t}_i$ that is reciprocal to all other spring wrenches $\mathbf{w}_j$ . (b) For a serial mechanism, a motion twist $v_i$ along the joint twist $\mathbf{t}_i$ yields a wrench $\mathbf{w}_i$ that is reciprocal to all other joint twists $\mathbf{t}_j$.

By duality, for an elastic behavior $\mathbf{C}$ realized with a serial mechanism with joint twists $\mathbf{t}_i$ ($i = 1, \dots, 6$), if we denote:

$$\mathbf{w}_i = \mathbf{K}\mathbf{t}_i, i = 1, 2, \dots, 6 \qquad (14)$$

then, $\mathbf{w}_i$ is the wrench caused by twist $\mathbf{t}_i$ being imposed on the body. Condition (11) indicates that a motion $\mathbf{v}_i$ along the $i$th joint twist $\mathbf{t}_i$ generates a wrench reciprocal to all other joint twists $\mathbf{t}_j$s. Thus, physically the wrench caused by $\mathbf{v}_i$ performs no work along any of the mechanism's other joint twists. For an elastic behavior, every pair of joint twists must satisfy the conditions of Ball's *conjugate screws of potential* [2] to realize the behavior. This property for serial mechanisms is illustrated in Fig. 1(b).

The reciprocal conditions (9) and (11) for parallel and serial mechanisms can be used to determine whether a given elastic behavior can be realized based on an evaluation of the mechanism kinematics alone.
If (9) and (11) are satisfied, the realization of the specified behavior is ensured if the non-negative spring coefficients of (10) or joint compliances of (12) can be physically attained.

### C. Dual Elastic Mechanisms

The set of elastic behaviors that can be realized with a parallel mechanism can provide understanding into a corresponding set of elastic behaviors that can be realized with a serial mechanism. The relationship between these two types of mechanisms is identified below.

If a wrench $\mathbf{w}$ is expressed in Plücker's ray coordinates and a twist $\mathbf{t}$ is expressed in Plücker's axis coordinates, then $\mathbf{w}$ and $\mathbf{t}$ are reciprocal [2] if and only if

$$\mathbf{w}^T\mathbf{t} = 0.$$

The reciprocal relationship between a wrench $\mathbf{w}$ and a twist $\mathbf{t}$ can be simply expressed as $\mathbf{w} \perp \mathbf{t}$.

For a parallel mechanism with six spring wrenches $\mathbf{w}_i$ (($i = 1, \dots, 6$)), we consider the six *unit* twists defined by:

$$\begin{array}{ccc} \mathbf{t}_1 & \perp & \text{span}\{\mathbf{w}_2, \mathbf{w}_3, \dots, \mathbf{w}_6\}, \\ \mathbf{t}_2 & \perp & \text{span}\{\mathbf{w}_1, \mathbf{w}_3, \dots, \mathbf{w}_6\}, \\ \vdots & & \vdots \\ \mathbf{t}_6 & \perp & \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_5\} \end{array}$$

i.e., $\mathbf{t}_i$ is reciprocal to all wrenches $\mathbf{w}_j$s except for $\mathbf{w}_i$. Since $\mathbf{t}_i$ is a unit twist and is reciprocal to five wrenches, each $\mathbf{t}_i$ is uniquely determined by the five wrenches. Thus, the set of six twists $\mathbf{t}_i$s is uniquely determined by the set of six wrenches $\mathbf{w}_i$s. Since a nonzero twist cannot be reciprocal to six independent wrenches, $\mathbf{t}_i^T\mathbf{w}_i \neq 0$ for $i = 1, 2, \dots, 6$. In summary, the six wrenches and six twists must satisfy:

$$\mathbf{t}_i^T\mathbf{w}_j \begin{cases} = 0, & \text{if } i \neq j, \\ \neq 0, & \text{if } i = j. \end{cases} \qquad (15)$$

This condition defines an interesting relationship between parallel and serial elastic mechanisms.

Let $\mathbf{W}$ be the spring wrench matrix for a parallel mechanism and let $\mathbf{T}$ be the joint twist matrix of a corresponding serial mechanism, i.e.,

$$\mathbf{W} = [\mathbf{w}_1, \dots, \mathbf{w}_6], \mathbf{T} = [\mathbf{t}_1, \dots, \mathbf{t}_6] \in \mathbb{R}^{6\times 6}.$$

We consider an elastic behavior described by $\mathbf{K}$ or its inverse $\mathbf{C}$. It can be proved that, if a set of six wrenches $\{\mathbf{w}_i\}$ and a set of six twists $\{\mathbf{t}_i\}$ satisfy (15), then

$$\mathbf{w}_i^T \mathbf{C} \mathbf{w}_j = 0 \Leftrightarrow \mathbf{t}_i^T \mathbf{K} \mathbf{t}_j = 0 \forall i \neq j \qquad (16)$$

and the space of elastic behaviors that can be realized by the mechanisms described by $\mathbf{W}$ and $\mathbf{T}$ are identical. For a parallel mechanism with six spring wrenches $\mathbf{w}_i$ and a serial mechanism with six joint twists $\mathbf{t}_i$, if every pair of $\mathbf{w}_i$ and $\mathbf{t}_i$ satisfies (15), then the two mechanisms are called *dual elastic mechanisms*. It can be seen that for any parallel mechanism with six independent spring wrenches, its dual serial mechanism is unique, and vice versa. Thus, an elastic behavior can be realized with the parallel mechanism if and only if it can be realized with its dual serial mechanism. Also, it can be proved that, if an elastic behavior is realized with a pair of dual mechanisms, the spring constants $k_i$ in the parallel mechanism and the joint compliances $c_i$ in the serial mechanism have the relationship:

$$k_i c_i = \frac{1}{(\mathbf{t}_i^T \mathbf{w}_i)^2}, i = 1,2, \dots, 6. \qquad (17)$$

The significance of dual elastic mechanisms can be interpreted in the geometry of the space of elastic behaviors. It is known that the space of all $6 \times 6$ symmetric PSD matrices is a 21-dimensional (21-D) cone. For a parallel mechanism at a given configuration ($\mathbf{w}_i, i = 1, \dots, 6$), the realizable space of elastic behaviors is a 6-D subcone inside the PSD cone of stiffness. The rank-1 matrices associated with six springs, $\mathbf{w}_i \mathbf{w}_i^T$, are the six edges of this subcone. If two stiffness matrices are in the subcone, then any *positive* combination of the two matrices is also in the subcone. In the 21-D cone of compliance, the realizable space for the same parallel mechanism configuration has a different representation. Condition (9) indicates that each of the 15 pairs of $\mathbf{w}_i$ and $\mathbf{w}_j$ defines a hyper-plane. The space of realizable compliances with the mechanism is the intersection of the 15 hyper-planes in the 21-D PSD cone of compliance. Thus, the realizable space of compliance matrices is also a 6-D subcone in the 21-D PSD cone of compliance. The six edges of this cone are associated with the six joint twists of the dual serial mechanism, $\mathbf{t}_i \mathbf{t}_i^T$. These concepts for a much lower dimensional space are illustrated in Fig. 2.

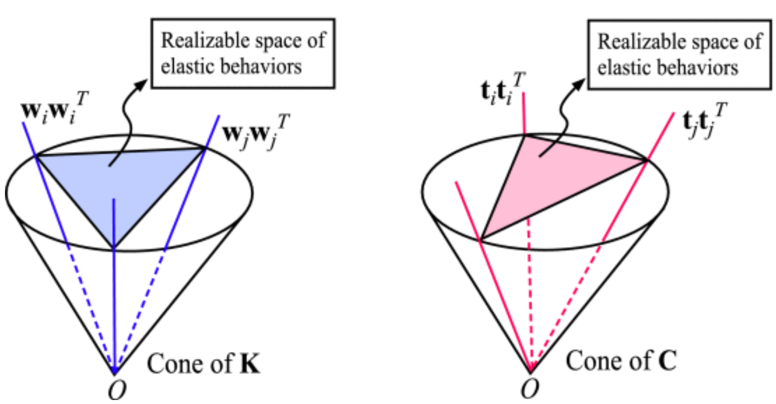


**Fig. 2.** Realizable space of elastic behaviors with a parallel mechanism is a cone having six edges $\mathbf{w}_i \mathbf{w}_i^T$ in the stiffness space. The space is also a cone having six edges in the compliance space. The spaces of realizable elastic behaviors for the two dual elastic mechanisms are identical.

## SECTION III.

## Spatial Stiffness Realization

In this section, a construction-based approach for realizing spatial elastic behavior is addressed. For a parallel mechanism, the realization of a compliant behavior involves finding a set of six spring wrenches $\mathbf{w}_i$s that satisfy the reciprocal conditions (9). Presented first below are some mathematical preliminaries useful in describing the subspace of elastic behaviors from which an elastic component helpful in realizing a specified compliance can be selected. A mathematical description of each subspace is presented. Then, a procedure for the synthesis of a parallel elastic mechanism is outlined. This construction-based approach allows one to choose each spring wrench from an allowable subspace based on its kinematics.

### A. Mathematical Preliminaries

For a given full-rank stiffness matrix, an infinite number of sets of six spring wrenches $\{\mathbf{w}_i\}$ capable of realizing the behavior can be identified using the reciprocal conditions (9). Below, for a full rank stiffness, properties of a set of spring wrenches $\mathbf{w}_i$ that satisfy the conditions are investigated. These properties are then used to realize an arbitrary stiffness.

We consider a set of $n$ screws $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ satisfying condition (9). Since the 6-D vectors are orthogonal about $\mathbf{C}$ in $\mathbb{R}^6$, the set of wrenches $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ must be linearly independent. Thus, the number of wrenches in the set, $n \le 6$.

For a pair of wrenches $\mathbf{w}_i$ and $\mathbf{w}_j$ in the set, since $\mathbf{w}_i^T\mathbf{C}\mathbf{w}_j = 0$, $\mathbf{w}_j$ must be in the null space of $\mathbf{w}_i^T\mathbf{C}$, or vice versa. This is expressed as follows:

$$\mathbf{w}_j \in \mathcal{N}(\mathbf{w}_i^T\mathbf{C})\text{or}\mathbf{w}_i \in \mathcal{N}(\mathbf{w}_j^T\mathbf{C}).$$

If a set $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ $(n \le 5)$ is determined, then a vector, $\mathbf{w}_{n+1}$, that is orthogonal to all $\mathbf{C}\mathbf{w}_i$ $(i \le n)$ can be found in the null space of the matrix $\mathbf{W}_n^T\mathbf{C}$:

$$\mathbf{w}_{n+1} \in \mathcal{N}(\mathbf{W}_n^T\mathbf{C}) \quad (18)$$

where $\mathbf{W}_n = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n] \in \mathbb{R}^{6\times n}$ describes the set of wrenches already selected. Thus, in realizing a given elastic behavior, if the first $n$ spring wrenches $\mathbf{w}_i$ satisfying (9) are determined, the $(n+1)$th spring wrench must be chosen from the linear space $\mathcal{N}(\mathbf{W}_n^T\mathbf{C})$.

For a given C and $\mathbf{W}_n$, the calculation of the space $\mathcal{N}(\mathbf{W}_n^T\mathbf{C})$ is messy. Below, it is shown that the null space of $\mathbf{W}_n^T\mathbf{C}$ is equal to the column space of a matrix involving $\mathbf{K}$, $\mathbf{C}$, and $\mathbf{W}_n$. Thus, the calculation of the null space is not necessary.

To show this, consider the symmetric matrix $\mathbf{K}_{n+1}$ defined as follows:

$$\mathbf{K}_{n+1} = (\mathbf{K} - \mathbf{W}_n\mathbf{H}_n\mathbf{W}_n^T) \in \mathbb{R}^{6\times 6} \qquad (19)$$

where $\mathbf{W}_n \in \mathbb{R}^{6\times n}$ has column vectors satisfying condition (9), and $\mathbf{H}_n \in \mathbb{R}^{n\times n}$ is the diagonal matrix:

$$\mathbf{H} = \text{diag}\left(\frac{1}{\mathbf{w}_1^T\mathbf{C}\mathbf{w}_1}, \dots, \frac{1}{\mathbf{w}_n^T\mathbf{C}\mathbf{w}_n}\right).$$

Below, it is proven that

$$\mathcal{N}(\mathbf{W}_n^T\mathbf{C}) = \mathcal{R}(\mathbf{K}_{n+1}). \qquad (20)$$

Since the columns of $\mathbf{W}_n$ satisfy $\mathbf{w}_i^T\mathbf{C}\mathbf{w}_j = 0$ for $i \neq j$

$$\mathbf{W}_n^T\mathbf{C}\mathbf{W}_n = \mathrm{diag}(\mathbf{w}_1^T\mathbf{C}\mathbf{w}_1, \dots, \mathbf{w}_n^T\mathbf{C}\mathbf{w}_n) = \mathbf{H}^{-1}$$

thus,

$$\begin{aligned}(\mathbf{W}_n^T\mathbf{C})\mathbf{K}_{n+1} &= \mathbf{W}_n^T\mathbf{C}(\mathbf{K} - \mathbf{W}_n\mathbf{H}\mathbf{W}_n^T) \\ &= \mathbf{W}_n^T - (\mathbf{W}_n^T\mathbf{C}\mathbf{W}_n)\mathbf{H}\mathbf{W}_n^T \\ &= \mathbf{W}_n^T - \mathbf{W}_n^T = 0\end{aligned}$$

which indicates

$$\mathcal{R}(\mathbf{K}_{n+1}) \subset \mathcal{N}(\mathbf{W}_n^T\mathbf{C}). \quad (21)$$

In addition,

$$\begin{aligned}\dim(\mathcal{N}(\mathbf{W}_n^T\mathbf{C})) &= 6 - \mathrm{rank}(\mathbf{W}_n^T\mathbf{C}) = 6 - n. \\ \dim(\mathcal{R}(\mathbf{K}_{n+1})) &= \mathrm{rank}\left(\mathbf{K} - \sum_{i=1}^{n} \frac{\mathbf{w}_i\mathbf{w}_i^T}{\mathbf{w}_i^T\mathbf{C}\mathbf{w}_i}\right) \\ &\geq \mathrm{rank}(\mathbf{K}) - \sum_{i=1}^{n} \mathrm{rank}(\mathbf{w}_i\mathbf{w}_i^T) = 6 - n.\end{aligned}$$

Thus, $\dim(\mathcal{N}(\mathbf{W}_n^T\mathbf{C})) \leq \dim(\mathcal{R}(\mathbf{K}_{n+1}))$. Due to relation (21), $\mathcal{N}(\mathbf{W}_n^T\mathbf{C}) = \mathcal{R}(\mathbf{K}_n)$. Also it is easy to verify that

$$\mathbf{K}_{n+1} = \mathbf{K}_{n+1}^T\mathbf{C}\mathbf{K}_{n+1}.$$

Since $\mathbf{C}$ is a symmetric PD matrix, $\mathbf{K}_{n+1}$ must be a PSD matrix.

In summary, we have the following proposition.

### Proposition 3:

For a given elastic behavior $\mathbf{K}$ ($\mathbf{C}$) and a set of $n$ ($n \leq 5$) wrenches $\mathbf{w}_i$ satisfying condition (9), the matrix $\mathbf{K}_{n+1}$ defined in (19) has the following properties:

1. $\mathbf{K}_{n+1}$ is PSD;
2. $\mathcal{R}(\mathbf{K}_{n+1}) = \mathcal{N}(\mathbf{W}_n^T\mathbf{C})$;
3. $\dim(\mathcal{R}(\mathbf{K}_{n+1})) = 6 - n$. □

Thus, in the construction-based realization process, if $n$ springs are determined, the $(n+1)$th spring wrench must be chosen from the column space of matrix $\mathbf{K}_{n+1}$ defined in (19).

B. Synthesis Procedure: Algebraic Approach

Below, a synthesis procedure to realize an arbitrary elastic behavior with a parallel mechanism is outlined. For a given elastic behavior, this procedure finds a set of spring wrenches $\mathbf{w}_i$ that satisfy condition (9). The selection of each spring wrench further restricts the space of allowable spring wrenches for use in subsequent selection. Each $\mathbf{w}_i$ can be chosen from the corresponding allowable linear space.

The procedure is outlined in the following steps.

1. Select the first spring wrench, $\mathbf{w}_1$: Since $\mathbf{K}$ is full-rank, the location in space, direction, and pitch can be selected arbitrarily. The available wrench space for the first spring, $\mathbb{W}_1$, is the entire space of $\mathbb{R}^6$:

$$\mathbb{W}_1 = \mathbb{R}^6.$$

   The spring constant associated with the selected spring wrench $\mathbf{w}_1$ is determined using (10):

$$k_1 = \frac{1}{\mathbf{w}_1^T \mathbf{C} \mathbf{w}_1}.$$

2. Select the second spring wrench, $\mathbf{w}_2$:
   a. Determine the space of available spring wrenches, $\mathbb{W}_2$. To do this first calculate

$$\mathbf{K}_2 = \mathbf{K} - \frac{\mathbf{w}_1 \mathbf{w}_1^T}{\mathbf{w}_1^T \mathbf{C} \mathbf{w}_1}.$$

   Then, $\mathbb{W}_2$ is a 5-D wrench space that is reciprocal to twist $\mathbf{t}_1$ and in the range of $\mathbf{K}_2$

$$\mathbb{W}_2 = \mathcal{R}(\mathbf{K}_2).$$

   b. Select a wrench in $\mathbb{W}_2$. Any unit wrench vector in $\mathbb{W}_2$ can be selected as the spring wrench of the second spring. The unit spring wrench $\mathbf{w}_2$can be selected from any combination of the columns of $\mathbf{K}_2$. When a wrench $\mathbf{w}_2$is selected, the spring constant can be determined using (10):

$$k_2 = \frac{1}{\mathbf{w}_2^T \mathbf{C} \mathbf{w}_2}.$$

3. In general, when $n$ springs have already been selected, the selection of $(n+1)$th spring can be made as follows:
   1. Determine the allowable space of $\mathbf{w}_{n+1}$, $\mathbb{W}_{n+1}$.

Calculate the symmetric matrix $\mathbf{K}_{n+1}$,

$$\mathbf{K}_{n+1} = \mathbf{K}_n - \frac{\mathbf{w}_n \mathbf{w}_n^T}{\mathbf{w}_n^T \mathbf{C} \mathbf{w}_n}.$$

Then, by Proposition 3,

$$\mathbb{W}_{n+1} = \mathcal{R}(\mathbf{K}_{n+1}), \dim(\mathbb{W}_{n+1}) = 6 - n.$$

2. Select a $\mathbf{w}_{n+1}$ in $\mathbb{W}_{n+1}$.

   Any unit wrench in $\mathbb{W}_{n+1}$ can be selected as the spring wrench of the $(n+1)$th spring. When a wrench $\mathbf{w}_{n+1}$ is selected, the spring constant can be determined using (10):

$$k_{n+1} = \frac{1}{\mathbf{w}_{n+1}^T \mathbf{C} \mathbf{w}_{n+1}}.$$

4. The procedure continues until all six springs are selected.

Note that in the synthesis procedure, the available space for the $(n+1)$th spring wrench, $\mathbb{W}_{n+1}$, depends on the previously selected $n$ spring wrenches $(\mathbf{w}_1, \dots, \mathbf{w}_n)$. For $n < 5$, $\dim(\mathbb{W}_{n+1}) \geq 2$ and the options available for selecting an appropriate $\mathbf{w}_{n+1}$ are not unique. Any wrench in the space can be selected. However, once five springs are chosen, the 6th spring wrench is uniquely determined. This is because $\mathbb{W}_6$ is a 1-D linear space in $\mathbb{R}^6$ and the unit wrench in this space is unique.

In this realization process, each spring wrench can be selected arbitrarily from the linear space $\mathbb{W}_i = \mathcal{R}(\mathbf{K}_i)$. There are infinitely many sets of wrenches that realize a given compliance. The only limitation of this algebraic construction-based procedure is that it does not explicitly take into account geometric properties of the spring and, therefore, yields limited insight into final mechanism geometry. In order to select each spring wrench based on its geometry, an understanding of the geometric characteristics of each space $\mathbb{W}_i$ is needed. The algebraic results derived in this section are used in generating the geometry of each of the allowable wrench spaces $\mathbb{W}_i$ in the following section.

# SECTION IV.

## Geometric Properties of Each Wrench Space $\mathbb{W}_i$

In this section, the geometric properties of each space $\mathbb{W}_i$ are presented. Using these properties, one can select each spring from the corresponding available space based on its geometry. As shown in Section III, when $n$ springs $\mathbf{w}_i$ are determined, the wrench space for the $(n+1)$th spring, $\mathbb{W}_{n+1}$, must be reciprocal to the $n$ twists $\mathbf{t}_i = \mathbf{C}\mathbf{w}_i$. Thus, the wrench space of $\mathbb{W}_{n+1}$ is a $(6-n)$-system.

Ball described general screw $n$-systems in [2]. Here, the properties of each $\mathbf{W}_n$ are investigated in more detail.

As stated previously, the first spring wrench can be arbitrarily chosen and the last spring wrench is unique. Below, the geometric properties of spaces $\mathbb{W}_2$ through $\mathbb{W}_5$ are discussed. The selection of a spring wrench in each space can be based on its pitch, location, and/or direction.

## A. $\mathbb{W}_2$: Acceptable Space for the Second Spring

When the first spring is determined, the acceptable space for the second spring wrench is a 5-D linear space reciprocal to $\mathbf{t}_1 = \mathbf{C}\mathbf{w}_1$. The necessary and sufficient condition for a screw $\mathbf{w}_2$ to be reciprocal to $\mathbf{t}_1$ [2] is as follows:

$$(p_{t1} + p_2)\cos\theta - d\sin\theta = 0 \qquad (22)$$

where $p_{t1}$ is the pitch of $\mathbf{t}_1$, $p_2$ is the pitch of $\mathbf{w}_2$, $\theta$ and $d$ are the angle and the distance between the axes of $\mathbf{t}_1$ and $\mathbf{w}_2$. Thus, any wrench $\mathbf{w}_2$ satisfying (22) can be selected as the spring wrench of the second spring.

Simple ways to select the second spring axis, $\mathbf{w}_2$, are to choose to have it:

1. intersect $\mathbf{t}_1$ at an arbitrary location at a right angle, then the pitch of $\mathbf{w}_2$ is arbitrary;
2. intersect $\mathbf{t}_1$ at an arbitrary location at an arbitrary angle, then select the pitch of $\mathbf{w}_2$ to be equal and opposite to that of $\mathbf{t}_1$;
3. be parallel to $\mathbf{t}_1$, then select the pitch of $\mathbf{w}_2$ to be equal and opposite to that of $\mathbf{t}_1$.

In addition to the simple selection process listed above (as shown in Fig. 3), the following three more-general approaches for spring selection are provided.

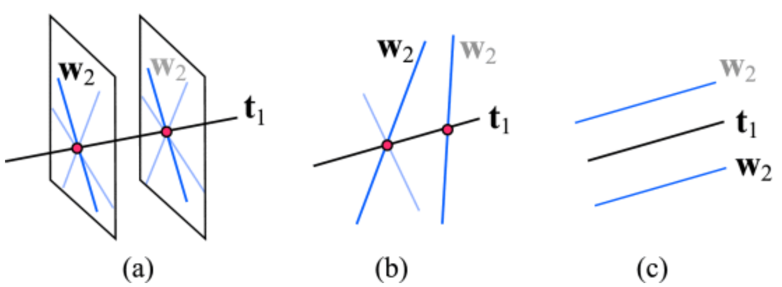


**Fig. 3**. Simple options for the selection of the second spring. (a) Any spring wrench that intersects and makes a right angle with $\mathbf{t}_1$. (b) Any spring wrench that intersects $\mathbf{t}_1$ at an arbitrary angle and has the equal and opposite pitch of twist $\mathbf{t}_1$. (c) Any wrench that is parallel to twist $\mathbf{t}_1$ and has the equal and opposite pitch of $\mathbf{t}_1$.

### 1) Wrench Along a Given Axis

For a given line of action (i.e., given both direction and location), the pitch of an acceptable wrench can be determined.

Suppose $\mathbf{w}_l$ is the unit line vector passing through a specified point in a given direction expressed in Plücker's ray coordinates, and $\mathbf{t}_l$ is the unit line vector for the line of action of the twist $\mathbf{t}_1 = \mathbf{C}\mathbf{w}_1$ expressed in Plücker's axis coordinates. Then, the two line vectors can be written as

$$\mathbf{w}_l = \begin{bmatrix} \mathbf{n}_w \\ \mathbf{r}_w \times \mathbf{n}_w \end{bmatrix}, \mathbf{t}_l = \begin{bmatrix} \mathbf{r}_t \times \mathbf{n}_t \\ \mathbf{n}_t \end{bmatrix}$$

where $\mathbf{n}_w$ and $\mathbf{n}_t$ are the unit vectors indicating the directions of $\mathbf{w}_l$ and $\mathbf{t}_l$, respectively, and $\mathbf{r}_w$ and $\mathbf{r}_t$identify the locations of the line vectors relative to the reference frame. The distance and angle between the two lines can be calculated using

$$d = \frac{|(\mathbf{r}_w - \mathbf{r}_t) \cdot (\mathbf{n}_w \times \mathbf{n}_t)|}{\|\mathbf{n}_w \times \mathbf{n}_t\|}, \theta = \cos^{-1}(\mathbf{n}_w \cdot \mathbf{n}_t).$$

The required pitch of the wrench is calculated using the reciprocal condition (22):

$$p = d\tan\theta - p_{t1}. \qquad (23)$$

Thus, for an arbitrarily selected line in space, there is a unique unit wrench in $\mathbb{W}_2$ along that line [see Fig. 4(a)]. The pitch of the wrench is determined by (23).

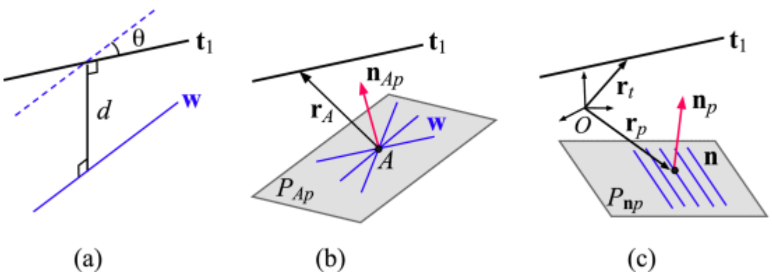


**Fig. 4**. Wrench space $\mathbb{W}_2$. (a) If a line of action is specified, there is one acceptable wrench in $\mathbb{W}_2$ along that line. (b) If a point in space is specified, for each pitch $p$, the set of acceptable spring wrenches corresponds to a pencil of lines in a plane $P_{Ap}$. (c) If the direction of wrench is specified, for each $p$, the set of acceptable spring wrenches corresponds to a family of parallel lines in a plane $P_{\mathbf{n}p}$.

2) Wrenches Passing Through a Given Point With a Specified Pitch

Consider the set of spring wrenches that pass through a given point $A$ in space. For every direction, there is a unique wrench in $\mathbb{W}_2$ passing through $A$ as described in Section IV-A. Here, the pitch of the spring wrench in $\mathbb{W}_2$ is specified. It can be shown that, for each $p \in (-\infty, +\infty)$, the space of wrenches in $\mathbb{W}_2$ that have pitch $p$ and pass through $A$ is a pencil of lines in a plane $P_{Ap}$. As such, any line passing through $A$ in the plane can be chosen as the axis of the second spring.

Suppose that $\mathbf{n}_t$ and $p_{t1}$ are the direction and pitch of twist $\mathbf{t}_1$, and that $\mathbf{r}_A$ is a vector from point $A$ to any point on the twist axis. We consider a direction vector defined by:

$$\mathbf{n}_{Ap} = \mathbf{r}_A \times \mathbf{n}_t + (p_{t1} + p)\mathbf{n}_t \qquad (24)$$

and consider the plane $P_{Ap}$ that passes through $A$ and has its normal along $\mathbf{n}_{Ap}$. First, it is proved that any wrench passing through $A$ and having pitch $p$ must be in plane $P_{Ap}$.

If a coordinate frame is chosen at point $A$, then any wrench passing through $A$ and having pitch $p$ can be expressed in Plücker's ray coordinates as follows:

$$\mathbf{w} = \begin{bmatrix} \mathbf{n} \\ p\mathbf{n} \end{bmatrix}$$

where $\mathbf{n}$ is the direction of the wrench. The twist $\mathbf{t}_1$ expressed in Plücker's axis coordinates at $A$ is

$$\mathbf{t}_1 = \begin{bmatrix} \mathbf{r}_A \times \mathbf{n}_t + p_{t1}\mathbf{n}_t \\ \mathbf{n}_t \end{bmatrix}.$$

By the reciprocal condition, $\mathbf{t}_1^T\mathbf{w} = 0$,

$$\mathbf{n}^T[\mathbf{r}_A \times \mathbf{n}_t + (p_{t1} + p)\mathbf{n}_t] = 0$$

which implies $\mathbf{n} \perp \mathbf{n}_{Ap}$. Thus, $\mathbf{w}$ must be in plane $P_{Ap}$. Also, it can be proved that any wrench passing through $A$ in the plane, if assigned pitch $p$ must be reciprocal to $\mathbf{t}_1$. Thus, the pencil of lines at $A$ in plane $P_{Ap}$ is the space of all spring wrench axes that pass through $A$ and have pitch $p$ [as shown in Fig. 4(b)].

Note that since reciprocal relations between two screws are invariant under co-ordinate transformation, the pencil of lines defined in plane $P_A$ is independent of the coordinate frame used to describe the wrenches and twists.

### 3) Wrenches With a Specified Direction and Pitch

As described in Section IV-A, if the direction of the wrench is given, then for every point in space, there is a unique wrench in $\mathbb{W}_2$ passing through it. Below, the subspace of wrenches in $\mathbb{W}_2$ that have the given direction $\mathbf{n}$ and pitch $p$ is determined.

Suppose in a frame the twist $\mathbf{t}_1 = \mathbf{C}\mathbf{w}_1$ and the wrench $\mathbf{w}$ having specified direction $\mathbf{n}$ and pitch $p$ are expressed as follows:

$$\mathbf{t}_1 = \begin{bmatrix} \mathbf{r}_t \times \mathbf{n}_t + p_t\mathbf{n}_t \\ \mathbf{n}_t \end{bmatrix}, \mathbf{w} = \begin{bmatrix} \mathbf{n} \\ \mathbf{r} \times \mathbf{n} + p\mathbf{n} \end{bmatrix}.$$

Due to the reciprocal condition, $\mathbf{t}_1^T\mathbf{w} = 0$,

$$(\mathbf{r} - \mathbf{r}_t) \cdot (\mathbf{n} \times \mathbf{n}_t) + (p + p_t)(\mathbf{n} \cdot \mathbf{n}_t) = 0. \qquad (25)$$

Let

$$\tilde{\mathbf{r}} = (p + p_t)(\mathbf{n} \cdot \mathbf{n}_t)(\mathbf{n} \times \mathbf{n}_t) \text{and} \mathbf{r}_p = \mathbf{r}_t - \tilde{\mathbf{r}}.$$

Then, (25) can be written as

$$(\mathbf{r}-\mathbf{r}_p)\cdot(\mathbf{n}\times\mathbf{n}_t)=0.$$

It can be seen that $\mathbf{w}$ lies on a plane $P_{\mathbf{n}p}$ having normal $\mathbf{n}_p=(\mathbf{n}\times\mathbf{n}_t)$ and passing through point $\mathbf{r}_p$. Thus, the wrenches in $\mathbb{W}_2$ having given direction $\mathbf{n}$ and pitch $p$ consist of a family of parallel lines on the plane $P_{\mathbf{n}p}$ as illustrated in Fig. 4(c).

## B. $\mathbb{W}_3$: Acceptable Space for the Third Spring

Since the third spring $\mathbf{w}_3$ is reciprocal to the two twists $\mathbf{t}_1=\mathbf{C}\mathbf{w}_1$ and $\mathbf{t}_2=\mathbf{C}\mathbf{w}_2$, $\mathbb{W}_3$ is a 4-system that is reciprocal to the 2-system formed by $\mathbf{t}_1$and $\mathbf{t}_2$, the cylindroid $\mathbb{T}_2$. General 2-systems are investigated in [2]. Suppose $\mathbf{e}_1$ and e2 are the two principal twists in $\mathbb{T}_2$ having pitch $p_{t1}$ and pt2. As shown in [2], $\mathbf{e}_1$ and e2 are reciprocal, orthogonal, and concurrent [2]. A new coordinate frame based on the two principal screws is used to facilitate the subsequent analysis.

We consider a wrench $\mathbf{w}$ in $\mathbb{W}_3$ that has pitch $p$. If $\mathbf{w}$ intersects the cylindroid (generic case), since $\mathbf{w}$ is reciprocal to all twists in $\mathbb{T}_2$ (a cubic surface), it must meet $\mathbb{T}_2$ at three points. At each intersection point, by the reciprocal condition (22), $\mathbf{w}$ must intersect the twist in $\mathbb{T}_2$ at a right angle or intersect the twist having pitch $-p$. Thus, among the three twists of $\mathbb{T}_2$ at the intersection points, one intersects $\mathbf{w}$ at a right angle and the other two have pitch $-p$. Since the pitches of all screws in $\mathbb{T}_2$ are bounded by $p_{t1}$ and $p_{t2}$, the pitch of the spring wrench, $p$, must be bounded by

$$min\{-p_{t1},-p_{t2}\}\le p\le max\{-p_{t1},-p_{t2}\}. \qquad (26)$$

Thus, if the pitch of $\mathbf{w}$ is in the range of (26), it must intersect two twists in $\mathbb{T}_2$ that have pitch $-p$. Also it can be seen that any wrench with pitch $p$ intersecting the two twists of pitch $-p$ in $\mathbb{T}_2$ must be reciprocal to all twists of $\mathbb{T}_2$, and thus must be in $\mathbb{W}_3$ . Therefore, for a $p$ in the range of (26), a screw in $\mathbb{W}_3$ has pitch $p$ *if and only if* it intersects the two twists in $\mathbb{T}_2$ that have pitch $-p$.

If the pitch $p$ of $\mathbf{w}$ is not in the range of (26), $(p+p_{t1})$ and $(p+p_{t2})$ have the same sign, and $\mathbf{w}$ does not intersect the cylindroid $\mathbb{T}_2$. Thus, $\mathbf{w}$ is parallel to the principal plane $P$ of $\mathbb{T}_2$ and the distance between $\mathbf{w}$ and the plane is greater than $\frac{1}{2}|p_{t1}-p_{t2}|$. Below, the space for $\mathbf{w}$ with pitch $p$ outside the range of (26) is determined.

Since the two principal twists of $\mathbb{T}_2$ are orthogonal and $\mathbf{w}$ is parallel to the principal plane of $\mathbb{T}_2$, the reciprocal conditions for $\mathbf{w}$ and the principal twists can be written as follows:

$$\begin{aligned}(p+p_{t1})\cos\theta-d\sin\theta&=0,\\(p+p_{t2})\sin\theta-d\cos\theta&=0\end{aligned} \qquad (27)(28)$$

where $\theta$ is the angle between $\mathbf{w}$ and $\mathbf{e}_1$ ($x'$-axis), and $d$ is the distance between $\mathbf{w}$ and the principal plane $P$.

Solving the two equations (27)–(28) yields

$$
\begin{aligned}
d &= \sqrt{(p + p_{t1})(p + p_{t2})}, \\
\theta &= \tan^{-1}\left(\frac{p+p_{t1}}{d}\right).
\end{aligned}
\quad (29)(30)
$$

Therefore, $\mathbf{w}$ is in one of the two planes $z' = \pm d$ and makes an angle $\theta$ with $\mathbf{e}_1$.

In the following, the wrench spaces for different geometric properties are discussed.

### 1) Wrenches With a Given Pitch

Suppose that the pitch of the third spring wrench is specified, we determine the wrench axis space in $\mathbb{W}_3$ . Based on the values of pitch $p$, two cases are considered.

1. If $p$ is in the range of (26), we consider the two twists $\mathbf{t}_{p1}$ and $\mathbf{t}_{p2}$ in $\mathbb{T}_2$ that have pitch $-p$. (The process to obtain the two twists in $\mathbb{T}_2$ having a specified pitch can be found in [2].) Then, the entire space of wrenches in $\mathbb{W}_3$ that have pitch $p$ is the collection of lines intersecting $\mathbf{t}_{p1}$ and $\mathbf{t}_{p2}$ as shown in Fig. 5(a). Any line that intersects the two twists can be selected.
2. If $p$ is not in the range of (26), the spring axis lines are two families of parallel lines in the two planes $P_1$ and $P_2$, both parallel to the principal plane $P$ of $\mathbb{T}_2$. The locations of the two planes are indicated by the distances to the principal plane $P$, $d$, which is given in (29). The angle $\theta$ between the parallel lines and the $x$-axis is given by (30). The two families are illustrated in Fig. 5(b).

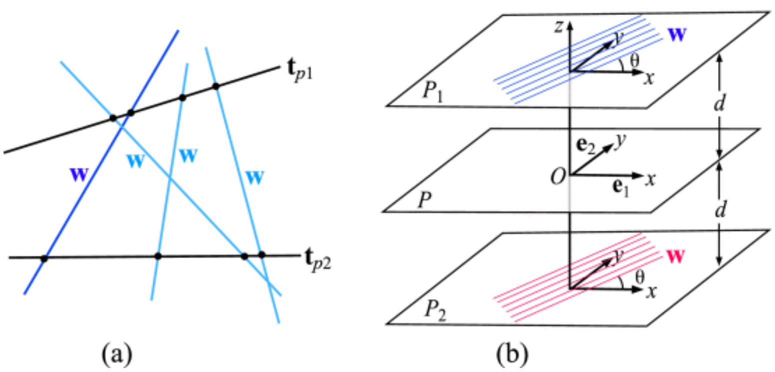


**Fig. 5.** Space of the third spring axes for a given pitch $p$. (a) If $p$ is in the range of (26), the space of acceptable spring axes is determined by the two twists $\mathbf{t}_{p1}$ and $\mathbf{t}_{p2}$ having pitch $-p$ in $\mathbb{T}_2$ . Any line that intersects the two twists can be selected. (b) For a $p$ outside the range of (26), the space consists of the two families of parallel lines in planes $P_1$ and $P_1$.

It can be seen that, for the third spring, any $p \in (-\infty, +\infty)$ can be selected.

### 2) Wrenches Passing Through a Given Point

We consider an arbitrary point $A$ $(x_0, y_0, z_0)$ in space. An infinite number of wrenches in $\mathbb{W}_4$ pass through the point. For a wrench passing through point $A$, consider the following two cases.

1. The wrench intersects the cylindroid $\mathbb{T}_2$ . Then, for every $p$ in the range of (26), there is only one wrench in $\mathbb{W}_4$ that passes through the given point. We consider the two twists $\mathbf{t}_{p1}$ and $\mathbf{t}_{p2}$ in $\mathbb{T}_2$ that have pitch $-p$. Then, we consider the two planes $P_{A1}$ and $P_{A2}$: $P_{A1}$ is constructed to contain $\mathbf{t}_{p1}$ and point $A$; and $P_{A2}$ is constructed to contain $\mathbf{t}_{p2}$ and point $A$. The intersection of the two planes is the axis of the acceptable wrench [illustrated in Fig. 6(a)].

2. The wrench does not intersect the cylindroid $\mathbb{T}_2$ . Then, the wrench must be parallel to the principal plane of $\mathbb{T}_2$ and the distance between $\mathbf{w}$ and the principal plane, $z_0$, must satisfy

$$|z_0| > \frac{1}{2}|p_{t1} - p_{t2}|.$$

In order to determine the pitch $p$ of $\mathbf{w}$, setting $d = |z_0|$ in (29) yields:

$$p_{1,2} = \tfrac{1}{2}(-(p_{t1} + p_{t2}) \pm \sqrt{(p_{t1} - p_{t2})^2 + 4z_0^2}). \qquad (31)$$

For each pi in (31), the angle $\theta$ between the wrench and principal axis $\mathbf{e}_1$ ($x$-axis) is obtained using (30):

$$\theta_i = \tan^{-1}(\tfrac{p_i + p_{t1}}{d}), i = 1,2. \qquad (32)$$

Therefore, there are two wrenches $\mathbf{w}$ and $\mathbf{w}'$ with pitches $p_1$ and $p_2$ in $\mathbb{W}_3$ that pass through the given point $A$ as illustrated in Fig. 6(b). It can be proved that the pitches obtained in (31) must be outside the range of (26), and the two lines $\mathbf{w}$ and $\mathbf{w}'$ associated with $\theta_1$ and $\theta_2$ of (32) are orthogonal.

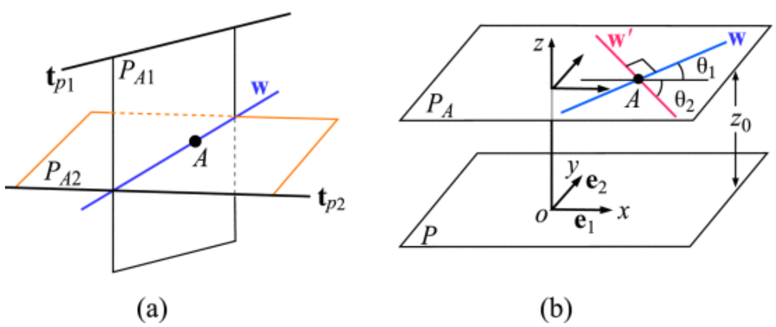


**Fig. 6.** Spring wrenches passing through a given point $A$. (a) If the wrench intersects $\mathbb{T}_2$ , for each pitch in the range of (26), there is a unique wrench determined by the two planes $P_{A1}$ and $P_{A2}$ . (b) If the wrench does not intersect $\mathbb{T}_2$ , there are two wrenches $\mathbf{w}$ and $\mathbf{w}'$ in plane $P_A$ passing through the given point.

Note that for a given point $A$ , and a given value of $p \in (-\infty, +\infty)$, there is only one wrench in $\mathbb{W}_3$ that passes through $A$ and has pitch $p$.

### 3) Wrenches With a Given Direction

Below, the wrenches in $\mathbb{W}_3$ along a given direction $\mathbf{n}$ are determined. The following two cases are considered.

1. If the direction $\mathbf{n}$ is not parallel to the principal plane of $\mathbb{T}_2$ , then wrenches along $\mathbf{n}$ intersect the cylindroid of $\mathbb{T}_2$ . Thus, the pitches of the wrenches must be in the range of (26). As shown below, for each $p$ in the range, the space $\mathbb{W}_3$ has a unique wrench having the given direction $\mathbf{n}$.

For a pitch $p$ in the range of (26), there are two twists $\mathbf{t}_{p1}$ and $\mathbf{t}_{p2}$ having pitch $-p$ in $\mathbb{T}_2$. Let $P_{\mathbf{n}1}$ be the plane containing both $\mathbf{t}_{p1}$ and $\mathbf{n}$, and let $P_{\mathbf{n}2}$ be the plane containing both $\mathbf{t}_{p2}$ and $\mathbf{n}$. Then the line defined by the intersection of the two planes $P_{\mathbf{n}1}$ and $P_{\mathbf{n}2}$ is the wrench axis that has the desired properties. As illustrated in Fig. 7(a), the obtained wrench $\mathbf{w}$ is unique.

2. If the direction $\mathbf{n}$ is parallel to the principal plane of $\mathbb{T}_2$, then any wrench along $\mathbf{n}$ cannot intersect the cylindroid $\mathbb{T}_2$ and the pitch of the wrench must be outside the range of (26). Below, the pitch and locations of the wrenches are determined for this case.

The angle $\theta$ between $\mathbf{n}$ and the principal axis ($\mathbf{e}_1$) can be calculated by $\cos\theta = \mathbf{n}\cdot\mathbf{e}_1$. The pitch and locations of the wrenches can be obtained by solving (27)–(28) for $p$ and $d$:

$$\begin{aligned} p &= \frac{p_{t2}\tan^2\theta - p_{t1}}{1-\tan^2\theta}, \\ d &= \sqrt{(p+p_{t1})(p+p_{t2})}. \end{aligned} \qquad (33)(34)$$

Thus, there are infinitely many wrenches along $\mathbf{n}$. The wrenches are in one of the two families of parallel lines in planes $P_1$ and $P_2$ offset from the principal plane of $\mathbb{T}_2$ by $d$ of (34) [as shown in Fig. 7(b)] and all wrenches have the same pitch $p$ determined by (33).

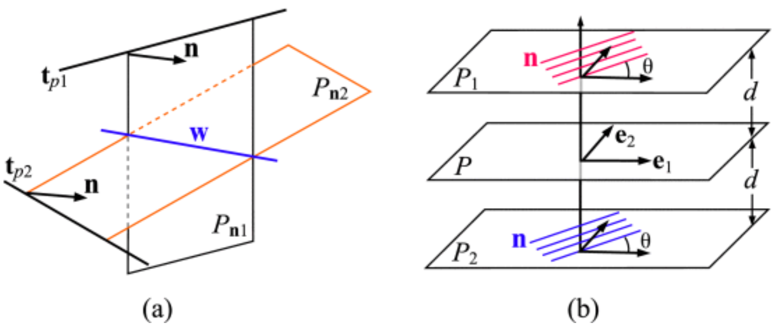


**Fig. 7**. Space of wrenches along a given direction $\mathbf{n}$. (a) When $\mathbf{n}$ is not parallel to the principal plane of $\mathbb{T}_2$, for every pitch in the range of (26), there is a unique wrench in $\mathbb{W}_3$. (b) When $\mathbf{n}$ is parallel to the principal plane of $\mathbb{T}_2$, wrenches have the same pitch and have parallel lines of action in planes $P_1$ and $P_2$.

### 4) Wrenches Intersecting Two Twists in $\mathbb{T}_2$

Suppose $\mathbf{t}'_1$ and $\mathbf{t}'_2$ are arbitrary twists in $\mathbb{T}_2$. We determine the wrenches in $\mathbb{W}_3$ that intersect the two twists. Suppose that the two twists have pitches $p'_{t1}$ and $p'_{t2}$. If $p'_{t1} = p'_{t2}$, then by (27) and (28), any wrench having pitch $-p'_{t1}$ and intersecting the two twists can be chosen. Below, the generic case $p'_{t1} \neq p'_{t2}$ is considered.

We consider a plane $P_1$ perpendicular to $\mathbf{t}'_1$ at point $A_1$. The plane intersects $\mathbf{t}'_2$ at point $A_2$. The two points $A_1$ and $A_2$ determine a line. As shown in Fig. 8(a), when $P_1$ moves along $\mathbf{t}'_1$, a family of lines is formed. Any wrench having pitch $-p'_{t2}$ and line of action in the family is reciprocal to $\mathbb{T}_2$ and intersect the two twists. Similarly, another family of lines can be obtained by considering plane $P_2$ perpendicular to $\mathbf{t}'_2$ and intersecting $\mathbf{t}'_1$ as shown in Fig. 8(b). For this family of lines, each wrench has a pitch $-p'_{t1}$.

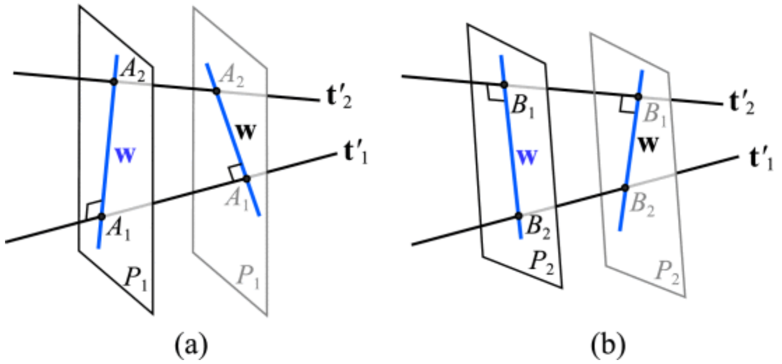


**Fig. 8.** Wrenches intersecting two given twists in $\mathbb{T}_2$ . (a) The wrench axes are the line family formed by moving plane $P_1$ along $\mathbf{t}'_1$ . All wrenches have pitch $-p'_{t2}$. (b) The wrench axes are the line family formed by moving plane $P_2$ along $\mathbf{t}'_2$. All wrenches have pitch $-p_{t1}$.

An easy way to obtain a subset of wrench axes in $\mathbb{W}_3$ is to consider the two twists associated with the first two spring wrenchest$_1 = \mathbf{C}\mathbf{w}_1$ and $\mathbf{t}_2 = \mathbf{C}\mathbf{w}_2$. Then, any wrench having its axis selected from the line families formed by translation of $P_1$ ($P_2$) and having equal and opposite pitches of $\mathbf{t}_2$ ($\mathbf{t}_1$) can be used as the third spring wrench.

## C. $\mathbb{W}_4$: Acceptable Space for the Fourth Spring

The space for the fourth spring axis is a 3-system reciprocal to the 3-system formed by twists $\mathbf{t}_1$, $\mathbf{t}_2$ and $\mathbf{t}_3$, $\mathbb{T}_3$. For the 3-system $\mathbb{W}_4$ , there are three principal wrenches $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ that are reciprocal, orthogonal, and intersect at a point [2]. These three principal wrenches can be uniquely determined from any three independent screws in $\mathcal{R}(\mathbf{K}_4)$ using the approaches presented in [25]–[27]. Since $\mathbb{W}_4$ and $\mathbb{T}_3$ are reciprocal, the three principal twists of $\mathbb{T}_3$ are coincident with the three principal wrenches of $\mathbb{W}_4$ but with opposite sign. The three principal wrenches can also be obtained from the three principal twists of $\mathbb{T}_3$ by changing the signs of their pitches.

Suppose that $\mathbf{u}_i$ are the three principal wrenches with pitch $p_i$. Then, in the principal frame, the three principal screws can be expressed as follows:

$$\mathbf{u}_i = \begin{bmatrix} \mathbf{n}_i \\ p_i\mathbf{n}_i \end{bmatrix}, i = 1,2,3 \quad (35)$$

where $\mathbf{n}_i$is the unit vector along the coordinate axes. Any wrench in $\mathbb{W}_4$ is a linear combination of the three principal wrenches:

$$\mathbf{w} = x\begin{bmatrix} \mathbf{n}_1 \\ p_1\mathbf{n}_1 \end{bmatrix} + y\begin{bmatrix} \mathbf{n}_2 \\ p_2\mathbf{n}_2 \end{bmatrix} + z\begin{bmatrix} \mathbf{n}_3 \\ p_3\mathbf{n}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \quad (36)$$

where $x, y, z$ are arbitrary scalars. The pitch of $\mathbf{w}$ can be calculated from (36):

$$p = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2} = \frac{x^2p_1+y^2p_2+z^2p_3}{x^2+y^2+z^2}. \quad (37)$$

It can be seen from (37) that the pitch $p$ is bounded by the greatest and least pitches of the three principal wrenches. If we suppose, without losing generality, $p_3 \le p_2 \le p_1$, then

$$p_3 \le p \le p_1. \qquad (38)$$

Equation (37) can be written as

$$(p_1 - p)x^2 + (p_2 - p)y^2 + (p_3 - p)z^2 = 0. \quad (39)$$

If $p$ satisfies (38), then the coefficients $(p_1 - p)$, $(p_2 - p)$, and $(p_3 - p)$ do not have the same sign. Thus, the surface of (39) is a cone. Since $(x, y, z)$ indicates the direction of $\mathbf{w}$ for any given value of pitch $p$, the directions of screws in $\mathbb{W}_4$ lie in the surface of the cone of (39). Since for a given direction, there is only one $p$ satisfying (38), once the direction of a wrench is specified, the pitch of the wrench is uniquely determined.

In general, a wrench in (36) does not pass through the origin of the principal coordinate frame. For a given $p$ the lines of action of wrenches in $\mathbb{W}_4$ lie on a quadric surface [2] defined by:

$$\begin{gathered}(p_1 - p)x^2 + (p_2 - p)y^2 + (p_3 - p)z^2 \\ +(p_1 - p)(p_2 - p)(p_3 - p) = 0.\end{gathered} \qquad (40)$$

For a $p$ in the range of (38), the above equation can be expressed in the principal frame as follows:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad (41)$$

Where

$$\begin{aligned} a &= \sqrt{|(p_2 - p)(p_3 - p)|}, \\ b &= \sqrt{|(p_1 - p)(p_3 - p)|}, \\ c &= \sqrt{|(p_1 - p)(p_2 - p)|}. \end{aligned} \qquad (42)(43)(44)$$

The quadric surface (41) is a hyperboloid of one sheet for any pitch $p$ in the range of (38). Every wrench having pitch $p$ must lie on the hyperboloid, or must be a generator of the hyperboloid. However, not every generator of the hyperboloid can be selected as a wrench axis.

Since the three principal twists of the reciprocal 3-system, $\mathbb{T}_3$, are coincident with the principal wrenches $\mathbf{u}_i$ but have pitches $-p_i$, surface (41) is also the hyperboloid for the reciprocal system $\mathbb{T}_3$ associated with pitch $-p$. Thus, every twist of pitch $-p$ in $\mathbb{T}_3$ also lies on the same hyperboloid.

For a hyperboloid (41), there are two families of generators $F_\lambda$ and $F_\mu$ as shown in Fig. 9. The screw representation of the two families are presented below.

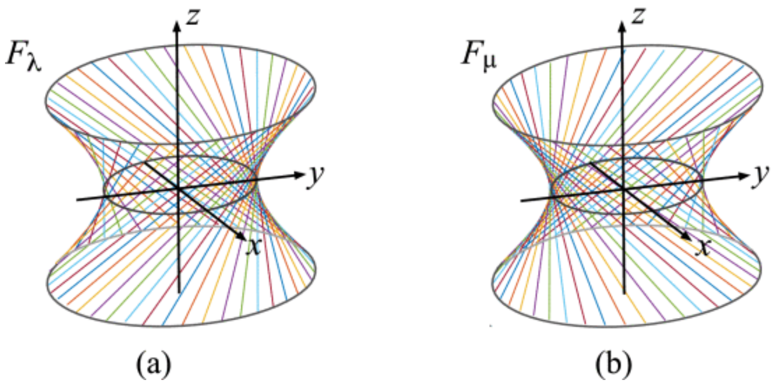


**Fig. 9.** Hyperboloid of one sheet associated with a given pitch $p$ There are two families of generators: (a) $F_\lambda$ and (b) $F_\mu$ . The fourth spring axis must be chosen from one of the two families, either $F_\lambda$ or $F_\mu$ .

Every line in $F_\lambda$ can be expressed in the form as follows:

$$F_\mu\colon \mathbf{s}_\mu = \begin{bmatrix} \mathbf{n}_\mu \\ \mathbf{r}_\mu \times \mathbf{n}_\mu \end{bmatrix}, \mu \in (-\infty, +\infty) \quad (45)$$

where

$$\begin{aligned} \mathbf{n}_\mu &= [-a(\mu^2-1), -2b\mu, c(\mu^2+1)]^T, \\ \mathbf{r}_\mu &= \frac{1}{1+\mu^2}[2a\mu, b(1-\mu^2), 0]^T. \end{aligned}$$

Every line in $F_\mu$ can be expressed in the form as follows:

$$F_\mu\colon \mathbf{s}_\mu = \begin{bmatrix} \mathbf{n}_\mu \\ \mathbf{r}_\mu \times \mathbf{n}_\mu \end{bmatrix}, \mu \in (-\infty, +\infty) \quad (46)$$

Where

$$\begin{aligned} \mathbf{n}_\mu &= [-a(\mu^2-1), -2b\mu, c(\mu^2+1)]^T, \\ \mathbf{r}_\mu &= \frac{1}{1+\mu^2}[2a\mu, b(1-\mu^2), 0]^T. \end{aligned}$$

The geometrical meaning of $\lambda$ and $\mu$ are not evident. To describe the wrench more geometrically, let $\lambda = \tan\theta$. Then, for $\lambda \in (-\infty, +\infty)$ , $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The direction of the line can be expressed as follows:

$$\mathbf{n}_\theta = \frac{1}{\cos^2\theta}[-a\cos 2\theta, b\sin 2\theta, c]^T.$$

The line passes through point $P_\theta$ in the $xy$ plane:

$$\mathbf{r}_\theta = [a\sin 2\theta, b\cos 2\theta, 0]^T.$$

Therefore, one family of lines can be expressed in the form as follows:

$$F_\theta : \mathbf{s}_\theta = [\begin{matrix}\mathbf{n}_\theta \\ \mathbf{r}_\theta \times \mathbf{n}_\theta\end{matrix}], \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}). \qquad (47)$$

Similarly, the lines of $F_\mu$ can be expressed as follows:

$$F_\phi : \mathbf{s}_\phi = [\begin{matrix}\mathbf{n}_\phi \\ \mathbf{r}_\phi \times \mathbf{n}_\phi\end{matrix}], \phi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \qquad (48)$$

where

$$\begin{aligned}\mathbf{n}_\phi &= \frac{1}{\cos^2\phi}[a\cos 2\phi, -b\sin 2\phi, c]^T, \\ \mathbf{r}_\phi &= [a\sin 2\phi, b\cos 2\phi, 0]^T.\end{aligned}$$

It can be proved that, of the two families of lines, $F_\theta$ ( $F_\lambda$ ) and $F_\phi$ ( $F_\mu$ ), one belongs to $\mathbb{W}_4$ and the other belongs to the reciprocal twist space $\mathbb{T}_3$ . Thus, if one wrench of pitch $p$ in $F_\theta$ is reciprocal to any twist $\mathbf{t}_i$, the entire family $F_\theta$ is in $\mathbb{W}_4$ . Otherwise, the entire family of $F_\phi$ is in $\mathbb{W}_4$ . Also, it can be seen that the parameters $\theta$ and $\phi$ indicate the location of the lines at the principal plane determined by the principal wrenches $\mathbf{u}_1$ and $\mathbf{u}_2$ .

In the following, acceptable spring wrenches in $\mathbb{W}_4$ are identified based on their geometric properties.

### 1) Wrenches With a Given Pitch

If the pitch of the wrench is specified, the value of the pitch $p$ must be in the range of (38).

1. Nongeneric case: Pitch $p$ is equal to the pitch of one of the principal wrenches. If $p$ is equal to the largest or the least pitch of the principal wrench, $p_1$ or $p_3$, then, as shown in Fig. 10(a), the wrench must be the corresponding principal wrench $\mathbf{u}_1$ or $\mathbf{u}_3$ .

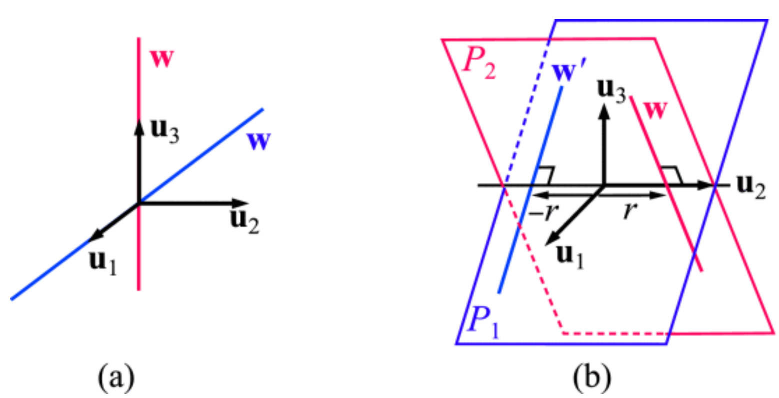

**Fig. 10.** (a) If the pitch is selected to be the largest or least pitch of the principal wrenches of $\mathbb{W}_4$ , $p_1$ or $p_3$ , the spring wrench must be the corresponding principal wrench $\mathbf{u}_1$ or $\mathbf{u}_3$ . (b) If the pitch is select to be the intermediate pitch of the principal wrenches, $p_2$, the spring wrench must be one of the three wrenches $\mathbf{u}_2$ , $\mathbf{w}$ or $\mathbf{w}'$ .

If $p$ is equal to the intermediate pitch $p_2$ , then the wrench lies on the surface:

$$(p_1 - p_2)x^2 - (p_2 - p_3)z^2 = 0$$

which is composed of two planes:

$$(\sqrt{p_1 - p_2})x \pm (\sqrt{p_2 - p_3})z = 0. \qquad (49)$$

However, not every line on the two planes is in $\mathbb{W}_4$ . It can be proved that there are three wrenches that are reciprocal to $\mathbb{T}_3$ [illustrated in Fig. 10(b)]. One wrench is the principal wrench $\mathbf{u}_2$. The other two can be viewed as the two lines in the $xz$ plane described in translated along the $y$-axis by:

$$r = \pm\sqrt{(p_1 - p)(p - p_3)} \qquad (50)$$

respectively.

2. Generic case: $p \neq p_i$. The space of wrench axes is either $F_\lambda$ of (45) or $F_\mu$ of (46). To determine which family is in $\mathbb{W}_4$ , a test of only one point is needed. We consider a wrench on family $F_\lambda$ with pitch $p$:

$$\mathbf{w}_\lambda = \begin{bmatrix} \mathbf{n}_\lambda \\ p\mathbf{n}_\lambda + \mathbf{r}_\lambda \times \mathbf{n}_\lambda \end{bmatrix}. \qquad (51)$$

If for a value of $\lambda$, $\mathbf{w}_\lambda$ is reciprocal to any $\mathbf{t}_i$ ($\mathbf{w}_\lambda^T \mathbf{t}_i = 0$), then the entire family of $F_\lambda$ is the space of wrench lines for the fourth spring. When $\lambda$ varies in $(-\infty, +\infty)$, $\mathbf{w}_\lambda$ gives all spring wrenches of pitch $p$ in $\mathbb{W}_4$ . If $\mathbf{w}_\lambda^T \mathbf{t}_i \neq 0$, then

$$\mathbf{w}_\mu = \begin{bmatrix} \mathbf{n}_\mu \\ p\mathbf{n}_\mu + \mathbf{r}_\mu \times \mathbf{n}_\mu \end{bmatrix} \qquad (52)$$

must be reciprocal to $\mathbf{t}_i$. Thus, for $\mu \in (-\infty, +\infty)$, $\mathbf{w}_\mu$ gives all spring wrenches of pitch $p$ in $\mathbb{W}_4$ .

### 2) Wrenches Passing Through a Given Point

Suppose a given point is expressed in the principal coordinate frame as $(x_0, y_0, z_0)$. We show that $\mathbb{W}_4$ contains one or three wrenches that pass through the point. The wrench pitches and directions are determined as follows:

1. Determine the pitches of the wrenches.
   Substituting the point $(x_0, y_0, z_0)$into (40) yields a third-order polynomial in $p$

$$(p - p_1)(p - p_2)(p - p_3) + (p - p_1)x_0^2 + (p - p_2)y_0^2 + (p - p_3)z_0^2 = 0. \tag{53}$$

   The cubic equation has three roots $\hat{p}_i$ $(i = 1,2,3)$, among which one or three must be real. It is easy to verify that each real root is in the range of (38) and can be selected as the pitch of a spring wrench in $\mathbb{W}_4$ .

2. For each pitch $\hat{p}_i$ selected in a), we determine the wrench direction. The hyperboloid associated with $\hat{p}_i$ is

$$F_i: (p_1 - \hat{p}_i)x^2 + (p_2 - \hat{p}_i)y^2 + (p_3 - \hat{p}_i)z^2 + (p_1 - \hat{p}_i)(p_2 - \hat{p}_i)(p_3 - \hat{p}_i) = 0 \tag{54}$$

   which can be expressed as in the form of (41).

   The two families of generators $F_\lambda$ and $F_\mu$ associated with the hyperboloid of (54) are obtained using (45) and (46). Since point $(x_0, y_0, z_0)$ is on the hyperboloid, the values of $\lambda$ and $\mu$ can be obtained by:

$$\lambda = \frac{x_0/a_i + z_0/c_i}{(1 + y_0/b_i)}, \mu = \frac{x_0/a_i - z_0/c_i}{(1 + y_0/b_i)}. \tag{55}$$

   Thus, two wrenches of pitch $\hat{p}_i$ passing through point $(x_0, y_0, z_0)$ are obtained using (51) and (52). Of the two wrenches, only one belongs to $\mathbb{W}_4$ , which can be determined by testing the reciprocal condition with any twist $\mathbf{t}_i$.

Note that, for each $\hat{p}_i$, $\mathbb{W}_4$ has only one wrench passing through the given point. The total number of the spring wrenches passing through the given point is either one or three depending on the number of real roots of (53).

### 3) Wrenches With a Given Direction

Suppose the direction is given as a unit vector in the principal coordinate frame as $\mathbf{n} = [n_x, n_y, n_z]^T$. As stated earlier, there is one wrench in $\mathbb{W}_4$ along that direction. The pitch and line of action of the screw can be determined as follows:

1. Determine the pitch of the wrench.

   By (37), the pitch of the screw can be calculated as follows:

$$p = n_x^2 p_1 + n_y^2 p_2 + n_z^2 p_3. \qquad (56)$$

2. Determine the location of the wrench. In the principal frame, the spring wrench is as follows

$$\mathbf{w} = \begin{bmatrix} n_x\mathbf{n}_1 + n_y\mathbf{n}_2 + n_z\mathbf{n}_3 \\ p_1 n_x\mathbf{n}_1 + p_2 n_y\mathbf{n}_2 + p_3 n_z\mathbf{n}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{n} \\ \mathbf{m} \end{bmatrix}.$$

Since $\mathbf{n}$ is a unit vector, the location of the wrench can be calculated using the following:

$$\mathbf{r} = \quad \mathbf{n} \times \mathbf{m} = (p_3 - p_2)n_y n_z \mathbf{n}_1 + (p_1 - p_3)n_x n_z \mathbf{n}_2 + (p_2 - p_1)n_x n_y \mathbf{n}_3$$

where $\mathbf{r}$ is the perpendicular vector from the coordinate frame origin to the axis of the wrench.

## D. $\mathbb{W}_5$: Acceptable Space for the Fifth Spring

The space for the fifth spring axis is a 2-D linear space reciprocal to $\mathbf{t}_i = \mathbf{C}\mathbf{w}_i, i = 1,2,3,4$. The 2-system $\mathbb{W}_5$ is a cylindroid formed by any two independent column vectors of $\mathbf{K}_5$.

Suppose that $p_1$ and $p_2$ are the two principal pitches of the cylindroid. Then for any wrench in $\mathbb{W}_5$, the pitch $p$ must satisfy:

$$min\{p_1, p_2\} \leq p \leq max\{p_1, p_2\}. \quad (57)$$

As shown in [2], every wrench in $\mathbb{W}_5$ must be parallel to the principal plane $P$ and intersect the $z$-axis at a right angle.

### 1) Wrenches With a Given Pitch

For any given pitch $p$ in the range of (57), there are two wrenches in $\mathbb{W}_5$ having that pitch. The two wrenches can be obtained using the method described in [2].

Since every wrench is parallel to the principal plane, the direction of the wrench is indicated by the angle between the wrench and the $x$-axis of the principal frame, $\theta$, which can be obtained by

$$\theta_{1,2} = \pm\sin^{-1}\left(\sqrt{\frac{|p-p_1|}{|p_1-p_2|}}\right) \quad (58)$$

and the corresponding locations in the principal frame of the two wrenches are as follows:

$$z_p = \pm\sqrt{(p - p_2)(p_1 - p)}. \qquad (59)$$

The two spring wrenches are illustrated in Fig. 11(a).

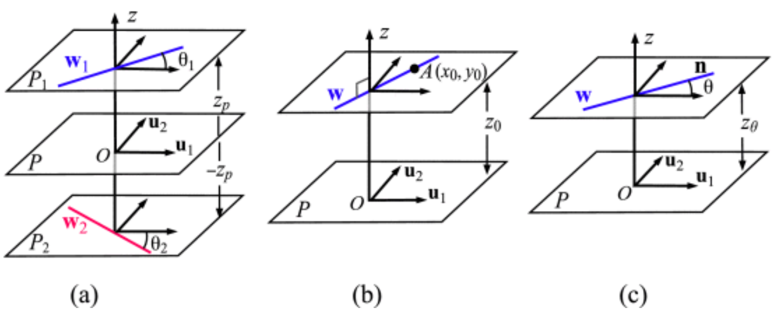


**Fig. 11**. Space of $\mathbb{W}_5$ . (a) For each pitch $p$ in the range of (57), there are two wrenches in the space. (b) For a given point $A$, there is only one wrench in the space. (c) For a given direction, there is only one wrench in the space.

## 2) Wrenches Passing Through a Given Point

The given point must be on the cylindroid and there is only one wrench in $\mathbb{W}_5$ passing through that point. Suppose the point $A$ is given in the principal frame as $(x_0, y_0, z_0)$ . The pitch of the wrench is calculated as follows [2]:

$$p = \frac{x_0^2 p_1 + y_0^2 p_2}{x_0^2 + y_0^2}. \qquad (60)$$

The wrench axis is along the line that passes through $A$ and intersects the $z$ -axis at a right angle as illustrated in Fig. 11(b).

## 3) Wrenches With a Given Direction

Since any wrench $\mathbb{W}_5$ must be parallel to the principal plane of the cylindroid, a given direction can be expressed as a unit vector in the principal coordinate frame in the form of $\mathbf{n} = [n_1, n_2, 0]^T$. Then, the angle $\theta$ between the wrench and $x$ -axis satisfies:

$$\cos\theta = n_1, \sin\theta = n_2.$$

The location of the wrench plane is:

$$z_\theta = (p_2 - p_1)\sin\theta\cos\theta. \qquad (61)$$

The pitch of the wrench is:

$$p = p_1\cos^2\theta + p_2\sin^2\theta. \qquad (62)$$

It can be seen that for a given direction parallel to the principal plane, there is a unique spring wrench in $\mathbb{W}_5$ . The spring wrench is illustrated in Fig. 11(c).

Special cases for the values of $p_1$ and $p_2$ yield the following results:

1. If $\mathbb{W}_5$ contains two wrenches having the same line of action but having different pitches, then all wrenches in $\mathbb{W}_5$ have that line of action and any screw along the line must be in $\mathbb{W}_5$.
2. If $p_1 = p_2$, then $\mathbb{W}_5$ becomes a plane determined by the two principal wrenches.
3. If $\mathbb{W}_5$ contains a wrench of infinite pitch, then at least one principal wrench has infinite pitch. If $\mathbb{W}_5$ contains two wrenches of infinite pitch, the two principal wrenches must have infinite pitch and all wrenches in $\mathbb{W}_5$ have infinite pitch, which means that the springs are torsional.

# SECTION V.

## Discussion

In each space $\mathbb{W}_i$ ($i \leq 5$), there are infinite number of spring wrenches available. In this section, observations related to the pitch and geometry of screws selected in the synthesis procedure for parallel and serial manipulators are presented.

### A. Simple Spring versus Screw Spring Realization

Depending on the value of the spring wrench pitch, two types of springs, line springs [8] and screw springs [12], are used for the realization. A line spring refers to a spring that only provides translational elastic coupling in one direction. A line spring wrench has a zero pitch and is readily realized with a prismatic joint. For those wrenches having finite nonzero pitch, a screw spring, which couples the translational and rotational elastic behaviors, must be used. The realization of a screw spring requires a helical joint with a specified pitch. Thus, physical realization of a zero pitch line spring is easier than that of a nonzero pitch screw spring.

In the processes used to select the first three springs ($\mathbb{W}_1$, $\mathbb{W}_2$, and $\mathbb{W}_3$), *any* value of pitch can be assigned. For $p = 0$, the subspaces of wrenches in $\mathbb{W}_2$ and $\mathbb{W}_3$ are identified in Sections IV-A and IV-B. However, wrenches with a zero pitch may not exist in spaces $\mathbb{W}_4$, $\mathbb{W}_5$ or $\mathbb{W}_6$.

As shown in Sections IV-C and IV-D, the wrench pitches in $\mathbb{W}_4$ and $\mathbb{W}_5$ are bounded by the largest and smallest pitches of the principal wrenches, $p_{max}$ and $p_{min}$. Therefore, $\mathbb{W}_4$ or $\mathbb{W}_5$ contains a zero-pitch wrench if and only if $(p_{min})(p_{max}) \leq 0$. For $\mathbb{W}_4$ and $\mathbb{W}_5$, since

$$\mathbb{W}_i = \mathcal{R}(\mathbf{K}_i), i = 4,5$$

where $\mathbf{K}_i$ is the PSD matrix defined in (19), it can be proved that the pitches of the principal wrenches of $\mathcal{R}(\mathbf{K}_i)$ have the same sign as the nonzero pitches of Eigenscrews of $\mathbf{K}_i$. Thus, for $i = 4,5,6$, $\mathbb{W}_i$ contains zero-pitch wrenches if and only if the nonzero eigenstiffnesses of $\mathbf{K}_i$ do not have the same sign.

A special case is when the given stiffness $\mathbf{K}$ satisfies the trace condition $\text{tr}(\mathbf{K\Delta}) = 0$. For this case, if the first three springs are selected to have zero pitch, the matrix $\mathbf{K}_4$ must also satisfy the trace condition. Since the eigenstiffnesses sum to zero, the eigenstiffnesses of $\mathbf{K}_4$ do not have the same sign. Then a hyperboloid associated with zero pitch can be constructed, and a zero-pitch wrench can be selected from the corresponding family of generators. Similarly, if the fourth spring wrench is selected to have zero pitch, there are two zero-pitch wrenches in $\mathbb{W}_5$. If the fifth spring is also selected to be a simple spring, the sixth spring wrench must also be isotropic. Therefore, in each wrench space $\mathbb{W}_i$, one can always select a zero-pitch spring wrench. The stiffness $\mathbf{K}$ can be realized with six simple springs.

If the space $\mathbf{W}_n$ contains no zero-pitch wrench, then $n \geq 4$ and all subsequent spaces $\mathbb{W}_i$ $(i \geq n)$ contain no zero-pitch wrenches. The selection process based on the geometry of springs does not guarantee that the minimum number of screw springs are achieved for the realization even if in each step an available zero-pitch wrench is selected.

## B. Concurrent Spring Axes

As shown in Sections IV-A–IV-C, in selecting the first four springs, each wrench can be chosen to pass through any point. Thus, the first four springs can always be chosen to each pass through an arbitrarily selected point. All six springs can pass through a single point only if that point is the stiffness center [18].

## C. Nonfull Rank Stiffness Realization

The realization procedure for full-rank stiffness can be modified to realize a stiffness matrix of arbitrary rank. We consider a stiffness matrix $\mathbf{K}$ having rank $m\ (< 6)$. Let $\tilde{\mathbf{W}} = [\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2, \dots, \tilde{\mathbf{w}}_{6-m}]$ be a set of orthogonal basis vectors of the null space of $\mathcal{R}(\mathbf{K})$, $\mathcal{N}(\mathcal{R}(\mathbf{K}))$. Then, the matrix

$$\tilde{\mathbf{K}} = \mathbf{K} + \tilde{\mathbf{W}}\tilde{\mathbf{W}}^T \qquad (63)$$

is full-rank, symmetric, and PD. The matrix can be expressed as follows:

$$\mathbf{K} = \tilde{\mathbf{K}} - \tilde{\mathbf{W}}\tilde{\mathbf{W}}^T.$$

Since $\tilde{\mathbf{K}}$ is full rank, it has inverse, $\tilde{\mathbf{K}}^{-1} = \tilde{\mathbf{C}}$. It can be verified that

R(K)=N(W~TC~).

Thus, the conditions of Proposition 3 are satisfied. The procedure presented in Section III-B can be used for the realization of the nonfull rank stiffness matrix $\mathbf{K}$.

The first spring wrench can be chosen arbitrarily for the space $\mathcal{R}(\mathbf{K})$. If $\mathbf{n}$ spring wrenches $\mathbf{W}_n = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n]$ have been selected, the $(n+1)$th spring wrench can be selected from the column space of $\mathbf{K}_{n+1}$ defined by

$$\mathbf{K}_{n+1} = \mathbf{K} - \mathbf{W}_n \tilde{\mathbf{H}}_n \mathbf{W}_n$$

where

$$\tilde{\mathbf{H}}_n = \text{diag}\left(\frac{1}{\mathbf{w}_1^T \tilde{\mathbf{C}} \mathbf{w}_1}, \frac{1}{\mathbf{w}_2^T \tilde{\mathbf{C}} \mathbf{w}_2}, \dots, \frac{1}{\mathbf{w}_n^T \tilde{\mathbf{C}} \mathbf{w}_n}\right).$$

The value of spring constant is calculated using:

$$k_i = \frac{1}{\mathbf{w}_i^T \tilde{\mathbf{C}} \mathbf{w}_i}, i = 1, \dots, m. \qquad (64)$$

The procedure continues until all $m$ spring are selected. For each spring wrench, depending on the rank $\mathbf{K}_i$, the geometric properties of the wrench space $\mathbb{W}_i$ are described in Section V. It is noted that the $6 \times (6-m)$ matrix $\tilde{\mathbf{W}}$ used in (63) is not unique, and as a consequence, the full-rank matrix $\tilde{\mathbf{K}}$ is nonunique. The spring constants $k_i$ calculated in (64), however, are independent of the choice of $\tilde{\mathbf{W}}$. They depend only on the stiffness $\mathbf{K}$ and the spring wrench selected.

### D. Coordinate Frame

In describing the properties of each space $\mathbb{W}_i$, the coordinate frames based on the principal wrenches of $\mathbb{W}_i$ or principal twists of the reciprocal space $\mathbb{T}_i$ are used. These coordinate frames, in general, are not the same as the frame originally used to describe the compliance behavior. The geometric properties of the spaces $\mathbb{W}_i$ (such as the cylindroid or hyperboloid), however, are invariant.

### E. Spring Selection Sequence

The space of the $\mathbf{n}$th spring depends on the previously selected $(n-1)$ springs. The first spring can be chosen arbitrarily, which allows one to select the most important geometric features. With more springs selected, the available space shrinks. Thus, in the spring selection process, one should realize the most important aspects first. The process presented in the paper does not guarantee that one can select all springs under all constraints, but provides the entire set of options for spring selection once some springs are already determined.

### F. Compliance Realization With Serial Mechanisms

By duality [13], the process for stiffness realization with parallel mechanisms can be modified for compliance realization with serial mechanisms. In the process, the stiffness matrix $\mathbf{K}$ is replaced with the compliance matrix $\mathbf{C}$, and the spring wrenches $\mathbf{w}_i$ are replaced with the joint twists $\mathbf{t}_i$. The realization of an elastic behavior can be achieved with a serial mechanism based on the geometric properties of joint twists. For each joint in the mechanism, one can choose a joint twist from the allowable space based on the value of the pitch, joint axis direction or location of the joint.

## SECTION VI.

### Example

An example is provided to demonstrate the construction-based realization of an elastic behavior. In a manipulator-based coordinate frame, the stiffness matrix to be realized is as follows:

$$\mathbf{K}=\begin{bmatrix}16 & 5 & 5 & 3 & 8 & 4\\ 5 & 14 & -1 & 14 & 0 & 4\\ 5 & -1 & 7 & -4 & 3 & -2\\ 3 & 14 & -4 & 22 & -1 & 6\\ 8 & 0 & 3 & -1 & 5 & 2\\ 4 & 4 & -2 & 6 & 2 & 4\end{bmatrix}.$$

Note that, because the trace condition ($\mathrm{tr}(\mathbf{K}\Delta)=0$) is not satisfied, at least one screw spring must be used in realizing $\mathbf{K}$. In selecting each spring, one can specify the pitch, direction or a particular point for the spring axis to pass through. A wrench with zero-pitch represents a line spring, which is relatively easy to realize. If available in the allowable space, a zero-pitch wrench is selected at every opportunity in this example.

### A. First Spring Selection

The first spring wrench can be selected arbitrarily. Here, a line spring passing through the origin is selected:

$$\mathbf{w}_1=[0,1,0,0,0,0]^T.$$

The spring constant is calculated using (10) to be

$$k_1 = \frac{1}{\mathbf{w}_1^T \mathbf{C} \mathbf{w}_1} = 2.$$

The twist associated with $\mathbf{w}_1$ is:

$$\mathbf{t}_1 = \mathbf{C}\mathbf{w}_1 = [-0.2774, 0.2774, -0.2774, 0, 0.8321, -0.5547]^T.$$

The pitch of $\mathbf{t}_1$ is:

$$p_{t1} = 0.3846.$$

B. Second Spring Selection

The second spring wrench must be selected to satisfy the reciprocal condition (9), i.e., to be reciprocal to twist $\mathbf{t}_1$. The second spring wrench $\mathbf{w}_2$ can be chosen arbitrarily from the linear space $\mathcal{R}(\mathbf{K}_2)$ which is calculated using (19):

$$\mathbf{K}_2 = \mathbf{K} - k_1 \mathbf{w}_1 \mathbf{w}_1^T = \begin{bmatrix} 16 & 5 & 5 & 3 & 8 & 4 \\ 5 & 12 & -1 & 14 & 0 & 4 \\ 5 & -1 & 7 & -4 & 3 & -2 \\ 3 & 14 & -4 & 22 & -1 & 6 \\ 8 & 0 & 3 & -1 & 5 & 2 \\ 4 & 4 & -2 & 6 & 2 & 4 \end{bmatrix}.$$

Here, a line spring passing through the origin can again be selected. By the theory presented in Section IV-A, $\mathbf{w}_2$ lies on the plane $P_O$ of zero-pitch wrenches that pass through the origin as shown in Fig. 4(b). The normal of $P_O$ is determined by (24):

$$\mathbf{n}_O = [-0.2774, 0.2774, -0.2774]^T.$$

Any line passing through the origin in this plane can be chosen as the axis of the second spring wrench. For this case, w2 is selected as follows:

$$\mathbf{w}_2 = [0.7071, 0.7071, 0, 0, 0, 0]^T.$$

The spring constant is calculated using (10):

$$k_2 = \frac{1}{\mathbf{w}_2^T \mathbf{C} \mathbf{w}_2} = 4.$$

The $\mathbf{t}_2$ associated with $\mathbf{w}_2$ and compliance $\mathbf{C}$ is:

$$\mathbf{t}_2 = \mathbf{C}\mathbf{w}_2 = [0.5237,0,0,-0.1746,-0.9602,0.2182]^T.$$

The pitch and location of $\mathbf{t}_2$ are as follows:

$$p_{t2} = -0.0914, \mathbf{r}_2 = [0,0.1143,0.5029]^T.$$

C. Third Spring Selection

The third spring wrench must be selected to be reciprocal to the two twists $\mathbf{t}_1$ and $\mathbf{t}_2$ . The space for the third spring wrench is $\mathbb{W}_3$ =R(K3). The matrix $\mathbf{K}_3$ is calculated using (19) to be

$$\mathbf{K}_3 = \mathbf{K}_2 - k_2 \mathbf{w}_2 \mathbf{w}_2^T = \begin{bmatrix} 14 & 3 & 5 & 3 & 8 & 4 \\ 3 & 10 & -1 & 14 & 0 & 4 \\ 5 & -1 & 7 & -4 & 3 & -2 \\ 3 & 14 & -4 & 22 & -1 & 6 \\ 8 & 0 & 3 & -1 & 5 & 2 \\ 4 & 4 & -2 & 6 & 2 & 4 \end{bmatrix}.$$

Any wrench in the linear space $\mathcal{R}(\mathbf{K}_3)$ can be selected. Again, zero-pitch line springs are considered.

Since the pitches of $\mathbf{t}_1$ and $\mathbf{t}_2$ have opposite sign, in the cylindroid $\mathbb{T}_2$ formed by $\mathbf{t}_1$ and $\mathbf{t}_2$ , two twists that have zero-pitch can be determined using the process described in [2] for the principal frame. The obtained two principal twists in the original frame are as follows:

$$\tilde{\mathbf{t}}_1 = [-0.4813,0.0553,-0.0553,0.1420,0.9470,-0.2882]^T,$$
$$\tilde{\mathbf{t}}_2 = [0.6452,0.6803,-0.6803,-0.4418,-0.3892,-0.8083]^T.$$

The two zero-pitch twists are located at:

$$\tilde{\mathbf{r}}_1 = [-0.0365,0.1466,0.4637]^T,$$
$$\tilde{\mathbf{r}}_2 = [0.8146,-0.8221,-0.0494]^T.$$

The axes of the two twists can be expressed as

$$L_{ti}: \tilde{\mathbf{r}} = \tilde{\mathbf{n}}_i q + \tilde{\mathbf{r}}_i, q \in (-\infty, +\infty), i = 1,2$$

where $\tilde{\mathbf{r}} = [x, y, z]^T$, and $\tilde{\mathbf{n}}_i$ is the direction of $\tilde{\mathbf{t}}_i$.

As described in Section IV-B, any line intersecting the two lines $L_{t1}$ and $L_{t2}$ can be chosen as the axis of the third spring. Here, we choose the line passing through the two points associated with $\tilde{\mathbf{r}}_1$ and $\tilde{\mathbf{r}}_2$. Then the direction of $\mathbf{w}_3$ is determined to be

$$\mathbf{n}_3 = \frac{\tilde{\mathbf{r}}_1 - \tilde{\mathbf{r}}_2}{\|\tilde{\mathbf{r}}_1 - \tilde{\mathbf{r}}_2\|} = [-0.6133, 0.6980, 0.3697]^T.$$

The spring wrench is given by

$$\mathbf{w}_3 = \begin{bmatrix} \mathbf{n}_3 \\ \tilde{\mathbf{r}}_1 \times \mathbf{n}_3 \end{bmatrix} = [-0.6133, 0.6980, 0.3697, -0.2694, -0.2709, 0.0644]^T.$$

The spring constant calculated using (10) is as follows:

$$k_3 = \frac{1}{\mathbf{w}_3^T \mathbf{C} \mathbf{w}_3} = 1.0693.$$

### D. Fourth Spring Selection

To satisfy the reciprocal condition (9), the fourth spring wrench must be selected from the linear space $\mathbb{W}_4 = \mathcal{R}(\mathbf{K}_4)$. The rank-3 matrix $\mathbf{K}_4$ is as follows:

$$\mathbf{K}_4 = \mathbf{K}_3 - k_3 \mathbf{w}_3 \mathbf{w}_3^T = \begin{bmatrix} 13.5978 & 3.4577 & 5.2425 & 2.8233 & 7.8224 & 4.0423 \\ 3.4577 & 9.4791 & -1.2760 & 14.2011 & 0.2022 & 3.9519 \\ 5.2425 & -1.2760 & 6.8538 & -3.8935 & 3.1071 & -2.0255 \\ 2.8233 & 14.2011 & -3.8935 & 21.9224 & -1.0780 & 6.0186 \\ 7.8224 & 0.2022 & 3.1071 & -1.0780 & 4.9215 & 2.0187 \\ 4.0423 & 3.9519 & -2.0255 & 6.0186 & 2.0187 & 3.9956 \end{bmatrix}.$$

To describe the geometry of $\mathbb{W}_4$, the three principal wrenches of the 3-system can be calculated from any three independent columns of $\mathbf{K}_4$ using the methods presented in [25]–[27]. The three principal wrenches are as follows:

$$[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = \begin{bmatrix} 0.7315 & 0.0506 & -0.6800 \\ 0.6778 & -0.1617 & 0.7172 \\ 0.0737 & 0.9855 & 0.1526 \\ 0.9360 & -0.4646 & 1.1033 \\ 0.3252 & -0.0365 & -0.6525 \\ 0.4241 & -0.6537 & -0.3565 \end{bmatrix}. \quad (65)$$

The three principal wrenches intersect at the center of the system:

$$\mathbf{r}_e = [-0.1246, 0.3701, 0.4106]^T. \qquad (66)$$

The three principal pitches are as follows:

$$p_1 = 0.9364, p_2 = -0.6618, p_3 = -1.2726.$$

Here, wrenches with pitch $p = 0$ are again considered. All zero-pitch wrenches in $\mathbb{W}_4$ form a hyperboloid of one sheet. Since $p$ is in the range of (38), using the process described in Section IV-C, the hyperboloid in the principal frame can be expressed as:

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ (67)$$

where $a = \sqrt{|p_2 p_3|} = 0.9177$, $b = \sqrt{|p_1 p_3|} = 1.0916$, and $c = \sqrt{|p_1 p_2|} = 0.7872$. The hyperboloid has two families of generators $F_\theta$ and $F_\phi$ (as shown in Fig. 9), but only one belongs to the space $\mathbb{W}_4$. It is easy to verify that the following family of lines provides all zero-pitch spring wrenches in $\mathbb{W}_4$:

$$F_\phi : \mathbf{s}_\phi = [\begin{matrix} \mathbf{n}_\phi \\ \mathbf{r}_\phi \times \mathbf{n}_\phi \end{matrix}], \phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

where

$$\begin{aligned} \mathbf{n}_\phi &= \frac{1}{\cos^2\phi}[0.9177, -1.0916\sin 2\phi, 0.7872\cos 2\phi]^T, \\ \mathbf{r}_\phi &= [0, 1.0916\cos 2\phi, 0.7872\sin 2\phi]^T. \end{aligned}$$

Here, we choose one wrench from $F_\phi$ by setting $\phi = 0$. The unit spring wrench associated with $F_\phi$ is as follows:

$$\tilde{\mathbf{w}}_4 = [0.7590, 0, 0.6511, 0.7107, 0, -0.8286]^T.$$

Note the spring wrench w~4 is expressed in the principal frame. The screw transformation matrix between the principal and original frames is

$$\mathbf{P} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{RT} & \mathbf{R} \end{bmatrix}$$

where $\mathbf{R}^T$ is the orthogonal (rotation) matrix composed by the first three rows [the three unit direction vectors of the principal wrenches of (65)], and $\mathbf{T} = [\mathbf{r}_e \times]$ is the antisymmetric cross product matrix associated with the position vector $\mathbf{r}_e$ in (66). The spring wrench expressed in the original frame is as follows:

$$\mathbf{w}_4 = \mathbf{P}\tilde{\mathbf{w}}_4 = [0.1125, 0.9814, 0.1553, 1.4288, -0.1780, 0.0898]^T.$$

The spring constant calculated using (10) is as follows:

$$k_4 = \frac{1}{\mathbf{w}_4^T \mathbf{C} \mathbf{w}_4} = 6.8359.$$

The twist associated with $\mathbf{w}_4$ is as follows:

$$\mathbf{t}_4 = \mathbf{C}\mathbf{w}_4 = [0.0001, 0, 0.0661, 0.0970, -0.0250, -0.0774]^T.$$

E. Fifth Spring Selection

When four springs are selected, the fifth spring wrench must be selected to be reciprocal to the twists associated with the four spring wrenches. The space of all allowable fifth spring wrenches, $\mathbb{W}_5$ is a 2-system determined by the rank-2 matrix:

$$\mathbf{K}_5 = \mathbf{K}_4 - k_4 \mathbf{w}_4 \mathbf{w}_4^T = \begin{bmatrix} 13.5113 & 2.7029 & 5.1231 & 1.7244 & 7.9593 & 3.9732 \\ 2.7029 & 2.8945 & -2.3176 & 4.6151 & 1.3965 & 3.3492 \\ 5.1231 & -2.3176 & 6.6890 & -5.4099 & 3.2960 & -2.1208 \\ 1.7244 & 4.6151 & -5.4099 & 7.9667 & 0.6606 & 5.1411 \\ 7.9593 & 1.3965 & 3.2960 & 0.6606 & 4.7049 & 2.1280 \\ 3.9732 & 3.3492 & -2.1208 & 5.1411 & 2.1280 & 3.9404 \end{bmatrix}.$$

The space $\mathbb{W}_5$ is a cylindroid formed by any two independent columns of $\mathbf{K}_5$. To describe the geometry of the cylindroid, the two principal wrenches of $\mathbf{K}_5$ are calculated [25]–[27] to be

$$\mathbf{u}_1 = [0.8615, 0.4917, -0.1268, 0.6893, 0.4809, 0.5999]^T,$$
$$\mathbf{u}_2 = [-0.3614, 0.4181, -0.8334, 0.8436, -0.2537, 0.4259]^T.$$

The two principal pitches are as follows:

$$p_1 = 0.7542, p_2 = -0.7659.$$

The two principal wrenches intersect at the center of the system:

$$\mathbf{r}_e = [-0.1272, 0.7348, -0.1091]^T.$$

For each pitch value $p \in (-0.7659, 0.7542)$, there are two wrenches in the cylindroid. Two zero-pitch wrenches can be selected from $\mathbb{W}_5$. This time, the spring wrench is chosen based on its direction $\mathbf{n}$. As shown in Section IV-D, only those directions that are parallel to the principal plane are available [as shown in Fig. 11(c)]. Suppose that the direction that makes an angle of $\theta = \frac{\pi}{4}$ with the principal wrench $\mathbf{u}_1$ is desired. Then, using (61), the distance between the wrench and the principal plane is calculated to be

$$d_z = (p_2 - p_1)\cos\theta\sin\theta = -0.76.$$

The wrench pitch is calculated using (62):

$$p_5 = p_1\cos^2\theta + p_2\sin^2\theta = -0.0058.$$

In the principal frame, the direction and location of the spring are as follows:

$$\mathbf{n}_5 = [\cos\frac{\pi}{4}, \sin\frac{\pi}{4}, 0]^T, \mathbf{r}_5 = [0,0,d_z]^T.$$

The unit spring wrench in the principal frame is as follows:

$$\tilde{\mathbf{w}}_5 = \begin{bmatrix} \mathbf{n}_5 \\ p_5\mathbf{n}_5 + \mathbf{r}_5 \times \mathbf{n}_5 \end{bmatrix} = [0.7071,0.7071,0,0.5333,-0.5416,0]^T.$$

The spring wrench expressed in the original frame is as follows:

$$\mathbf{w}_5 = \mathbf{u}_1\cos\frac{\pi}{4} + \mathbf{u}_2\sin\frac{\pi}{4} = [0.8647,0.0520,0.4996,-0.1091,0.5194,0.1230]^T.$$

The spring constant is calculated using (10) to be

$$k_5 = \frac{1}{\mathbf{w}_5\mathbf{C}\mathbf{w}_5} = 16.2465.$$

F. Sixth Spring Identification

Once five springs are selected, the sixth spring is uniquely determined. The spring wrench $\mathbf{w}_6$ can be obtained from any nonzero column of $\mathbf{K}_6$, where

$$\mathbf{K}_6 = \mathbf{K}_5 - k_5\mathbf{w}_5\mathbf{w}_5^T = \begin{bmatrix} 1.3643 & 1.9721 & -1.8956 & 3.2567 & 0.6623 & 2.2452 \\ 1.9721 & 2.8505 & -2.7398 & 4.7072 & 0.9575 & 3.2452 \\ -1.8956 & -2.7398 & 2.6336 & -4.5245 & -0.9202 & -3.1193 \\ 3.2567 & 4.7072 & -4.5245 & 7.7734 & 1.5811 & 5.3590 \\ 0.6623 & 0.9575 & -0.9202 & 1.5811 & 0.3215 & 1.0900 \\ 2.2452 & 3.2452 & -3.1193 & 5.3590 & 1.0900 & 3.6946 \end{bmatrix}.$$

The unit spring wrench is calculated to be

$$\mathbf{w}_6 = [0.4463, 0.6452, -0.6201, 1.0654, 0.2167, 0.7345]^T.$$

The spring constant is calculated using (10) to be

$$k_6 = \frac{1}{\mathbf{w}_6^T \mathbf{C} \mathbf{w}_6} = 6.8484.$$

The pitch of the spring is as follows:

$$p_6 = 0.1598.$$

To ensure that the obtained $\mathbf{w}_i$ and $k_i$ realize the given stiffness matrix $\mathbf{K}$, it is verified that

$$\mathbf{K} = \sum_{i=1}^{6} k_i \mathbf{w}_i \mathbf{w}_i^T.$$

# SECTION VII.

## Summary

In this paper, the realization of an arbitrary spatial elastic behavior using parallel and serial mechanisms is addressed. A set of necessary and sufficient conditions for a mechanism to realize a given spatial compliance is presented and physical interpretations of the realization conditions are provided. The methods presented in this paper allow one to synthesize any compliant behavior by selecting each elastic component in a parallel or serial mechanism based on the geometry of the corresponding space of allowable candidates. Since the space available for each spring is determined by the kinematic restrictions alone, the requirements on the mechanism geometry are independent of the selection of joint stiffnesses. If the synthesis procedure is followed, all joint stiffnesses must be positive. As such, the mechanism kinematic properties and the joint stiffness properties are decoupled.

There are an infinite number of mechanisms that realize a specified compliance. The selection of mechanism design parameters depends on the kinematic constraints and specific application. The approach presented in this paper provides possible geometric options so that one can select each element based on context specific needs and task requirements. The methods identified can be used for mechanisms having VSAs to realize a desired compliance behavior by changing the mechanism configuration and joint stiffnesses.

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