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Fiaz Ahmad Bhatti
National College of Business Administration and Economics

Gholamhossein G. Hamedani
Marquette University, gholamhoss.hamedani@marquette.edu

Azeem Ali
University of Veterinary and Animal Sciences

Sedigheh Mirzaei Salehabadi
St. Jude Childrens' Research Hospital

Munir Ahmad
National College of Business Administration and Economics

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CHARACTERIZATIONS AND RELIABILITY MEASURES OF THE GENERALIZED LOG BURR XII DISTRIBUTION

Fiaz Ahmad Bhatti[1], G. G. Hamedani[2], Azeem Ali[3], Sedigheh Mirzaei Salehabadi[4] and Munir Ahmad[1]

[1]National College of Business Administration and Economics
Lahore, Pakistan
e-mail: fiazahmad72@gmail.com
munirahmaddr@yahoo.co.uk

[2]Marquette University
Milwaukee, WI 53201-1881, U. S. A.
e-mail: g.hamedani@mu.edu

[3]University of Veterinary and Animal Sciences
Lahore, Pakistan
e-mail: azeem.ali@uvas.edu.pk

[4]St. Jude Children’s Research Hospital
Memphis, TN, U. S. A.
e-mail: sedigheh.mirzaei@stjude.org

Abstract

In this paper, we derive the generalized log Burr XII (GLBXII) distribution [2] from the generalized Burr-Hatke differential equation. We characterize the GLBXII distribution via innovative techniques.

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We derive various reliability measures (series and parallel). We also authenticate the potentiality of the GLBXII model via economics applications. The applications of characterizations and reliability measures of the GLBXII distribution in different disciplines of science will be profitable for scientists.

1. Introduction

The GLBXII distribution is flexible due to its applications in areas such as survival analysis, economics and environmental sciences. The study of the GLBXII distribution focuses on the following motivations: (i) to derive some statistical properties such as (a) characterizations and (b) reliability measures (series and parallel). (ii) We reveal the adequacy of the GLBXII model among its class and competitor models using a real data set.

The contents of the article are as follows: Section 2 derives the GLBXII distribution [2] from the generalized Burr-Hatke differential equation (GBHE). Section 3 provides certain characterizations of the GLBXII distribution via innovative techniques. In Section 4, we derive various reliability measures. Section 5 considers an application of the GLBXII distribution to real data set. In Section 6, we conclude the article.

2. The GLBXII Distribution

We derive the GLBXII distribution with a new method such as GBHE.

2.1. Burr-Hatke differential equation [3]

Here, we derive the GLBXII distribution from the GBHE. The GBHE is

$$\frac{d[\ln F(x)]}{dx} = \zeta(x)[F(x)]^{m-1} + \upsilon(x)[F(x)]^{n-1}, \tag{1}$$

where $F(x)$ is the cumulative distribution function (cdf).

Taking $m = 2,\ n = 1,\ \zeta(x) = -\omega[x,\ F(x)],\ \upsilon(x) = \omega[x,\ F(x)],$ we obtain

$$F(x) = \frac{\exp\left\{\int \omega[x,\ F(x)]dx\right\}}{1 + \exp\left\{\int \omega[x,\ F(x)]dx\right\}}. \tag{2}$$

Taking $\omega(x,\ F(x)) = \dfrac{\dfrac{2ap}{bx}\left(\dfrac{\ln x}{b}\right)^{2a-1}\left[1 + \left(\dfrac{\ln x}{b}\right)^{2a}\right]^{-1}}{1 - \left[1 + \left(\dfrac{\ln x}{b}\right)^{2a}\right]^{-p}}$ in (2), we arrive at

$$F(x) = 1 - \left[1 + \left(\frac{\ln x}{b}\right)^{2a}\right]^{-p}, \quad x \geq 1, \tag{3}$$

which is cdf of the GLBXII model. The probability density function (pdf) of the GLBXII model is

$$f(x) = \frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{(2a-1)}\left[1 + \left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}, \ x > 1,\ a > 0,\ b > 0,\ p > 0, \tag{4}$$

where a, b and p are parameters.

If $X \sim GLBXII(a,\ b,\ p),$ reverse failure rate "$r_F(x) = \dfrac{f(x)}{F(x)}$" and Mills ratio "$m(x) = \dfrac{1 - F(x)}{f(x)}$" of X are given, respectively, by (for $x > 1$)

$$r_F(x) = \frac{\dfrac{2ap}{bx}\left(\dfrac{\ln x}{b}\right)^{(2a-1)}\left[1 + \left(\dfrac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}}{1 - \left[1 + \left(\dfrac{\ln x}{b}\right)^{2a}\right]^{-p}}$$

and

$$m(x) = \frac{bx}{2ap}\left(\frac{b}{\ln x}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right].$$

3. Characterizations

In order to develop a stochastic function for a certain problem, it is necessary to know whether the function fulfills the theory of specific underlying probability distribution. It is required to study the characterizations of specific probability distribution. Different characterization techniques have been developed.

In this section, the GLBXII distribution is characterized via: (i) doubly truncated moments; (ii) reverse hazard function and (iii) Mills ratio.

3.1. Doubly truncated moment

Here, we characterize the GLBXII distribution via doubly truncated moment of a function of *X*.

Proposition 3.1.1. *Let* $X : \Omega \to (1, \infty)$ *be a continuous random variable* (*r.v.*) *with cdf* $F(x)$. *Then for* $p > 1$, X *has cdf* (4) *if and only if*

$$E\left[\left(\frac{\ln X}{b}\right)^{2a} \middle| x \leq X \leq y\right]$$

$$= \frac{p}{p-1}\left[\frac{1}{p} + \frac{r(x)r(y)(\ln x/b)^{2a}}{(h(y)r(x)-h(x)r(y))\overline{F}(y)} - \frac{r(x)r(y)(\ln y/b)^{2a}}{(h(y)r(x)-h(x)r(y))\overline{F}(x)}\right]. \quad (5)$$

Proof. The hazard rate and the reverse hazard rate functions satisfy the following equations:

$$h(x) = \frac{f(x)}{1-F(x)},\ h(x) = \frac{f(x)}{\overline{F}(x)},\ r(x) = \frac{f(x)}{F(x)},\ \frac{r(x)}{h(x)} = \frac{F(x)}{\overline{F}(x)},\ \frac{h(x)}{r(x)} = \frac{\overline{F}(x)}{F(x)}.$$

The following lemmas can be stated in view of Nofal [4]:

L.1. $F(y) - F(x) = F(y)\overline{F}(x) - F(x)\overline{F}(y)$,

L.2. $\dfrac{F(y) - F(x)}{\overline{F}(x)\overline{F}(y)} = \dfrac{h(y)r(x) - h(x)r(y)}{r(x)r(y)}$,

L.3.

$$\frac{xf(x) - yf(y)}{\overline{F}(y)\overline{F}(x)}$$

$$= \frac{[xh(x)r(x)r(y) + xh(y)h(x)r(x) - yh(y)r(x)r(y) - yh(y)h(x)r(y)]}{r(x)r(y)}.$$

If X has pdf (5), then

$$E\left[\left(\frac{\ln X}{b}\right)^{2a} \middle| x \le X \le y\right] = \frac{\int_x^y uf(u)du}{F(y) - F(x)},$$

$$E\left[\left(\frac{\ln X}{b}\right)^{2a} \middle| x \le X \le y\right] = [\overline{F}(x) - \overline{F}(y)]^{-1} \int_x^y uf(u)du,$$

$$E\left[\left(\frac{\ln X}{b}\right)^{2a} \middle| x \le X \le y\right]$$

$$= [\overline{F}(x) - \overline{F}(y)]^{-1} \int_x^y \left(\frac{\ln x}{b}\right)^{2a} \frac{2ap}{bu}\left(\frac{\ln u}{b}\right)^{2a-1}\left[1 + \left(\frac{\ln u}{b}\right)^{2a}\right]^{-(p+1)} du.$$

Upon integration and simplification, we have

$$E\left[\left(\frac{\ln X}{b}\right)^{2a} \middle| x \le X \le y\right]$$

$$= \frac{p\left(\left(\frac{\ln y}{b}\right)^{2a} \overline{F}(y) - \left(\frac{\ln x}{b}\right)^{2a} \overline{F}(x)\right) - (\overline{F}(x) - \overline{F}(y))}{(1 - p)(\overline{F}(x) - \overline{F}(y))},$$

$$E\left[\left(\frac{\ln X}{b}\right)^{2a}\middle| x \le X \le y\right]$$

$$= \frac{p\left(\left(\frac{\ln y}{b}\right)^{2a}\overline{F}(y) - \left(\frac{\ln x}{b}\right)^{2a}\overline{F}(x)\right)}{(1-p)(\overline{F}(x) - \overline{F}(y))} - \frac{1}{(1-p)}.$$

Using the above lemmas, we have

$$E\left[\left(\frac{\ln X}{b}\right)^{2a}\middle| x \le X \le y\right]$$

$$= \frac{p}{p-1}\left[\frac{1}{p} + \frac{r(x)r(y)(\ln x/b)^{2a}}{(h(y)r(x) - h(x)r(y))\overline{F}(y)} - \frac{r(x)r(y)(\ln y/b)^{2a}}{(h(y)r(x) - h(x)r(y))\overline{F}(x)}\right].$$

Conversely, if (5) holds, then

$$\int_x^y \left(\frac{\ln u}{b}\right)^{2a} f(u)du = \frac{p}{(p-1)}\left(\left(\frac{\ln y}{b}\right)^{2a}\overline{F}(y) - \left(\frac{\ln x}{b}\right)^{2a}\overline{F}(x)\right)$$

$$+ \frac{1}{(1-p)}(\overline{F}(y) - \overline{F}(x)).$$

Differentiating the above equation with respect to y, we obtain the solution as

$$f(y)\left[\left(\frac{\ln y}{b}\right)^{2a} - p\left(\frac{\ln y}{b}\right)^{2a} + 1 + p\left(\frac{\ln y}{b}\right)^{2a}\right]$$

$$= \left[\frac{2ap}{by}\left(\frac{\ln y}{b}\right)^{2a-1}(1 - F(y))\right].$$

Integrating the above equation, we have

$$F(y) = 1 - \left[1 + \left(\frac{\ln y}{b}\right)^{2a}\right]^{-p}, \quad y \ge 1,\ a > 0,\ b > 0,\ p > 0.$$

3.2. Characterization based on reverse hazard function

Here, we characterize the GLBXII distribution via reverse hazard function of X.

Definition 3.2.1. Let $X : \Omega \to (1, \infty)$ be a continuous random variable with twice differentiable cdf $F(x)$, pdf $f(x)$ and the reverse hazard function $r_F(x)$ satisfying the differential equation

$$\frac{d}{dx}[\ln f(x)] = \frac{r'_F(x)}{r_F(x)} + r_F(x). \tag{6}$$

Proposition 3.2.2. *The continuous random variable* $X : \Omega \to (1, \infty)$ *has pdf* $f(x)$ *if and only if its reverse hazard function* $r_F(x)$ *satisfies the differential equation*

$$r'_F(x)\frac{bx}{2ap}\left(\frac{\ln x}{b}\right)^{-2a+1} = \frac{r_F(x) + \frac{(2a-1)}{x\ln x} - \frac{2a(p+1)(\ln x)^{2a-1}}{x(b^{2a} + (\ln x)^{2a})} - \frac{1}{x}}{\left[1 + \left(\frac{\ln x}{b}\right)^{2a}\right]^{(p+1)} - 1 - \left(\frac{\ln x}{b}\right)^{2a}}. \tag{7}$$

Proof. If X has pdf (4), then (7) surely holds. Now, if (7) holds, then

$$\frac{d}{dx}\left[\left(1 - \left[1 + \left(\frac{\ln x}{b}\right)^{2a}\right]^{-p}\right) r_F(x)\right]$$

$$= \frac{d}{dx}\left(\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1 + \left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}\right),$$

or

$$r_F(x) = \frac{\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1 + \left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}}{1 - \left[1 + \left(\frac{\ln x}{b}\right)^{2a}\right]^{-p}},$$

which is the reverse hazard function for the GLBXII distribution.

3.3. Characterization based on Mills ratio

Here, we characterize the GLBXII distribution via Mills ratio.

Definition 3.3.1. Let $X : \Omega \to (1, \infty)$ be a continuous random variable with twice differentiable cdf $F(x)$, pdf $f(x)$ and Mills ratio, $m(x)$, satisfying the equation

$$\frac{d[\ln f(x)]}{dx} + \frac{m'(x)+1}{m(x)} = 0. \tag{8}$$

Proposition 3.3.2. *Let* $X : \Omega \to (1, \infty)$ *be a continuous random variable. Then the pdf of X is* (4) *if and only if the Mills ratio satisfies the first order differential equation*

$$m'(x) + \left\{\frac{-1}{x} + \frac{(2a-1)}{x\ln x} - \frac{2a(p+2)(\ln x)^{2a-1}}{b^{2a}x} \times \left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-1}\right\} m(x) + 1 = 0. \tag{9}$$

Proof. If X has pdf (4), then (9) surely holds. Now, if (9) holds, then

$$\frac{d}{dx}\left[m(x)\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}\right]$$

$$= \frac{d}{dx}\left[\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-p}\right],$$

or

$$m(x) = \frac{x}{2ap}\frac{b^{2a}}{(\ln x)^{2a-1}}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right],$$

which is Mills ratio of the GLBXII distribution.

4. Multicomponent Stress Strength Reliability System Model

A system which has at least one component functioning is called a *multicomponent system*. A model in which the probability of the strengths X_i, $i = 1, 2, \ldots, \kappa$ of κ identical components is more than the stress Y imposed on the components, is called the multicomponent stress-strength reliability model. We have

$$R_{s,\kappa} = P[\text{strengths}\,(X_i,\ i = 1, 2, \ldots, \kappa) > \text{stress}\,(Y)],$$

$$R_{s,\kappa} = P[\text{at least minimum "}s\text{" of } (X_i,\ i = 1, 2, \ldots, \kappa) \text{ exceed } Y],$$

$$R_{s,\kappa} = \sum_{l=s}^{\kappa} \binom{\kappa}{\ell} \int_0^{\infty} [1 - F(y)]^l [F(y)]^{\kappa - l} dG(y)$$

(Bhattacharyya and Johnson [1]).

We study the two types of multicomponent system: (i) series system and (ii) parallel system.

4.1. Multicomponent stress strength-series reliability system model

A system in which all the components are interconnected in a series is called the *series system*. The whole system will not function if any of its components fails.

4.1.1. Multicomponent stress strength-series reliability system model for *n* identical components

Let X_i, $i = 1, 2, \ldots, n$ be the strengths and Y be the stress. Let $X_i \sim \text{GLBXII}(a, b, p_1)$ and $Y \sim \text{GLBXII}(a, b, p_2)$ with unknown shape parameters p_1 and p_2. The strengths X_i, $i = 1, 2, \ldots, n$ and the stress Y are independently distributed (i.d.). Multicomponent stress strength-series reliability system model for n identical components for the GLBXII distribution is

$$R_s = \int_{-\infty}^{\infty} (1 - F_X(y))^n f_Y(y) dy,$$

$$R_{ns} = \int_1^{\infty} \frac{2ap_2}{by}\left(\frac{\ln y}{b}\right)^{2a-1}\left[1 + \left(\frac{\ln y}{b}\right)^{2a}\right]^{-np_1 - p_2 - 1} dy.$$

Integrating the above equation, we have

$$R_{ns} = \frac{p_2}{(np_1 + p_2)}.$$

4.1.2. Multicomponent stress strength-series reliability system model for *n* different components

Let X_i, $i = 1, 2, \ldots, n$ be the strengths and Y be the stress. Let $X_i \sim$ GLBXII(a, b, p_1) and $Y \sim$ GLBXII(a, b, p), with unknown shape parameters $p_1, p_2, \ldots, p_n, p$ and common scale parameter b. The strengths X_i, $i = 1, 2, \ldots, n$ and the stress Y are i.d. Multicomponent stress strength-series reliability system model for n different components for the GLBXII distribution is

$$R_s = \int_{-\infty}^{\infty} (1 - F_{X_1}(y))(1 - F_{X_2}(y)) \cdots (1 - F_{X_n}(y)) f_Y(y) dy$$

$$R_s = \int_1^{\infty} \frac{2ap}{by}\left(\frac{\ln y}{b}\right)^{2a-1}\left[1 + \left(\frac{\ln y}{b}\right)^{2a}\right]^{-p - p_1 - p_2 - p_3 \cdots - p_n - 1} dy.$$

Integrating the above equation, we have

$$R_s = \frac{p}{(p + p_1 + p_2 + p_3 + \cdots + p_k)}.$$

4.2. Multicomponent stress strength-parallel reliability system model

A system in which all the components are interconnected in parallel is called the *parallel system*. The whole system will work if at least one of its components works.

4.2.1. Multicomponent stress strength-parallel reliability system model for *n* identical components

Let $X_i,\ i = 1, 2, ..., n$ be the strengths and Y be the stress. Let $X_i \sim$ GLBXII$(a, b, p_1), i = 1, 2, ..., n$ and $Y \sim$ GLBXII(a, b, p_2) with unknown shape parameters b and p_2. The strengths $X_i,\ i = 1, 2, ..., n$ and the stress Y are i.i.d. Multicomponent stress strength-parallel reliability system model for n identical components for GLBXII distribution is

$$R_\pi = 1 - \int_{-\infty}^{\infty} (F_X(y))^n dF_Y(y)$$

$$R_\pi = 1 - p_2 \int_1^{\infty} \left(1 - \left[1 + \left(\frac{\ln y}{b}\right)^{2a}\right]^{-p_1}\right)^n$$

$$\cdot \left(\frac{2a}{by}\left(\frac{\ln y}{b}\right)^{2a-1}\left[1 + \left(\frac{\ln y}{b}\right)^{2a}\right]^{-p_2-1}\right) dy.$$

Integrating the above equation, we have

$$R_\pi = 1 - \frac{p_2}{p_1} B\left(n + 1, \frac{p_2}{p_1}\right).$$

4.2.2. Multicomponent stress strength-parallel reliability system model for two different components

Let $X_i,\ i = 1, 2$ be the strengths and Y be the stress. Let $X_i \sim$ GLBXII$(a, b, p_i), i = 1, 2$ and $Y \sim$ GLBXII(a, b, p), with the unknown shape parameters p_1, p_2 and p. The strengths X_i and the stress Y are i.d. Multicomponent stress strength-parallel reliability system model for two different components for the GLBXII distribution is

$$R_{2\pi} = 1 - \int_{-\infty}^{\infty} [F_{X_1}(y) F_{X_2}(y)] dF_Y(y),$$

$$R_{2p} = 1 - \int_1^{\infty} \left(1 - \left[1 + \left(\frac{\ln y}{b}\right)^{2a}\right]^{-p_1}\right)\left(1 - \left[1 + \left(\frac{\ln y}{b}\right)^{2a}\right]^{-p_2}\right)$$

$$\cdot \left(\frac{2ap}{by}\left(\frac{\ln y}{b}\right)^{2a-1}\left[1 + \left(\frac{\ln y}{b}\right)^{2a}\right]^{-p-1}\right).$$

Integrating the above equation, we have

$$R_{2p} = p\left[\frac{1}{(p + p_1)} + \frac{1}{(p + p_2)} - \frac{1}{(p + p_1 + p_2)}\right].$$

5. Statistical Inference

First, we adopt the maximum likelihood estimation technique for the GLBXII parameters. We explain the utility of the GLBXII model among its family and class using a real data set.

Let $\xi = (a, b, p)^T$ be the unknown parameter vector. Then the log-likelihood function $\ell(\xi)$ for the GLBXII distribution is

$$\ell = \ell(\xi) = n \ln 2 + n \ln 2a + n \ln \pi - 2na \ln b - \sum_{i=1}^{n} \ln x_i$$

$$+ (2a - 1)\sum_{i=1}^{n} \ln(\ln x_i) - (p + 1)\sum_{i=1}^{n} \ln\left[1 + \left(\frac{\ln x_i}{b}\right)^{2a}\right]. \quad (10)$$

We can compute the MLEs of a, b and p by solving equations $\frac{\partial \ell}{\partial a} = 0,$ $\frac{\partial \ell}{\partial b} = 0$ and $\frac{\partial \ell}{\partial p} = 0$ simultaneously, either directly or using quasi-Newton procedures in R software.

5.1. Data applications for selection and comparison

We consider an application to consumption expenditure to authenticate the potentiality of the GLBXII distribution. We compare the GLBXII distribution with the sub-models such as log Burr XII (LBXII), log Lomax (LLM), log-logistic (Log-log) and competitive models such as generalized Burr XII (GBXII), Burr XII (BXII), modified Burr XII (MBXII) and modified Burr III (MBIII) distributions. For selection of the optimum distribution, we compute the estimate of various model selection criteria such as "likelihood ratio statistics $(-2\hat{\ell})$, Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC)" and goodness of fit statistics (GOFs) such as "Cramer-von Mises (W^*), Anderson-Darling (A^*) and Kolmogorov-Smirnov statistics (K-S)" with p-values for all competing models. We estimate the MLEs for the parameters and their standard errors (SEs) in parentheses.

Data set I. Pakistan household consumption expenditures data: Pakistan household consumption expenditures from 1967 to 2016 data are: 227.12, 216.85, 197.98, 186.87, 185.00, 173.33, 141.42, 133.20, 139.25, 118.71, 106.58, 84.26, 72.69, 61.48, 54.07, 55.15, 55.74, 47.66, 44.82, 46.75, 46.15, 43.91, 36.90, 37.17, 34.06, 31.01, 29.51, 28.99, 28.69, 25.05, 25.27, 25.53, 25.29, 23.19, 25.26, 22.97, 19.69, 16.52, 14.35, 12.00, 10.74, 9.601, 7.360, 4.952, 7.359, 8.437, 8.118, 6.947, 6.259, 6.004.

Table 1 reports the MLEs (SEs in parentheses) and measures such as W^*, A^*, K-S (p-values). Table 2 displays the values of statistics $-2\hat{\ell}$, AIC, CAIC, BIC, and HQIC.

We infer from the application results (Tables 1 and 2) that our GLBXII model is the best fitted, with the smallest values for all criteria and maximum p-value. Figure 1 concludes that the GLBXII distribution is closely fitted to empirical data.

Table 1. MLEs (SEs) and W^*, A^*, K-S (p-values) for consumption expenditure data

Model	a	b	p	c	W^*	A^*	K-S (p-value)
GLBXII	1.8774 (0.2147)	12.6256 (13.8387)	79.8037 (315.6443)	…	0.0582	0.4841	**0.0733 (0.9332)**
LBXII	8.0464 (28.4313)	1	0.05106 (0.1804)	…	0.2545	1.5042	0.3609 (2.434e-06)
LLM	1	11094.00 (1042.5468)	3124.06 (529.1556)	…	0.0849	0.6304	0.3763 (7.197e-07)
GBXII	1.4608 (0.4358)	44.0855 (42.5064)	1.2649 (1.13745)	…	0.0641	0.5697	0.0772 (0.9041)
BXII	5.1847 (18.8125)	1	0.0543 (0.1971)	…	0.0849	0.6303	0.3763 (7.188e-07)
MBXII	0.0029 (0.0007)	1.7999 (0.1804)	…	0.0045 (0.0014)	0.0700	0.6084	0.1283 (0.3526)
MBIII	91.7318 (154.0701)	1.3410 (0.3850)	…	51.8560 (142.8354)	0.0652	0.5703	0.0823 (0.8599)

Table 2. $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC for consumption expenditure data

Model	$-2\hat{\ell}$	AIC	CAIC	BIC	HQIC
GLBXII	**502.9096**	**508.9096**	**509.4313**	**514.6456**	**511.0939**
LBXII	596.5412	600.5412	600.7965	604.3652	601.9974
LLM	581.9256	585.9256	586.181	589.7497	587.3819
GBXII	507.0756	513.0757	513.5974	518.8117	515.26
BXII	581.9124	585.9123	586.1676	589.7364	587.3685
MBXII	508.6754	514.6754	515.1971	520.4115	516.8597
MBIII	506.7518	512.7518	513.2735	518.4879	514.9361

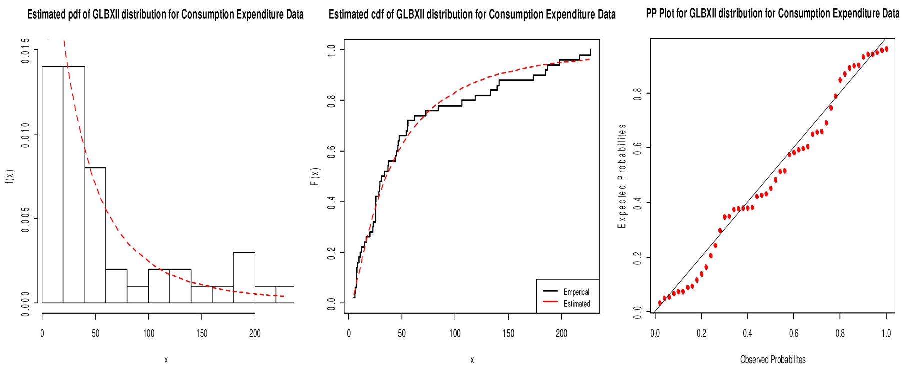

Figure 1. Fitted (left) pdf, (center) cdf and (right) PP plot of the GLBXII distribution for consumption expenditure data.

6. Conclusion

We derive new and interesting statistical properties of the GLBXII distribution such as characterizations and reliability measures. We demonstrate the potentiality and utility of the GLBXII distribution by considering an application to consumption expenditure data. We apply various model selection criteria and graphical tools to examine the adequacy of the GLBXII distribution. We infer that the GLBXII model is empirically suitable for consumption expenditure data. Therefore, the GLBXII model is a flexible, reasonable and parsimonious to other existing distributions. Further, as a perspective of future projects, we may consider several intensive subjects: (i) statistical inferences using different sampling schemes such as simple random sampling (SRS) and rank set sampling (RSS); (ii) reliability analysis using SRS and RSS; (iii) Bayesian estimation of the GLBXII parameters via SRS and RSS under different loss functions; and (iv) the study of the complexity of the GLBXII via Bayesian methods.

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References

[1] G. K. Bhattacharyya and R. A. Johnson, Estimation of reliability in a multicomponent stress-strength model, J. Amer. Statist. Assoc. 69 (1974), 966-970.

[2] F. A. Bhatti, A. Ali, G. G. Hamedani and M. Ahmad, On generalized log Burr XII distribution, Pakistan Journal of Statistics and Operation Research 14 (2018), 615-643.

[3] A. I. Maniu and V. G. Voda, Generalized Burr-Hatke equation as generator of a homographic failure rate, Journal of Applied Quantitative Methods 3 (2008), 215-222.

[4] Z. M. Nofal, Characterization of beta and gamma distribution based on the doubly truncated mean function, J. Appl. Statist. Sci. 19(2) (2011), 159-168.