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Abstract
This paper presents a method for the design of PID-type controllers, including those augmented by a filter on the D element, satisfying a required gain margin and an upper bound on the (complementary) sensitivity for a finite set of plants. Important properties of the method are: (i) it can be applied to plants of any order including non-minimum phase plants, plants with delay, plants characterized by quasi-polynomials, unstable plants and plants described by measured data, (ii) the sensors associated with the PI terms and the D term can be different (i.e., they can have different transfer function models), (iii) the algorithm relies on explicit equations that can be solved efficiently, (iv) the algorithm can be used in near real-time to determine a controller for on-line
modification of a plant accounting for its uncertainty and closed-loop specifications, (v) a single plot can be generated that graphically highlights tradeoffs among the gain margin, (complementary) sensitivity bound, low-frequency sensitivity and high-frequency sensor noise amplification, and (vi) the optimal controller for a practical definition of optimality can readily be identified.

Keywords
PID control, Robustness, Gain and phase margin, Sensitivity

1. Introduction
Methods for tuning PI and PID controllers have been reported widely and active research continues due to the extensive use of such controllers in industry. The tuning methods can be divided into two main categories, those emphasizing gain and phase margin specifications and those emphasizing sensitivity specifications. Design techniques based on gain and phase margin specifications include those by Ho, Hang, and Cao (1995a) and Ho, Hang, and Zhou (1995b). They developed simple analytical formulae to tune PI and PID controllers for commonly used first-order and second-order plus dead time plant models to meet gain and phase margin specifications. Ho, Hang, and Zhou (1997) and Ho, Lim, and Xu (1998) presented tuning formulae for the design of PID controllers that satisfy both robustness and performance requirements. Crowe and Johnson (2002) presented an automatic PI control design algorithm to satisfy gain and phase margins based on a converging algorithm. Suchomski (2001) developed a tuning method for PI and PID controllers that can shape the nominal stability, transient performance, and control signal to meet gain and phase margins.

Although gain and phase margin specifications are classical measures of robustness, they may fail to guarantee a reasonable bound on the sensitivity. This point was considered by several researchers. Ogawa (1995) used the QFT-framework to propose a PI design technique that satisfies a bound on the sensitivity for an uncertain plant. Poulin and Pomerleau (1999) developed a PI design methodology for integrating processes that bounds the maximum peak resonance of the closed-loop transfer function. The peak resonance constraint is equivalent to bounding the complementary sensitivity, which can be converted to bounding the sensitivity. Cavicchi (2001) presented a design method for bounding the sensitivity while achieving desired steady-state performance. Although the method can be applied to measured data, plant uncertainty is not considered, and the procedure fits only simple compensation structures. Crowe and Johnson (2001) reported a design approach to find a PI/PID controller that bounds the sensitivity while satisfying a phase margin condition. Kristiansson and Lennartson (2002) emphasized the need to bound the sensitivity and complementary sensitivity. They suggested the use of an optimization routine to design PI/PID controllers with low-pass filters on the derivative gain to optimize for control efforts, while rejecting disturbances and bounding the sensitivity. They also gave tuning rules for non-oscillatory stable plants or plants with a single integrator. Astrom, Panagopoulos, and Hagglund (1998) and Panagopoulos, Astrom, and Hagglund (2002) described a numerical method for designing PI controllers based on optimization of load disturbance rejection with constraints on sensitivity and weighting of the setpoint response.

Other tuning methods have been proposed. Yeung, Wong, and Chen (1998) presented a non-trial and error graphical design technique for controller design of the lead-lag structure that enables simultaneous fulfillment of gain margin, phase margin and crossover frequency specifications. Guillermo, Silva, and Bhattacharyya (2002) developed a theorem to calculate all stabilizing PID controllers for first-order delayed plants. However, uncertainty, sensitivity and margins were not discussed.

These papers and others apply gain and phase margin constraints in finding PI and PID controller designs. Some add limitations on the (complementary) sensitivity. However, there are several differences between approaches reported in the literature and the idea proposed here. First, the approach presented here bounds the sensitivity
of the closed-loop transfer function for all frequencies, not just at the crossover frequencies where the gain and phase margins are satisfied. (It is possible that the gain and phase margin conditions are met with a given PI/PID design, but the sensitivity can be very high.) Second, the approach accounts for plant uncertainty, with the controller design satisfying the specifications for a set of plants. Third, the algorithm can be applied to plants of any order including plants with pure delay, unstable plants, and plants given by measured data. Fourth, it allows for different sensor models for the PI terms and the D term. Fifth, the approach relies on explicit equations, rather than optimization routines, to determine the set of all possible controllers. Sixth, since the algorithm uses explicit equations that can be solved efficiently, it is very fast and suitable for near-real time implementation. Seventh, it is possible to extend the method to design cascaded loop and other control structures.

2. Problem statement and motivation

Consider the open-loop transfer function, \( L(s) \),

\[
L(s) = a[P_1(s) + bP_2(s)].
\]

If \( P_1(s) = (1 + k_i/s)P(s) \) and \( P_2(s) = sP(s) \), (1) corresponds to the open-loop transfer function of plant, \( P(s) \), with a PID controller

\[
C(s) = ak_i s + a + abs.
\]

It is possible to use different sensors for the D term and for the PI terms. In this case, if the transfer function of the sensor associated with the D term is taken as \( H(s) \) (with \( H(s) \) being a delay and/or a low-pass filter to decrease noise) and the transfer function of the sensor for the PI terms is unity, then \( P_1(s) = (1 + k_i/s)P(s) \), \( P_2(s) = sH(s)P(s) \), and the controller can be written as

\[
C(s) = ak_i s + a + absH(s).
\]

The gain and phase margin conditions, the typical measures of robustness, are replaced by a condition on the closed-loop sensitivity inequality,

\[
11 + kL(s) \leq M \text{ for } s= j\omega, \forall \omega \geq 0, k \in [1, K],
\]

where the sensitivity bound \( M > 1 \) and the gain uncertainty of the plant, \( k \), is in the interval \([1, K]\). Yaniv (1999) shows that (4) guarantees the following margins

\[
\begin{align*}
GM & = 20\log_{10}(K) + 20\log_{10} MM - 1, \\
PM & = 2\arcsin[(2M)^{-1}].
\end{align*}
\]

Inequality (4) is a more encompassing measure of robustness than gain and phase margin. It places a bound on the sensitivity at all frequencies, not just at the two frequencies associated with the gain and phase margins.

Two design problems are considered here:

1. Determine all \((a, b)\) pairs that satisfy (4) where the pair \((P_1, P_2)\) is uncertain in the sense that it belongs to a finite set of pairs \( (P_{1m}, P_{2m}) \), \( m = 1, \ldots, n \), and, in particular, extract an optimal pair \((a_0, b_0)\) for a given optimality criterion.

2. Replacing \( L(s) \) in (4) by

\[
L(s) = a_1 + k_i sP_1(s) + bP_2(s)H(s), = a[PP_1(s) + bPP_2(s)]
\]
determine all $a, b, k_i$ and $H(s)$ of a given structure that satisfy (4), and, in particular, extract an optimal solution $a_0, b_0, k_i$ and $H_0(s)$.

These two problems for the special case of $P_2^\prime\bar{P}_1s$, without considering plant and gain uncertainty, were solved by Astrom et al. (1998) and Panagopoulos et al. (2002) to determine a single controller for the case of maximum $ka$ or $a$, where the solutions were not obtained explicitly but numerically (with MathWorks’ Matlab 5 Optimization Toolbox).

3. Design methodology

In order to determine the $(a, b)$ values for which the closed-loop system is stable and (4) is satisfied, consider first the special case of no gain uncertainty, i.e., $K=1$, and a single plant pair $(P_1(s), P_2(s))$.

Substituting (1) into (4) after splitting $P_1(s)$ and $P_2(s)$ for $s=j\omega$ into real and imaginary parts,

$$P_1(j\omega)=A_1(\omega)+jB_1(\omega), P_2(j\omega)=A_2(\omega)+jB_2(\omega)$$

gives,

$$(6)\ (1+aA_1+abA_2)^2+(aB_1+abB_2)^2-1/M^2\geq0.$$  

For an $(a, b)$ pair which is on the boundary region of the allowed $(a, b)$ values, an $\omega$ exists such that (6) is an equality. Moreover, since at that particular $\omega$, (6) is minimum, its derivative (with respect to $\omega$) at the same $\omega$ is zero. (This observation was also used in Astrom et al. (1998) to design a controller for maximum $a$.)

Thus, 

$$(7)\ (1+aA_1+abA_2)(\dot{A}_1+b\dot{A}_2)+(aB_1+abB_2)(\dot{B}_1+b\dot{B}_2)=0.$$  

Solving (7) for $a$ gives

$$a=-2(\dot{A}_1+b\dot{A}_2)\dot{D}_1+\dot{D}_3b+\dot{D}_2b^2,,$$

where

$$D_1=A_1^2+B_1^2, D_2=A_2^2+B_2^2, D_3=2(A_1A_2+B_1B_2).$$

Substituting (8) into the equality of (6) gives a fourth-order equation for each $\omega$.

$$(9)\ x_4b^4+x_3b^3+x_2b^2+x_1b+x_0=0,$$

where

$$x_4=-2E_2\dot{D}_2+D_2F_2+QH_4,$$

$$x_3=D_2F_1-2E_3\dot{D}_2+QH_3+4D_3\dot{A}_2^2-2E_2\dot{D}_3,$$

$$x_2=-2E_3\dot{D}_3+QH_2+D_2F_0+4D_1\dot{A}_2^2+8D_3\dot{A}_1\dot{A}_2-2E_1\dot{D}_2-2E_2\dot{D}_1,$$

$$x_1=4D_3\dot{A}_1^2-2E_3\dot{D}_1+QH_1-2E_1\dot{D}_3+8D_1\dot{A}_1\dot{A}_2,$$

$$x_0=-2E_1\dot{D}_1+QH_0+4D_1\dot{A}_1^2,$$

$$Q=1-M^2, E_1=2\dot{A}_1A_1, E_2=2\dot{A}_2A_2,$$

$$E_3=\dot{A}_1A_2+\dot{A}_2A_1, F_0=4\dot{A}_1^2, F_1=8\dot{A}_1\dot{A}_2, F_2=4\dot{A}_2^2,$$
The allowed \((a,b)\) region for a given \(M\) value can be calculated as follows: For a given \(\omega\), solve \((9)\) for \(b\). Noting that \(b\) has four solutions (for a given \(\omega\)), pick the positive real solution(s) and use \((8)\) to find their corresponding \(a\). Select the \((a,b)\) pairs for which the resulting closed-loop system is stable and \((4)\) is satisfied. Searching over a range of frequencies, \(\omega\), gives two vectors that are a function of \(\omega\), \((a(\omega),b(\omega))\) which lie on the boundary of the allowed \((a,b)\) region. Note that for an \((a,b)\) on the boundary, one of the following conditions can occur: (i) increasing \(a\) is inside the region, (ii) decreasing \(a\) is inside the region, or (iii) neither increasing nor decreasing \(a\) is inside the region. Thus, for two points, \((a_1,b)\) and \((a_2,b)\), on the boundary, any \(a\in[a_1,a_2]\) and \(b\) is a pair within the region only if (i) increasing \(a_1\) is within the region, (ii) decreasing \(a_2\) is within the region, and (iii) there exist no \((a,b)\) points on the boundary for any \(a\in(a_1,a_2)\). Since, as will be shown later, the optimal pair lies on the boundary of the \((a,b)\) region, internal points are not of interest.

### 3.1. Example

Consider an armature-controlled DC motor with the input being motor current and the output being position. The motor transfer function is \(P(s)=e^{-0.001s}/s^2\). It is required to find the region of the \((a,b)\) pairs such that the complementary sensitivity \(M\leq 1.46\), which allows for gain uncertainty \(k\in[1,K]\) and the pair for which \(a\) is maximum. This is equivalent to at least 40° phase margin and \([10+20\log(K)]\)dB gain margin. The plant is

\[
(10) \quad P_1(s)=1+k_isP(s), \quad P_2(s)=sP(s)
\]

**Fig. 1** depicts the boundary of the allowed \((a,b)\) pairs for \(k=80\) and \(K=1\) (the \((a,b)\) values fall in both shaded regions). **Fig. 1** can also be used to find the \((a,b)\) values which satisfy any gain uncertainty constraint \(k\in[1,K]\). For example, if 6dB gain uncertainty is desired \((K=2)\), then for any \(b\) the allowed \(a\) values should be 6dB less in order to cope with the increase of \(a\) between 0 and 6dB. The \((a,b)\) region will therefore be the lower shaded region depicted in **Fig. 1** where the upper curve is shifted down by 6dB. The maximum \(a\) for \(K=1\) occurs at \((a,b)=(94.2dB, 0.063)\) and for \(K=2\) occurs at \((a,b)=(84.9dB, 0.011)\), giving the controller designs corresponding to lowest sensitivity at low frequencies. Qualitatively, this means that the price of protecting the system from a possible gain uncertainty of 6dB is increasing the low-frequency sensitivity by 9.5dB while the high-frequency noise decreases by 48%. The open-loop Nichols plot for maximum \(a\) and \(K=2\) is shown in **Fig. 2** for verification.

![Fig. 1](image-url)  
**Fig. 1.** Region of \((a,b)\) values for \(M=1.46\), equivalent to 40° phase margin or greater and 10dB gain margin or greater. Lower shaded region is with additional 6dB plant gain uncertainty \((K=2)\) for a total of 16dB or greater.
3.2. Extension to complementary sensitivity specs

It is possible to replace the sensitivity margin constraint (4) by the complementary sensitivity,

\[ kL(j\omega)1+kL(j\omega)\leq M, \forall \omega \geq 0, k \in [1,K]. \]

The following lemma shows that \( L = L_0 \) satisfies (4) if and only if \( L = M^2/(M^2-1)L_0 \) satisfies (11).

Lemma 1

The pair \((a,b)\) solves problem 1 (2) stated in Section 2 if and only if the pair \(((M^2-1)/M^2)a,b)\) solves problem 1 (2) where (4) is replaced by (11).

4. Optimization

The answer to the question “Which is the best \((a,b)\) pair?” of course depends on the optimization criterion. Seron and Goodwin (1995) note that “In general, the process noise spectrum is typically concentrated at low frequencies, while the measurement noise spectrum is typically more significant at high-frequencies”. It follows that an optimal controller can be found by weighting the performance at low frequencies and of noise at high frequencies. Since the high-frequency noise is proportional to \(ab\) and low-frequency performance to \(1/a\), a practical optimal criterion can be \( J = \alpha(1/a) + \beta(ab) \) whose solution must lie on the boundary of the \((a,b)\) curve. When \(\beta\) is small enough or zero (meaning the sensor noise is neglected), the optimal solution is the maximum possible \(a\). This is the criterion corresponding to the Nichols plots of the example.

5. Design methodology for PID controllers with low pass on D term

In the previous sections, a design methodology of a PID controller whose three parameters are \((a,b,k)\) was given assuming the \(k_i\) term is known \((ak_i\) is the I term of the PID, see (5)). The extension to include a filter, \(H\), on the D parameter of the PID is again by searching over both the \(k_i\) and \(H(s)\) (see Eq. (5)). The idea is to choose the structure of the filter \(H(s)\), for example, \(H(s)=p/(s+p)\) or \(H(s)=p^2/(s^2+ps+p^2)\), and search over the parameter \(p\).

The question then is how best to choose the \(p\) values for the search. Since the reason for introducing the filter \(H(s)\) is to limit the sensor noise amplification of the D term and/or reduce high-frequency resonances, it is recommended to perform an iterative search on \(p\) as follows: starting with very large \(p\), measure the noise and if it is too large decrease \(p\). When reaching an acceptable noise level a refined search can be conducted around the satisfactory \(p\).

A reasonable range for this search on \(p\) for first- or second-order filters can be calculated as follows: If \(\omega_0\) is the largest frequency where the open-loop phase is \(-180^\circ\) for a given \(k\) where \(H_0(s)=1\), the search should not exceed...
about \( p=10\omega_0 \) because above that value the low-pass filter phase at frequencies larger than \( \omega_0 \) is neglected (less than 5°).

The answer to the question of how best to choose the \( k_i \) values for the search is based on the following equation: the PID controller is

\[
(12) \quad a_0 1 + k_i s + bs = a_0 (1 + bs)1 + k_i / s1 + bs
\]

(if \( P_2 \neq P_2 \) this is an approximation). Let \((a_0,b_0)\) denote the optimal solution for \( k=0 \) and \( \omega_0 \) its lowest crossover frequency. Find the range of \( k_i \) values whose phase contribution to \((12)\) at \( \omega_0 \) is between −45° and about −1°, that is, the \( k_i \) values for which

\[
(13) \quad \arg a_0 1 + k_i / (j\omega_0)1 + b_j\omega_0 = -45°, -1°.
\]

Use these two \( k_i \) values as the largest and lowest values for the search on \( k_i \).

**Remark 1**

There may exist control applications where the plant has a high-frequency resonance that must be attenuated because (i) if it is not attenuated the achievable closed-loop performance is restricted and (ii) the resonance might generate non-tolerable aliasing phenomena in discrete systems. This attenuation can be done using a notch filter or a low-pass filter on the D term or on both the D and P terms. The algorithm proposed here supports this kind of filter where \( H(s) \) can be applied either on D or on the D and P terms.

### 5.1. Continuation of example for designing \( H(s) \)

It is next of interest to evaluate the tradeoff between high-frequency noise, that is a quantity proportional to the multiplication \( ab \) by the sensor's noise of the D term filtered by \( H(s) \). For simplicity, it is assumed that this noise is \( abp \) and the tradeoff is considered for \( k=100 \). The frequency \( \omega_0 \) is 1500 rad/s, and thus the search is in the range \( p \in [15,000,1500] \). Fig. 3 depicts the boundary of the allowed \((a_k,abp)\) pairs for different \( p \)'s.

Using a low-pass filter with \( p=5000 \) instead of \( p=15,000 \) decreases the noise by a factor of three, while decreasing the performance by 2.1 dB. If the optimal criterion is that the high-frequency noise must be less than \( 1.34 \times 10^6 \), then \( p=5000 \) gives the best performance. Its controller parameters are \( k_i=30.9 \text{dB}, abp=1.34 \times 10^6 \) and \( k_i=100 \). Its open-loop Nichols plot is shown in [Fig. 4](#).

![Fig. 3. Region of \((a_k,abp)\) values for \( M=1.46 \) and \( K=1 \).](#)
6. Extension to plants given by measured data

If a plant identified at a list of frequencies is given and it is not possible to find a state-space model or the accuracy of a chosen model is not good enough, it is still possible to design a controller. One option is to interpolate and/or spline fit the plant pair(s) corresponding to the known frequencies and replace the derivatives appearing in (7) by a numerical derivative. Another option is based on the fact that any \(a, b\) pair on the boundary of the \((a, b)\) region must satisfy \(|1+L(j\omega)|=1/M\), from which the following observations can be made: (i) using the notation \(L(j\omega)=x+jy\),

\[
L(j\omega)=a[A_1(\omega)+bA_2(\omega)]+j[B_1(\omega)+bB_2(\omega)],
\]

then \(|1+L(j\omega)|=1/M\) if and only if \((x-1)^2+y^2=M^2\) or equivalently \(x=-1+\cos \theta/M\) and \(y=\sin \theta/M\), where \(\theta \in [0,2\pi]\), (ii) any \(x+jy\) on the circle \((x-1)^2+y^2=M^2\) must satisfy \(|y/x|\leq 1/(M^2-1)\) and \(x<0\) and (iii) solving (14) for \(b\) gives

\[
b=B_1 x-A_1 y A_2 y-B_2 x=B_1-A_1 y/x A_2 y/x-B_2.
\]

Based on the above three observations, the proposed design method is:

1. Pick a dense list of \(y/x\) in the interval \(|y/x|\leq 1/(M^2-1)\).

2. Solve for \(b\) using (15). From all possible \(b\) values, pick only the positive \(b\)'s for which the sign of \(x\), that is, of \(a(A_1(\omega)+bA_2(\omega))\) is negative.

3. Substitute \(b\) in (6) to get the following quadratic equality on \(a(Q=1-1/M^2)\)

\[
a^2[(A_1+bA_2)^2+(B_1+bB_2)^2]+a[A_1+bA_2]Q=0
\]

and solve for \(a\).

4. All the \((a,b)\) pairs for which the closed-loop system is stable and (6) is satisfied at all measured frequencies lie on the boundary of the \((a,b)\) region.

5. Repeat the above steps for all measured data points to get a list of controllers, that is a list of \((a,b)\) pairs. Apply the optimal criterion on this list to determine the optimal solution.

The drawback of this technique, compared to the one using a model, is that the computation time can be much longer.
7. Conclusions
In this paper, explicit equations are provided for determining controllers of the classical form, i.e., PI, PID, and PID with D filtered, that stabilize a given set of plants and satisfy both gain margin constraints and a bound on the (complementary) sensitivity. The algorithm fits any plant dimension including pure delay, unstable plants, continuous plants, discrete plants, and plants given by measured data.

The outcome of applying the algorithm is a list of controllers from which an optimal controller can be extracted for many practical optimization criteria. Moreover, tradeoffs among high-frequency sensor noise, low-frequency sensitivity, the parameters of the PID and the filter on D are directly presented by design graphs (a single plot suffices for a single filter). Since the algorithm uses explicit equations, it can be executed very fast, and as such the controller design can be updated in near real-time to reflect changes in plant uncertainty and/or closed-loop specifications.

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