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Coarse Topologies in Nonstandard Extensions via Separative ultrafilters

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BY

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0. Introduction

Let \( \mathcal{M} \) be a nonstandard extension (\( \omega_1 \)-saturated will do) of a suitably large ground model \( \mathcal{M} \). If \( A \in \mathcal{M} \) then \( \ast A \) will denote the image of \( A \) in \( \mathcal{M} \) under the canonical embedding, and \( \ast[A] \) will denote the set \( \{ \ast a : a \in A \} \). If \( (X, \tau) \) is a topological space in \( \mathcal{M} \) then \( \ast[\tau] \) is in general no longer a topology but is a basis for what we call the coarse topology on \( \ast X \). This is one of two natural topologies one could put on \( \ast X \) (the other, generated by \( \ast \tau \), is the "Q-topology" (see [1], [2], [4], [7], [8]) and is much finer) and is closely related to the "S-topology" (see [6], [8]) used in monad constructions in the setting of uniform spaces.

Our interest here is centered on the question of when \( \ast X \) (always with the coarse topology) enjoys some of the usual separation properties. As an example, if \( (\mathbb{R}, \nu) \) denotes the real line with its usual topology then \( \ast \mathbb{R} \) can never be a \( T_0 \)-space when \( \mathcal{M} \) is \( \omega_1 \)-saturated. In fact, if \( \ast \mathcal{M} \) is an enlargement (e.g. \( \ast \mathcal{M} \) is \( |\mathcal{M}|^+ \)-saturated) then \( \ast X \) is never \( T_0 \) for infinite \( X \).

As far as we know, it is an open question whether \( \ast X \) can be \( T_0 \) when \( X \) is infinite and \( \mathcal{M} \) is \( \omega_1 \)-saturated. However, with the help of extra set theory (notably Martin’s Axiom (MA) and the Continuum Hypothesis (CH)), we can construct extensions \( \mathcal{M} \) in which \( \ast X \) can be \( T_0 \) (even Tikhonov) for a large class of spaces \( X \).

To begin with, we confine our attention to ultrapower extensions \( \mathcal{M} = \Pi_D(\mathcal{M}) \) where \( D \) is a free (that is, nonprincipal) ultrafilter on a countable set \( I \). Then \( \mathcal{M} \) is automatically \( \omega_1 \)-saturated (since \( D \) is countably incomplete) and its elements are equivalence classes \([f] = [f]_D\) of functions \( f \in \mathcal{M} \); 
\[
[f] = \{ g \in \mathcal{M} : \{ i \in I : g(i) = f(i) \} \in D \}.
\]

In the case of the nonstandard real line, for example, we have \([f] \prec [g] \) iff \( \{ i \in I : f(i) < g(i) \} \in D \). For topological spaces \( (X, \tau) \), if \( U \in \tau \) then \( \ast U = \{ [f] : \{ i : f(i) \in U \} \in D \} \). (Note that \( X \) and \( \ast[X] \) are naturally homeomorphic \( (x \mapsto \ast x \) is a homeomorphism) and that \( \ast[X] \subseteq \ast X \) is a dense subset. This is true for any extension \( \mathcal{M} \).

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The ultrafilters of special interest to us for the purposes of separation properties are the so-called "separative" ultrafilters of B. Scott [10] and will be discussed starting in §2. The reader is assumed to be conversant with some of the more well known properties of ultrafilters on a countable set (e.g., selective ultrafilters, P-points, etc.) as well as the Rudin-Keisler order $\leq_{RK}$ (see [3], [5], [9]). Our set theoretic notation is standard: $|A|$ is the cardinality of $A$, $^B A$ is the set of functions $f : B \to A$, and cardinals are initial ordinals (which are the sets of their ordinal predecessors). Thus $\alpha^\beta = |^\beta \alpha|$ for cardinals $\alpha, \beta$. As usual, $\omega = \{0, 1, 2, \ldots\}$, and $\mathfrak{c} = 2^\omega$.

1. General Properties of $^*_X$

We will assume always that $^*_\mathcal{M}$ is an $\omega_1$-saturated extension (e.g., a countably incomplete ultrapower extension) and that $(X, \tau)$ is an infinite topological space. By way of an introductory remark, it is easy to see that $^*[\tau]$ is not in general a topology, even though it is in natural one-one correspondence with $\tau$. Indeed, let $D$ be a free ultrafilter on $\omega$ (in terms of the Stone-Čech functor $\beta$, $D \in \beta(\omega) \setminus \omega$, where $\omega$ has the discrete topology), and let $\tau$ be the discrete topology on $X = \omega$. Then

$$\bigcup \{^*\{x\} : x \in X\} = \not\in ^*[\tau]$$

since $^*[X]$ is countably infinite and members of $^*[\tau]$ are either finite or of cardinal $\mathfrak{c}$. (If $A \in \mathcal{M}$ is a countably infinite set then $|^* A| = |\Pi_D(A)| = \mathfrak{c}$, by a well known property of ultraproducts (see, e.g., [5]).) Thus $^*[\tau]$ is not closed under arbitrary unions. It is also worthy of note that $^*[\sigma]$ needn’t basically generate $^*[\tau]$ when $\sigma$ is a basis for $\tau$. For let $\langle X, \tau \rangle$ be as above and let $\sigma = \{\{x\} : x \in X\}$. Then $\sigma$ is a basis for the discrete topology; however any union of members of $^*[\sigma]$ will be a subset of $^*[X]$. (The situation is quite different in the case of the $Q$-topology: $^*\sigma$ is always a basis for $^*\tau$ when $\sigma$ is a basis for $\tau$.)

Our first proposition is an easy consequence of the definitions involved.

1.1 Proposition. $^*_X \setminus ^*[X]$ is nonempty and self-dense (i.e., without isolated points). Thus $^*_X$ is never discrete.

The following lemma is true for general $\omega_1$-saturated extensions, and is a basic result of model theory (see, e.g., [5]).

1.2 Lemma. Let $\langle A_n : n < \omega \rangle$ be a sequence of subsets of $X$ and let $m < \omega$. If $|\cap_{n<k} A_n| \geq m$ for each $k < \omega$ then $|\cap_{n<\omega} A_n| \geq m$.

1.3 Theorem. $^*_X$ is a nonmetrizable Baire space which is Lindelöf just in case it is compact.
Proof. This kind of argument has been employed before (see [1], [2], [5]), so we will only sketch it here.

To see that \( *X \) is Baire, let \( \langle M_n : n < \omega \rangle \) be a family of dense open subsets of \( *X \), and let \( U \in \tau \) be nonempty. We show \( *U \cap (\bigcap_{n<\omega} M_n) \neq \emptyset \). First find nonempty \( U_0 \in \tau \) such that \( *U_0 \subseteq *U \cap M_0 \). Using induction, we can find nonempty \( U_{n+1} \in \tau \) such that \( *U_{n+1} \subseteq *U_n \cap (\bigcap_{k<n} M_k) \). By (1.2), \( \emptyset \neq \bigcap_{n<\omega} *U_n \subseteq \bigcap_{n<\omega} M_n \).

\( *X \) is nonmetrizable: for if \( y \in *X \setminus *X \) and if \( \langle U_n : n < \omega \rangle \) is such that \( y \in \bigcap_{n<\omega} *U_n \), then for each \( k < \omega \), \( \cap_{n<k} U_n \) is infinite (since \( *X \setminus *X \) is self-dense). Therefore by (1.2), \( |\cap_{n<\omega} *U_n| \geq 2 \); so in fact, \( *X \) fails strongly to be first countable.

Finally, suppose \( *X \) is Lindelöf, and let \( \nu \) be an open cover of \( *X \). We can assume \( \nu \) is countable and consists of basic open sets \( \langle *U_n : n < \omega \rangle \). Let \( A_n = X \setminus U_n \). Then \( \cap_{n<\omega} A_n = \emptyset \), so by (1.2) there is a \( k < \omega \) with \( \cap_{n<k} A_n = \emptyset \). That is, \( \langle *U_n : n < k \rangle \) is a finite subcover of \( \nu \).

A point \( x \in X \) is a weak-P-point if \( x \) is not in the closure of any countable subset of \( X \setminus \{x\} \). \( X \) is a weak-P-space if every point is a weak-P-point, i.e., if all countable subsets are closed. Clearly, a weak-P-space is \( T_1 \) and “anticompact” (i.e., no infinite subset is compact); and a \( P \)-space which is \( T_1 \) is a weak-P-space. We will be concerned with these classes of spaces in the next section; for now we record the following.

1.4 Proposition. \( *X \) is not a weak-P-space.

Proof. Let \( A \subseteq X \). Then \( *[A] \) is dense in \( *A \subseteq *X \). If \( *[A] \) were closed in \( *X \) then \( *[A] \) would equal \( *A \), whence \( A \) would be finite.

1.5 Proposition. If \( X \) is compact then \( *X \) is compact but not \( T_1 \).

Proof. Let \( \langle *U_i : i \in I \rangle \) be a basic open cover of \( *X \). Then \( \langle U_i : i \in I \rangle \) is an open cover of \( X \). If \( U_1, \ldots, U_n \) is a finite subcover then \( *X = *(U_1 \cup \cdots \cup U_n) = *U_1 \cup \cdots \cup *U_n \), so \( *X \) is compact.

For each \( x \in X \), let \( \mu(x) = \cap \{ *U : x \in U \in \tau \} \) denote the “monad” of \( x \). If \( *X \) were \( T_1 \) then \( \mu(x) \) would be \( \{ *x \} \) for each \( x \in X \); hence given \( y \in *X \setminus *X \) and \( x \in X \) there would be a neighborhood \( U \) of \( x \) with \( y \notin *U \). By compactness, then, there would be a finite subcollection of the \( U \)'s covering \( X \); consequently \( y \notin *X \), an absurdity.

1.6 Corollary. If \( *X \) is \( T_1 \) then \( X \) is anticom pact.

Proof. Suppose \( A \subseteq X \) is compact. Then \( *A \) is compact \( T_1 \), whence \( A \) is finite by (1.5).
2. Separative Ultrapower Extensions

For the rest of the paper, we assume \( *\mathcal{M} = \Pi_D(\mathcal{M}) \) for some \( D \subseteq \beta(I) \setminus I \), \( I \) countable (discrete).

2.1 Proposition. Suppose \( *X \) is \( T_0 \). Then \( D \) is "separative": for each pair of functions \( f, g \in \mathcal{I} \) which are distinct (mod \( D \)) (i.e., \( \{ i \in I : f(i) \neq g(i) \} \subseteq D \) there is a \( J \in D \) such that \( f[J] \cap g[J] = \emptyset \).

Proof. Separative ultrafilters are introduced and studied in [10]. Suppose \( f, g \in \mathcal{I} \) are distinct (mod \( D \)) and let \( h : I \to X \) be one-one. Then \( h \circ f, h \circ g \) are distinct (mod \( D \)). Since \( *X \) is \( T_0 \), it is easy to see that there is a set \( J \in D \) such that \( (h \circ f)[J] \cap (h \circ g)[J] = \emptyset \). Thus \( f[J] \cap g[J] = \emptyset \).

2.2 Remark. It is straightforward to show that \( D \subseteq \mathcal{I} \) is separative iff whenever \( f, g \in \mathcal{I} \) are distinct (mod \( D \)) then their Stone-Cech liftings disagree at \( D \) (i.e., \( \beta(f)(D) \neq \beta(g)(D) \)). (This condition is in fact the definition of separativity used in [10].)

We record the basic facts about separative ultrafilters which will be of use to us here.

2.3 Theorem (B. Scott [10]). (i) Selective ultrafilters are separative (hence MA implies the existence of separative ultrafilters).

(ii) Separativity and being a P-point are not simply related.

(iii) If \( D \) is separative and \( E \leq_{rk} D \) then \( E \) too is separative.

(iv) If \( D \) and \( E \) are separative P-points and there is no \( F \subseteq \mathcal{I} \) with \( F \leq_{rk} D \) and \( F \leq_{rk} E \) then

\[
D \cdot E = \{ R \subseteq I \times I : \{ i : \{ j : (i, j) \in R \} \in E \} \in D \}
\]

is separative (but not a P-point since \( D \cdot E \) is not minimal in the Rudin-Frolik ordering: \( D <_{RF} D \cdot E \)).

(v) \( D \cdot D \) is not separative.

By (2.1, 2.3(v)) we know immediately that \( *X \) is not \( T_0 \) whenever \( X \) is infinite and \( *\mathcal{M} = \Pi_D(\mathcal{M}) \). Since there is no known proof in ZFC that separative ultrafilters exist, we do not know "absolutely" that coarse topologies can ever have any reasonable separation properties. But, given that \( D \) is a separative ultrafilter, quite a lot can be said in this connection.

2.4 Proposition. If \( D \) is separative and \( X \) is a Hausdorff P-space then \( *X \) is Hausdorff.

Proof. Let \([f], [g] \) be distinct and let \( J \in D \) be such that \( f[J] \cap g[J] = \emptyset \). Since \( X \) is a Hausdorff P-space and \( f[J], g[J] \) are countable,
there are disjoint open sets $U, V \subseteq X$ with $f[J] \subseteq U, g[J] \subseteq V$. Thus $[f] \in ^*U, [g] \in ^*V$, and $^*U \cap ^*V = \emptyset$. Let $X$ be any $T_1$-space and let $w(X)$ denote the Wallman compactification of $X$ (see [11]). Points of $w(X)$ are ultrafilters of closed subsets of $X$, and basic open sets are of the form $U^* = \{p \in w(X) : U$ contains a member of $p\}$ for $U \in \tau$. We identify $x \in X$ with the fixed ultrafilter $Px$ of closed supersets of $\{x\}$ and define $\varphi : \beta(\omega) \times ^*X \to w(X)$ by

\[
\varphi(D, f) = \{A : A \subseteq X \text{ is closed and } f^{-1}[A] \in D\}
\]

(easily seen to be a member of $w(X)$).

2.5 Lemma. Let $\varphi_D : ^*X \to w(X)$ be given by $\varphi_D(f) = \varphi(D, f)$. If $X$ is a weak-P-space and $D$ is a separative ultrafilter then $\varphi_D$ induces an embedding of $^*X$ into $w(X)$ which leaves the points of $X$ fixed (i.e., $\varphi_D(x) = Px$).

Proof. Let $f, g : \omega \to X$ be equal (mod $D$). If $A \in \varphi_D(f)$ then $f^{-1}[A] \in D$. Now $g^{-1}[A] \supseteq f^{-1}[A] \cap \{n : f(n) = g(n)\} \in D$, so $A \in \varphi_D(g)$. Thus $\varphi_D$ is well defined on $^*X$. Let $U \in \tau$. Then $[f] \in \varphi_D^{-1}(U^*)$ iff there is closed $A \subseteq U$ such that $f^{-1}[A] \in D$ iff there is a closed $A \subseteq U$ such that $[f] \in ^*A$ iff $[f] \in ^*U$, since $f[\omega]$ is countable hence closed. Thus $\varphi_D$ is continuous.

To show $\varphi_D([U]) = \varphi_D([X]) \cap U^*$, we note that $\varphi_D([f]) \in \varphi_D([U])$ iff there is a closed $A \subseteq U$ such that $f^{-1}[A] \in D$ iff $\varphi_D([f]) \in U^*$, again since countable sets are closed.

We need to show $\varphi_D$ is one-one. Suppose $f, g : \omega \to X$ are distinct (mod $D$) and let $J \in D$ be such that $f[J] \cap g[J] = \emptyset$. Then $f[J] \in \varphi_D([f])$ and $g[J] \in \varphi_D([g])$, whence these ultrafilters of closed sets are also distinct.

Finally, it is easy to see that points of $X$ are fixed by $\varphi_D$, so the proof is complete.

2.6 Theorem. Let $D$ be a separative ultrafilter.

(i) If $X$ is a weak-P-space then $^*X$ is $T_1$.

(ii) If $X$ is a normal weak-P-space then $^*X$ is Tichonov.

(iii) If $X$ is a normal P-space then $^*X$ is "strongly 0-dimensional" (i.e., disjoint zero sets are separable via clopen sets; equivalently, $\beta(X)$ is "0-dimensional" in the sense of weak inductive dimension).

(iv) If $X$ is an extremally disconnected normal weak-P-space then $^*X$ is extremally disconnected.

Proof. (i) By (2.5), $^*X$ embeds in $w(X)$, a compact $T_1$-space.

(ii) If $X$ is normal then $w(X) = \beta(X)$.

(iii) Regular $P$-spaces are strongly 0-dimensional, hence their Stone-
Čech compactifications are 0-dimensional. Now we can make believe that $X \subseteq ^*X \subseteq \beta(X)$. Thus $\beta(^*X) = \beta(X)$, whence $^*X$ is strongly 0-dimensional.

(iv) $\beta(X)$ is extremally disconnected and $^*X$ is a dense subspace.
2.7 Question. Can \( \ast X \) ever be Lindelöf \( T_0 \)?

2.8 Theorem. Let \( X \) be a normal weak-P-space such that \( \ast X \) is Lindelöf \( T_0 \). Then \(|X| > c\).

Proof. Since \( \ast X \) is \( T_0 \), \( D \) is separative. Thus we can consider \( X \subseteq \ast X \subseteq \beta(X) \). By (1.3), \( \ast X \) is compact, hence equal to \( \beta(X) \). Let \( A \subseteq X \) be countable discrete. Then \( A \) is closed in \( X \), hence \( C^* \)-embedded there (see [11]). This says that \( A \) is a countable \( C^* \)-embedded subset of \( \beta(X) \); whence the closure of \( A \) in \( \beta(X) \) is homeomorphic to \( \beta(\omega) \), whose cardinality is well known to be \( 2^c \). Thus \( |\ast X| \geq 2^c \), so \(|X| > c\).

2.9 Question. Is it possible for \( \ast X \) to be normal? Paracompact?

Motivated by this question, we now turn to the special case of spaces \( \ast X \) where \( X \) is countable discrete (\( X = \omega \)). First of all notice that by (2.1) and (2.6), \( D \) is separative iff \( \ast \omega \) is \( T_0 \) iff \( \ast \omega \) is an extremally disconnected strongly 0-dimensional space iff \( \beta(\ast \omega) = \beta(\omega) \). (In [1] it is proved by contrast that topological ultraproducts which are not discrete can never be extremally disconnected unless their cardinalities exceed a measurable cardinal.) A weak affirmative answer to (2.9) is the following.

2.10 Theorem (CH). Let \( D \) be a separative P-point (e.g., a selective ultrafilter). Then \( \ast \omega \setminus \ast[\omega] \) is hereditarily paracompact.

Proof. We first note that in the embedding \( \varphi_D : \ast \omega \to \beta(\omega) \), the image of \( \varphi_D \) is precisely \( \{ E \in \beta(\omega) : E \leq_{RK} D \} \). (Indeed, \( J \in \varphi_D([f]) \) iff \( f^{-1}[J] \in D \), so \( \varphi_D([f]) \leq_{RK} D \). On the other hand, if \( E \leq_{RK} D \) then there is some \( f : \omega \to \omega \) such that \( J \in E \) iff \( f^{-1}[J] \in D \). Hence \( E = \varphi_D([f]) \).) Thus if \( D \) is a P-point as well as being separative then \( \ast \omega \setminus \ast[\omega] \) is a P-space. Now \( \ast \omega \) has an open basis of cardinality \( \omega_1 \), so every subset of \( \ast \omega \setminus \ast[\omega] \) is a P-space which is "\( \omega_1 \)-Lindelöf" (i.e., every open cover has a subcover of cardinality less than or equal to \( \omega_1 \)). We are done once we prove the claim (also proved in [1]): If \( X \) is an \( \omega_1 \)-Lindelöf regular P-space then every open cover of \( X \) refines to an open partition of \( X \). To see this, simply take an open cover \( \nu \) which we can assume to consist of clopen sets and to have cardinality \( \omega_1 \); say \( \nu = \{ U_\xi : \xi < \omega_1 \} \). Let \( V_\xi = U_\xi \setminus (\bigcup_{\eta \leq \xi} U_\eta) \). Then \( \langle V_\xi : \xi < \omega_1 \rangle \) is an open refinement of \( \nu \), the members of which are pairwise disjoint.

We close this section with a simple observation about covering properties in \( \ast \omega \) for separative \( D \).

2.11 Proposition. If \( D \) is separative then \( \ast \omega \) is anticom pact, and neither \( \ast \omega \) nor \( \ast \omega \setminus \ast[\omega] \) is Lindelöf.

Proof. A compact subset of \( \ast \omega \) is closed in \( \beta(\omega) \), and infinite closed
subsets of \( \beta(\omega) \) are well known to have cardinality \( 2^\omega \). Since \( \ast \omega \) is uncountable, no infinite subset can be compact.

Now one of \( \ast \omega \), \( \omega \setminus \ast \omega \) is Lindelöf just in case the other is. If \( \ast \omega \) were Lindelöf, it would, by (1.3), be compact. Impossible.

**3. Iterated Ultrapowers**

Suppose \( D, E \) are free ultrafilters on (countable) sets \( I, J \) respectively and let \( \mathcal{M} \) be given. Then, letting \( \langle D \rangle \mathcal{M} \) denote \( \Pi_D(\mathcal{M}) \) (to avoid confusion, we replace asterisks with the ultrafilter in brackets) we can iterate the extension process and ask whether \( \langle D \rangle \mathcal{M} \) is an ultrapower extension of \( \mathcal{M} \). The answer is well known to be “yes”; \( \langle D \rangle \mathcal{M} \) is naturally isomorphic (as a membership structure) to \( \langle D \rangle \mathcal{M} \). The isomorphism is defined as follows. First define \( \psi : \langle D \rangle \mathcal{M} \to I \times J \mathcal{M} \) by \( \psi(f)((i, j)) = f(i)(j) \). One can then check quite easily that \( \psi \) induces an isomorphism

\[
\overline{\psi} : \langle D \rangle \mathcal{M} \to \langle D \rangle \mathcal{M}, \quad \text{where } \overline{\psi}([f]_D) = [\psi(f)]_{D,E}.
\]

Now a natural question to ask is whether \( \overline{\psi} \) further induces homeomorphisms between corresponding coarse topologies (as is the case with the \( \mathcal{Q} \)-topology (see [1])). It is easy to see that, for \( U \in \tau, \overline{\psi}^{-1}([D \langle E \rangle U]) = [D \langle E \rangle U], \) so \( \overline{\psi} \upharpoonright \langle D \rangle \mathcal{X} \) is a continuous bijection onto \( \langle D \rangle \mathcal{X} \). In answer to the question of whether \( \overline{\psi} \upharpoonright \langle D \rangle \mathcal{X} \) is always a homeomorphism, we have the following.

**3.1 Proposition.** Let \( D, E \) be free ultrafilters on \( \omega \). Then \( \overline{\psi} \upharpoonright \langle D \rangle \mathcal{X} \) is not an open map.

**Proof.** Since \( \omega \) has the discrete topology, \( \langle E \rangle \omega = \cup_{n<\omega} \langle E \rangle \{n\} \) is open in \( \langle E \rangle \omega \), hence \( \langle D \rangle \langle E \rangle \omega \) is open in \( \langle D \rangle \omega \). Let \( f : \omega \to \langle E \rangle \omega \) be given by \( f(m) = \langle f \rangle \omega = m \). Then \( \{m : f(m) \in \langle E \rangle \omega\} = \omega = D, \) so \( [f]_D \in \langle D \rangle \langle E \rangle \omega \). Now \( \overline{\psi}([f]_D) = [g]_{D,E} \) where \( g(m, n) = m, \) and \( [h]_{D,E} \in \overline{\psi}([D \langle E \rangle \omega]) \) iff

\[
\{m : \{n : h(m, n) = p\} \in E \text{ for some } p \} \in D.
\]

If \( \langle D \rangle \langle E \rangle J \) is any basic open set containing \( [g]_{D,E} \), then \( J \in D \), hence \( J \) is infinite. Since both \( D \) and \( E \) are free ultrafilters, we can find \( k : \omega \times \omega \to J \) such that \( \{m : \{n : k(m, n) = p\} \in E \text{ for some } p \} \notin D \). Thus \( \langle D \rangle \langle E \rangle J \not\subseteq \overline{\psi}([D \langle E \rangle \omega]) \), hence \( \overline{\psi}([D \langle E \rangle \omega]) \) is not an open set.

**3.2 Lemma.** Let \( D, E \) be free ultrafilters on \( \omega \). Then \( \langle D \rangle \mathcal{X} \) is not a regular space.

**Proof.** Look at the proof of (3.1) above. If \( U \) is any basic neighborhood of \( [f]_D \) which is contained in \( \langle D \rangle \mathcal{X} \) then \( U \) must be of the form \( \langle D \rangle \langle E \rangle [J] \) for some \( J \in D \). The closure of this set in \( \langle D \rangle \mathcal{X} \) is easily seen to be \( \langle D \rangle \langle E \rangle J \). Again, since both \( D \) and \( E \) are free, we can find \( [g]_D \in \langle D \rangle \langle E \rangle J \) such that
\{n : g(n) \text{ is not constant (mod } E)\} \in D. \text{ Thus } (D\times E)[\omega] \text{ is an open set containing } [f]_D \text{ which does not contain the closure of any open set containing } [f]_D. \quad \blacksquare

The following shows that, under CH, \( (D\times E)X \) and \( (D-E)X \) can have easily distinguishable topological types.

3.3 Theorem (CH). \textit{There are ultrafilters } D, E \text{ on } \omega \text{ such that } (D-E)\omega \text{ is regular, but } (D\times E)\omega \text{ is not regular.}

\textit{Proof.} Using CH and Theorem (9.13) of [5] there are nonisomorphic selective ultrafilters \( D, E \) on \( \omega \). Since both are minimal in the Rudin-Keisler ordering, they satisfy the hypothesis of (2.3 (iv)). Thus \( D \cdot E \) is separative; so by (2.6), \( (D-E)\omega \) is a Tichonov space. However, by (3.2), \( (D\times E)\omega \) fails to be even regular. \quad \blacksquare

3.4 Remark. Under the CH, the converses of (2.4) and (2.6 (i)) fail: there is a Hausdorff space \( X \), not a weak-P-space, and an ultrafilter \( D \in \beta(\omega) \setminus \omega \) such that \( (D)X \) is Hausdorff. For let \( D, E \) be as in (3.3), and let \( X = (E)\omega \). Since \( (D-E)\omega \) is Tichonov, hence Hausdorff, and the natural bijection \( \overline{\psi} \upharpoonright (D\times E)\omega \) is continuous, we know that \( (D)X \) is also Hausdorff. But \( X \) fails to be a weak-P-space by (1.4).

\begin{thebibliography}{10}

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