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Sub-Independence: An Expository Perspective

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Abstract

Limit theorems as well as other well-known results in probability and statistics are often based on the distribution of the sums of independent random variables. The concept of sub-independence, which is much weaker than that of independence, is shown to be sufficient to yield the conclusions of these theorems and results. It also provides a measure of dissociation between two random variables which is much stronger than uncorrelatedness.

Keywords

Characteristics function, Independence, Limit theorems, Sub-independence

1. Introduction

Limit theorems as well as other well-known results in probability and statistics are often based on the distribution of the sums of independent (and often identically distributed) random variables rather than the joint distribution of the summands. Therefore, the full force of independence of the summands will not be required. In other words, it is the convolution of the marginal distributions which is needed rather than the joint distribution of the summands, which in the case of independence, is the product of the marginal distributions.

The concept of sub-independence can help to provide solution for some modeling problems where the variable of interest is the sum of a few components. Examples include household income, the total profit of major firms in an industry, and a regression model $Y = g(X) + \epsilon$ where $g(X)$ and ϵ are uncorrelated but may not be independent. For example, in Bazargan et al. (2007), the return value of significant wave height (Y) is modeled by the sum of a cyclic function of random delay D, $\hat{g}(D)$, and a residual term $\hat{\varepsilon}$. They found that the two components are at least uncorrelated but not independent and used sub-independence to compute the distribution of the return value.

Let X and Y be two random variables (rv's) with joint and marginal cumulative distribution functions (cdf's) $F_{X,Y}$, F_X , and F_Y, respectively. Then X and Y are said to be independent if and only if

(1.1)

$$F_{X,Y}(x,y) = F_X(x)F_Y(y), for\ all \quad (x,y) \in \mathbb{R}^2,$$

or equivalently if and only if

(1.2)

$$\varphi_{X,Y}(s,t) = \varphi_X(s)\varphi_x(t), for\ all \quad (s,t) \in \mathbb{R}^2$$

where $\varphi_{X,Y}(s,t)$, $\varphi_X(s)$, and $\varphi_x(t)$, are, respectively, the corresponding joint and marginal characteristic functions (cf's). Note that (1.1) and (1.2) are also equivalent to

(1.3)

$$P(X \in A and Y \in B) = P(X \in A)P(Y \in B),$$

for all Borel sets $A,\ B$.

The concept of sub-independence, as far as we have gathered, was formally introduced by Durairajan (1979) stated as follows: The rv's X and Y with cdf's F_X and F_Y are sub-independent (s.i.) if the cdf of $X + Y$ is given by

(1.4)

$$F_{X+Y}(z) = (F_X * F_Y)(z) = \int_{\mathbb{R}} F_X(z-y)dF_Y(y)\,, \qquad z \in \mathbb{R},$$

or, equivalently, if and only if

(1.5)

$$\varphi_{X+Y}(t) = \varphi_{X,Y}(t,t) = \varphi_X(t)\varphi_y(t), \qquad for\ all\ t \in \mathbb{R}.$$

The drawback of the concept of sub-independence in comparison with that of independence has been that the former does not have an equivalent definition in the sense of (1.3) which some believe to be the natural definition of independence. We believe to have found such a definition now which is stated below. We will give two separate definitions, one for the discrete case (Definition 1.1) and the other for the continuous case (Definition 1.2).

Let $(X,\ Y){:}\,\Omega\ \rightarrow\ \mathbb{R}^2$ be a discrete random vector with range $\Re(X,\ Y)\ =\ (x\,i\ ,\ y\,j\ ){:}\ i,\ j\ =\ 1,2,\ldots$ (finitely or infinitely countable). Consider the events

$$A_i = \{\omega \in \Omega{:}\, X(\omega) = x_i\}, \qquad B_j = \{\omega \in \Omega{:}\, Y(\omega) = y_j\}$$

And

$$A^z = \{\omega \in \Omega{:}\, X(\omega) + Y(\omega) = z\}, \qquad z \in \Re(X+Y).$$

Definition 1.1

The discrete rv's X and Y are s.i. if for every $z \in \Re(X+Y)$

(1.6)

$$P(A^z) = \sum_{i,j,}\ \sum_{x_i+y_j=z} P(A_i)P(B_j).$$

To see that (1.6) is equivalent to (1.5), suppose X and Y are s.i. via (1.5), then

$$\sum_i \sum_j e^{it(x_i+y_j)} f\left(x_i + y_j\right) = \sum_i \sum_j e^{it(x_i+y_j)} f_X(x_i) f_Y(y_j),$$

where f, f_X, and f_Y are probability functions of $(X,\ Y)$, X, and Y, respectively. Let $z \in \Re(X+Y)$, then

$$e^{itz} \sum_{i,j,}\ \sum_{x_i+y_j=z} \mathrm{f}\left(x_i + y_j\right) = e^{itz} \sum_{i,j,}\ \sum_{x_i+y_j=z} f_X(x_i) f_Y\left(y_j\right),$$

which implies (1.6). For the converse, assume (1.6) holds and reverse the above last two steps to arrive at (1.5).

For the continuous case, we observe that the half-plane $H\ =\ (x,\ y){:}\ x\ +\ y\ <\ 0$ can be written as a countable disjoint union of rectangles:

$$H = \bigcup_{i=1}^{\infty} E_i \times F_i,$$

where E_i and F_i are intervals. Now, let $(X,\ Y){:}\,\Omega\ \rightarrow\ \mathbb{R}^2$ be a continuous random vector and for $c\ \in\ \mathbb{R}$ let

$$A_c = \{\omega \in \Omega : X(\omega) + Y(\omega) < c\}$$

and

$$A_i^{(c)} = \left\{\omega \in \Omega{:}\, X(\omega) - \frac{c}{2} \in E_i\right\}, \qquad B_i^{(c)} = \left\{\omega \in \Omega{:}\, Y(\omega) - \frac{c}{2} \in F_i\right\}$$

Definition 1.2

The continuous rv's X and Y are s.i. if for every $c\ \in\ \mathbb{R}$

(1.7)

$$P(A_c) = \sum_{i=1}^{\infty} P\left(A_i^{(c)}\right) P\left(B_i^{(c)}\right)$$

To see that (1.7) is equivalent to (1.4), observe that (LHS of (1.7))

(1.8)

$$P(A_c) = P(X + Y < c) = P\big((X, Y) \in H_c\big),$$

where $H_c = (x, y){:}\ x + y < c$. Now, if X and Y are s.i. then

$$P(A_c) = (P_X \times P_Y)(H_c),$$

where P_X , P_Y are probability measures on $\mathbb{R}$ defined by

$$P_X(B) = P(X \in B) \text{ and } P_Y(B) = P(Y \in B),$$

and $P_X \times P_Y$ is the product measure.

We also observe that (RHS of (1.7))

(1.9)

$$\begin{aligned} \sum_{i=1}^{\infty} P\left(A_i^{(c)}\right) P\left(B_i^{(c)}\right) &= \sum_{i=1}^{\infty} P\left(X - \frac{c}{2} \in E_i\right) P\left(Y - \frac{c}{2} \in F_i\right) \\ &= \sum_{i=1}^{\infty} P\left(X \in E_i + \frac{c}{2}\right) P\left(Y \in F_i + \frac{c}{2}\right) \\ &= \sum_{i=1}^{\infty} P_X \times P_Y \left(E_i + \frac{c}{2}\right) \times \left(F_i + \frac{c}{2}\right). \end{aligned}$$

Now, (1.8) and (1.9) will be equal if $H_c = \cup_{i=1}^{\infty} \left\{\left(E_i + \frac{c}{2}\right) \times \left(F_i + \frac{c}{2}\right)\right\}$, which is true since the points in H_c are obtained by shifting each point in H over to the right by $\frac{c}{2}$ units and then up by $\frac{c}{2}$ units.

Remark 1.1

(i) Note that H can be written as a union of squares and triangles. The triangles are congruent to $0 \le y < x, 0 \le x < 1$ which in turn can be written as a disjoint union of squares. For example, take $[0,1/2) \times [0,1/2)$ then $[1/2,3/4) \times [0,1/4)$ and so on. (ii) The discrete rv's X, Y, and Z are s.i. if (1.6) holds for any pair and

(1.10)

$$P(A^s) = \sum_{i,j,k,} \sum_{x_i + y_j + z_k = s} \sum P(A_i) P(B_j) P(C_k).$$

For n variate case we need $2^n - n - 1$ equations of the above form. (iii) The representation (1.7) can be extended to the multivariate case as well. (iv) For the sake of simplicity of the computations, we use (1.5) and its extension to the multivariate case as our definition of sub-independence throughout this work.

We may in some occasions have asked ourselves if there is a concept between "uncorrelatedness" and "independence" of two random variables. It seems that the concept of "sub-independence" is the one: it is much stronger than uncorrelatedness and much weaker than independence. The notion of sub-independence seems important in the sense that under usual assumptions, Khintchine's Law of Large Numbers and Lindeberg-Lévy's Central Limit Theorem as well as other important theorems in probability and statistics hold for a sequence of s.i. random variables. While sub-independence can be substituted for independence in many cases, it is difficult (in general) to find conditions under which the former implies the latter. Even in the case of two discrete identically distributed rv's *X* and *Y* the joint distribution can assume many forms consistent with sub-independence. In order for two random variables *X* and *Y* to be s.i. the probabilities

$$p_i = P(X = x_i), i = 1,2, \dots, n$$

and

$$q_{ij} = P(X = x_i, Y = x_i), \qquad i, j = 1.2 \dots., n,$$

must satisfy the following conditions.

1. $\sum(q_{ij} - p_i p_j) = 0$, where the sum extends for all values of i and j for which $x_i + x_j = z$ and *z* takes all the values in the set $min\left(x_i + x_j\right), \dots, max\left(x_i + x_j\right)$.
2. $p_i = \sum_{j=1}^{n} q_{ij} = \sum_{j=1}^{n} q_{ji}$, $i = 1, 2, \dots, n$.

This linear system in n^2 variables q_{ij} is considerably underdetermined for all but the smallest value of n specially if a large number of points (x_i, x_j) lie on the line $x + y = z$. On the other hand, the only q_{ij} consistent with independence is $q_{ij} = p_i p_j$.

If X and Y are s.i. then unlike independence, X and αY are not necessarily s.i. for any real $\alpha \neq 1$ as the following simple examples (discrete and continuous cases respectively) show.

Example 1.1

Let X and Y be identically distributed rv's with support on the integers 1,2,3 and joint probabilities:

$$\mathrm{p}_{11} = \mathrm{p}_{22} = \mathrm{p}_{33} = \frac{1}{9}, \quad \mathrm{p}_{21} = \mathrm{p}_{32} = \mathrm{p}_{13} = \frac{2}{9}$$
$$\mathrm{p}_{12} = \mathrm{p}_{23} = \mathrm{p}_{31} = 0.$$

Then X and Y are s.i. but X and $-Y$ are not.

Example 1.2

Let *X* and *Y* have the joint cf given by

$$\varphi_{X,Y}(t_1, t_2) = \exp\{-\left(t_1^2 + t_2^2\right)/2\}\left[1 + \beta t_1 t_2 (t_1 - t_2)^2, \times \exp\left\{\left(t_1^2 + t_2^2\right)/4\right\}\right], (t_1, t_2) \in \mathbb{R}^2,$$

where β is an appropriate constant. (The corresponding joint probability density function (pdf) is given by

$$f(x,y) = \frac{1}{2\pi}\exp\{-(x^2+y^2)/2\}[1-16\beta p(x,y)\times\exp\{-(x^2+y^2)/2\}],$$
$$(x,y)\in\mathbb{R},$$

where $p(x,\ y)\ =\ \{6xy\ -\ 2x^2\ -\ 2y^2\ +\ 4x^2\,y^2\ -\ 2x^3\,y\ -\ 2xy^3\ +\ 1\})$.

Then X and Y are s.i. standard normal rv's and hence $X\ +\ Y$ is normal with mean 0 and variance 2, but X and $-Y$ are not s.i. and, consequently, $X\ -\ Y$ does not have a normal distribution.

It is clear that one is interested to know under what conditions sub-independence implies independence. Durairajan (1979) posed this question and gave two specific examples, one for the discrete case and one for the continuous case in which he claimed the given conditions (different for each example) are sufficient for the s.i. rv's to be independent. Although his discrete example with the given conditions worked nicely, his example for the continuous case did not. Here is his example: Let X and Y be two continuous rv's such that their joint distribution function is $F(\sigma x,\ y)$ for $\sigma\ \in\ \mathbb{R}^+$ and $x\ >\ 0,\ y\ >\ 0$ with marginal distribution functions as $F_X\ (\sigma x)$ and $F_Y\ (y)$. If X and Y are s.i. for all $\sigma\ \in\ \mathbb{R}^+$, then X and Y are independent. It is not hard to see that under the stated conditions, the rv X will have to be degenerate at 0; hence, X will be independent of any rv Y. We will revisit the above mentioned question later in Sec. 2.

The concept of sub-independence defined by (1.5) can be extended to $n\ (>\ 2)$ rv's as follows.

Definition 1.3

The rv's $X_1,\ X_2, \ldots,\ X_n$ are s.i. if for each subset $X_{\alpha_1},\ X_{\alpha_2}, \ldots,\ X_{\alpha_r}$ of $\{X_1,\ X_2, \ldots,\ X_n\}$

(1.11)

$$\varphi_{X_{\alpha_1},\ldots,X_{\alpha_r}}(t,\ldots,t) = \prod_{i=1}^{r}\varphi_{X_{\alpha_1}}(t), \qquad for\ all\ \mathrm{t}\in\mathbb{R}$$

As we mentioned before, Durairajan (1979) formally introduce the concept of sub-independence and pointed out that the Khintchine's law of large numbers and Lindeberg-Levy's central limit theorem hold for a sequence of s.i. rv's. The reason we used the word “formally” in the previous sentence is that Lukacs (1970) had used (1.5) implicitly in proving certain results such as Cramér's, based on cf's but for independent rv's. We will mention them as we go along. As we mentioned earlier, Durairajan (1979) tried to find conditions under which sub-independence and independence are equivalent. It was pointed out by Hamedani (1983) that Durairajan conditions forced one of the random variables involved to be degenerate which of course is a trivial case. To show how weak the concept of sub-independence is in comparison with that of independence, even in the case involving normal distribution, Hamedani (1983) gave the following examples.

Example 1.3

Consider the joint cf

$$\varphi_{X,Y}(t_1,t_2) = \exp\{-(t_1^2+t_2^2)/2\} + \frac{1}{32}t_1,t_2(t_1^2-t_2^2)\times\exp\{-(t_1^2+t_2^2)/4\},$$
$$(t_1,t_2)\in\mathbb{R}^2.$$

Then, $X,\ Y,\ X\ +\ Y$, and $X\ -\ Y$ are all normal, which imply (1.5) (the cf version of the definition of sub-independence) holds for X and Y as well as X and $-Y$, but X and Y are not independent.

We can generalize Example 1.3 as follows.

Example 1.4

Given (a_k, b_k): $k = 1,2,\ldots, N$ a finite set in $\mathbb{R}^2$. Consider the joint cf

$$\varphi_{X,Y}(t_1,t_2) = \exp\{-(t_1^2+t_2^2)/2\} \\ + t_1, t_2(t_1^2 - t_2^2) \\ \times \exp\left\{-\frac{1}{2}[c_1 - c_2(t_1^2+t_2^2)]\right\}\prod_{k=1}^{N}(b_k^2 t_1^2 - \alpha_k^2 t_2^2),\ \ (t_1,t_2) \in \mathbb{R}^2,$$

where c_1 and c_2 are suitable constants. Then X and Y are standard normal rv's, X and Y as well as X and $-Y$ are s.i. and more

$$\varphi_{X,Y}(a_k t, b_k t) = \varphi_X(a_k t)\varphi_Y(b_k t), \qquad \text{for all } t \in \mathbb{R}, k = 1,2,\ldots, N,$$

i.e., $a_k X$ and $b_k Y$, $k = 1, 2, \ldots, N$ are s.i. and of course $a_k X + b_k Y$, $k = 1, 2, \ldots, N$ are all normally distributed but X and Y are not independent.

Remark 1.2

The set $\{(a_k, b_k)$: $k = 1, 2, \ldots, N\}$ in Example 1.4 cannot be taken to be infinitely countable. Hamedani and Tata (1975) showed that two normally distributed rv's X and Y are independent only if they are uncorrelated and $a_k X$ and $b_k Y$, $k = 1, 2, \ldots$ are s.i., i.e.,

$$\varphi_{X,Y}(a_k t, b_k t) = \varphi_X(a_k t)\varphi_Y(b_k t), \qquad \text{for all } t = \mathbb{R}, k = 1,2,\ldots,$$

where $\{(a_k, b_k)$: $k = 1, 2, \ldots\}$ is a distinct sequence in $\mathbb{R}^2$.

In the next section we present the results based on the concept of sub-independence from 1979, the starting point, to 2011 as far as we have been able to gather. We hope this article will be a good starting point for those who are interested in the concept of sub-independence and may be leaning towards using this notion in their works.

2. Results

The results reviewed and established in this section will all be based on the concept of sub-independence. We will divide this section to a number of subsections each of which will be dealing with a specific distribution and/or subject. The results in each subsection will be presented in the chronological order of appearances and not that of their importance.

2.1. Characterizations of Normal Distribution and Related Results

We start this subsection with the s.i. version of Cramér's famous theorem (Theorem 1, 1936) which appeared in Hamedani and Walter (1984b).

Theorem 2.1 (Cramér)

If the sum $X + Y$ of the rv's X and Y is normally distributed and these rv's are s.i., then each of X and Y is normally distributed.

Remark 2.1

The proof of Theorem 2.1 can be deduced from Lukacs (1970). The important lemma used in the proof involves showing that the cf of each rv is an entire function. It is somewhat surprising that this lemma is true under a much weaker hypothesis about the relation of the two variables. Hamedani and Walter (1984b) presented a proof of this assertion (see Theorem 2.2 below) which, however, requires an auxiliary condition.

Theorem 2.2

If the sum of two rv's is normally distributed and if the cdf G of their difference satisfies the condition

$$1 - G(w) = O(\exp\{-|w|(1+\varepsilon)\}) = G(-w), \qquad \text{for all } \varepsilon > 0 \text{ as } w \to \infty,$$

then the cf of each rv is an entire function.

Remark 2.2

It is clear that if $X - Y$ has a compact support or has a normal distribution (as in Example 1.3) then the auxiliary condition of Theorem 2.2 is satisfied.

The following simple characterization of normal distribution, given in Hamedani and Walter (1984b), is based on the concept of sub-independence which strengthen the characterizations given in Chung (1974) under the assumption of independence.

Proposition 2.1

Let X and Y be s.i.i.d. (sub-independent and identically distributed) rv's with mean 0 and variance 1 such that:

i. *X and $-Y$ are s.i. and*
ii. *$X + Y$ and $X - Y$ are s.i.*

Then both X and Y have standard normal distributions.

Remark 2.3

We note that the hypotheses of Proposition 2.1 do not imply that X and Y are independent (see Example 1.3 and Theorem 1 of Hamedani and Tata (1975) for further details), nor that they be jointly normal. This proposition is in spirit close to Maxwell-Kac-Berstein Theorem (see Feller, 1971, p. 78).

As with independence, distinct linear combinations of sub-independent rv's need not be sub-independent. However, if they are normal, the following holds (Hamedani and Walter, 1984b).

Proposition 2.2

Let X and Y as well as X and $-Y$ be s.i. normally distributed rv's with the same variance.
Then $X + Y$ and $X - Y$ are s.i.

Ahsanullah and Hamedani (1988) made the following observation: If X and Y are i.d. (identically distributed) with mean 0 such that X and $-Y$ are s.i. and $(X - Y)^2/2 \sim \chi^2(1)$ (chi-square with 1 degree of freedom), then X and Y have standard normal distributions. The following example shows that in the absence of sub-independence, X and Y may not be standard normal variables.

Example 2.1

Let X and Y be jointly normally distributed with means equal to μ and variances equal to $(2/(1 - \rho))^{1/2}$, where ρ is their correlation coefficient. Then $(X - Y)^2/2 \sim \chi^2(1)$.

Remark 2.4

It can easily be seen that if X and Y are s.i.i.d. and if $X + Y$ is symmetric (about 0), then $(X + Y)^2/2 \sim \chi^2(1)$ if and only if X and Y are standard normal.

Ahsanullah et al. (1991) presented two characterizations of normal distribution based on chi-square distribution and the notion of sub-independence stated in Theorems 2.3 and 2.4 below.

Theorem 2.3

Let X and Y be s.i.i.d. non degenerate rv's. If X^2 and $\frac{1}{2}(X+Y)^2$ are i.d. chi-square with one degree of freedom, then the common distribution of X and Y is standard normal.

Theorem 2.4

Let $X_1,\ X_2, \dots,\ X_n$ be s.i.i.d. non degenerate rv's. If k, $\overline{X}_k^2, \overline{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i$, is distributed as chi-square with one degree of freedom for two positive integers m_1 and m_2, then $X_i's$ are normally distributed.

2.2. Characterizations of Reciprocal Distribution and Related Results

A rv X (or its pdf f_X) is called reciprocal if its cf is a multiple of a pdf. It is called self-reciprocal if there exist constants *A* and α such that $Af_X\ (\alpha t)$ is the cf of X. It is strictly self-reciprocal if $(2\pi)^{1/2}\ f_X\ (t)$ is the cf of X. Using the concepts of reciprocal, self-reciprocal, strictly self-reciprocal, and sub-independence Hamedani and Walter (1985) reported the following observations (Propositions 2.3–2.5 and Theorem 2.5 below).

Proposition 2.3

Let X and Y be s.i. reciprocal rv's. Then $X\ +\ Y$ is reciprocal.

Proposition 2.4

Let X and Y be s.i.i.d. rv's with bounded pdf. Then $X\ -\ Y$ is reciprocal.

The following proposition gives a characterization of the normal distribution which is based on the concepts of sub-independence and self-reciprocal.

Proposition 2.5

Let X be the standard normal rv and Y be any infinitely divisible rv s.i. of X. Then $X\ +\ Y$ is self-reciprocal if and only if Y is normally distributed.

We can weaken the infinitely divisible hypothesis, but at the expense of considerable work, Theorem 4.3 of Hamedani and Walter (1985) stated below.

Theorem 2.5

Let X be the standard normal rv and Y be strictly self-reciprocal and s.i. of X. Then X + Y is self-reciprocal if and only if it is normally distributed.

The following characterization of strictly self-reciprocal distribution based on sub-independence is due to Hamedani and Walter (1987).

Proposition 2.6

Let X be the standard normal rv and Y a symmetric (about 0) rv s.i. of X. Then Y is strictly self-reciprocal if and only if the cf ϕ of the rv $X\ +\ Y$ satisfies the functional equation

$$\varphi(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\{(s+it)^2/2\}\varphi(s)ds, \qquad \text{for all } t \in \mathbb{R}.$$

2.3. Characterizations of Stable Distribution

Let $X, X_1, X_2, \dots, X_n$ be s.i.i.d. rv's. If X is normally distributed with mean zero, then $\sum_{i=1}^{n} X_i$ and $\sqrt{n}X$ are i.d. Hamedani et al. (2004) raised the following question: Are there other rv X for which properties similar to the one mentioned on the above two lines hold? Lukacs (1956) proved the following result (restated here in terms of cf ϕ of X) for the i.i.d. case.

Theorem 2.6

Let ϕ be a cf such that for every n and every choice of real numbers $a_1, a_2, \dots, a_n$,

$$\prod_{i=1}^{n} \varphi(a_i t) = \varphi\left(\gamma_n^{1/\alpha} t\right), \qquad t \in \mathbb{R},$$

where $\gamma_n = \sum_{i=1}^{n} |a_i|^{\alpha}$. Then ϕ is cf of a symmetric stable distribution of order α.

Laha and Lukacs (1965) improved Theorem 2.6 for the special case of normal distribution showing that if a cf ϕ satisfies

$$\varphi(t) = \left(\varphi(\sqrt{n}t)\right)^{1/n}, t \in \mathbb{R},$$

then it must be the cf of a normal distribution with mean zero.

Eaton (1966) improved Theorem 2.6 for particular fixed choices of the $a_i's$ and for fixed values of n under additional assumption that the rv X is symmetric about zero as follows.

Theorem 2.7

Let $X, X_1, X_2, \dots, X_n$ be i.i.d. symmetric rv's and let m and n be integers $2 \le m < n$ such that $\log m / \log n$ is irrational. If

(2.3.1)

$$\varphi(t) = \left(\varphi\left(m^{1/\alpha} t\right)\right)^{1/m} = \left(\varphi\left(n^{1/\alpha} t\right)\right)^{1/n}, \qquad t \in \mathbb{R},$$

where $0 < \alpha \le 2$, then X has a symmetric stable distribution of order α.

Remark 2.5

It should be noted that the proofs of Theorem 2.6, the result of Laha and Lukacs (1965), and similar result by Kagan et al. (1973) depend on the Levy-Khinchine representation of a characteristic function. Eaton's proof does not employ the Levy-Khinchine representation due to the fact that the random variable X is assumed to be symmetric. Hamedani et al. (2004) considered functional equation of type (2.3.1) and investigated the solutions of the equation. Since the functions that they considered are not cf's they did not have Lévy-Khinchine representation at their disposal. Nor they were able to assume that their solutions are real-valued as in the case considered by Eaton (1966). Instead, they extended Theorem 2.7 to $\phi: \mathbb{R} \to \mathbb{C}$, with appropriate constants multiple of $X_i's$ assumed to be s.i. and then directed their attention to the case, when the solution of their general equation is a cf (see Secs. 2–4 of Hamedani et al., 2004).

2.4. Characterizations of Sub-Independent Random Variables

Mohammadpour and Safe (2002) considered, among other things, a characterization of s.i. $S\alpha S$ rv's (symmetric α-stable) with discrete spectral measure. This work was presented in a conference and their main result is now included in a (2010) article by Mohammadpour which will be discussed later in this subsection.

Mohammadpour (2004) considered jointly (symmetric) Cauchy rv's defined as: the rv's $X_1, X_2, \dots, X_n$ are said to be jointly (symmetric) Cauchy, if their joint cf has the form

(2.4.1)

$$\varphi_{\mathrm{X}}(\mathbf{t}) = \exp\left\{\int_{S_n} |\mathbf{t}'\mathbf{s}|\Gamma(d\mathbf{s}) + i\mathbf{t}'\boldsymbol{\mu}\right\},$$

where $\boldsymbol{X} = (X_1, X_2, \dots, X_n)'$, $\boldsymbol{t} = (t_1, t_2, \dots, t_n)' \in \mathbb{R}^n$, Γ is a finite Borel symmetric measure on the unite sphere $S_n = \{\boldsymbol{s} = (s_1, s_2, \dots, s_n)' | \boldsymbol{s}'\boldsymbol{s} = 1\}$ of $\mathbb{R}^n$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n)' \in \mathbb{R}^n$. The measure Γ is unique and is called the spectral measure of the random vector $\boldsymbol{X}$. He also called rv's $X_1, X_2, \dots, X_n$ associated if for any functions f and $g: \mathbb{R}^n \to \mathbb{R}$ non-decreasing in each argument, $Cov(f(\boldsymbol{X}), g(\boldsymbol{X})) \geq 0$ whenever the covariance exists. He then presented two characterizations (Theorems 2.8 and 2.9 below) of the concept of s.i. based on jointly Cauchy distributed rv's. His theorems are clearly in different direction than the previously stated results and are based on the specific underlying distribution. These theorems, however, may have interesting applications.

Theorem 2.8

Let $X_1, X_2, \dots, X_n$ be jointly Cauchy rv's with joint cf (2.4.1). Then $X_1, X_2, \dots, X_n$ are s.i. if and only if the spectral measure Γ satisfies the condition

(2.42)

$$\Gamma(S_n^{\pm}) = \Gamma(S_n),$$

Where $S_n^{\pm} = \{(s_1, s_2, \dots, s_n) \in S_n | s_i \geq 0 \; for \; all \; i \; or \; s_i \leq 0 \; for \; all \; i\}$.

Theorem 2.9

Let $X_1, X_2, \dots, X_n$ be jointly Cauchy rv's with joint cf (2.4.1). Then sub-independence is a necessary and sufficient condition for association of $X_1, X_2, \dots, X_n$.

Remark 2.6

It was shown, via an example in Mohammadpour (2004), that condition (2.4.2) is not a necessary and sufficient condition for sub-independence of α-stable ($\alpha \neq 1$) random variables.

Mohammadpour (2010) stated the following definitions: A random vector $\boldsymbol{X}$ is said to be a Lévy stable, α-stable, $\alpha - S$, random vector in $\mathbb{R}^n$ if there are parameters $0 < \alpha \leq 2$, $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n)' \in \mathbb{R}^n$, positive definite matrix A of order *n* and a finite measure Γ on the unit sphere S^n of $\mathbb{R}^n$ such that

(2.4.3)

$$-\ln\varphi_{\mathbf{X}}(\mathbf{t}) = \begin{cases} \int_{S_n} |\mathbf{t}'\mathbf{s}| \left(1 - i\, sgn\,(\mathbf{t}'\mathbf{s}) \tan\left(\frac{\pi\alpha}{2}\right)\right) \Gamma(d\mathbf{s}) - i\mathbf{t}'\boldsymbol{\mu}, & 0 < \alpha \neq 1 < 2, \\ \int_{S_n} |\mathbf{t}'\mathbf{s}| \left(1 + i\,\frac{2}{\pi}\, sgn\,(\mathbf{t}'\mathbf{s}) \ln\, |\mathbf{t}'\mathbf{s}|\right) \Gamma(d\mathbf{s}) - i\mathbf{t}'\boldsymbol{\mu}, & \alpha = 1, \\ \mathbf{t}'A\mathbf{t} - i\mathbf{t}'\boldsymbol{\mu}, & \alpha = 2 \end{cases}$$

where as before $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)'$. Equation (2.4.3) takes a slightly different form for Lévy stable, α-stable, $S\,\alpha\,S$ (symmetric α-stable), and $\alpha - S$ random vectors (see, Mohammadpour, 2010, for the details). He then presented the following theorem.

Theorem 2.10

Let $0 < \alpha < 2$. A stable random vector $\boldsymbol{X}$ with $-ln$ cf (2.4.3) is s.i. if and only if for each set $\{l_1, l_2, \ldots, l_r\} \subseteq \{1, 2, \ldots, n\}$,

(2.4.4)

$$\int_{S_n} \left|\sum_{j=1}^{r} s_{l_j}\right|^{\alpha} \Gamma(d\mathbf{s}) = \int_{S_n} \sum_{j=1}^{r} \left|s_{l_j}\right|^{\alpha} \Gamma(d\mathbf{s}),$$

and

(2.4.5)

$$\begin{cases} \displaystyle\int_{S_n} \left|\sum_{j=1}^{r} s_{l_j}\right|^{\alpha} sgn\left(\sum_{j=1}^{r} s_{l_j}\right) \Gamma(d\mathbf{s}) = \int_{S_n} \sum_{j=1}^{r} \left|s_{l_j}\right|^{\alpha} sgn\left(s_{l_j}\right) \Gamma(d\mathbf{s}), & \alpha \neq 1, \\ \displaystyle\int_{S_n} \ln\left|\sum_{j=1}^{r} s_{l_j}\right| \sum_{j=1}^{r} s_{l_j}\, \Gamma(d\mathbf{s}) = \int_{S_n} \sum_{j=1}^{r} \ln\left|s_{l_j}\right| s_{l_j}\, \Gamma(d\mathbf{s}), & \alpha = 1. \end{cases}$$

2.5. Characterizations of Symmetric Distribution and Related Results

Behboodian (1989) reported the following result: Let X and Y be independent and $X + Y$ symmetric. If X is symmetric with cf $\phi_X(t) \neq 0$, for all t, then Y must be symmetric. Hamedani (1995) improved this result as follows (see Theorems 2.11 and 2.12 below).

Theorem 2.11

Let X and Y be s.i.i.d. rv's whose sum $X + Y$, is symmetric. Then X and Y are symmetric rv's.

Remark 2.7

If X and Y are s.i. symmetric rv's then clearly $X + Y$ is symmetric, however, the symmetry of $X + Y$ alone does not imply that the sub-independent (in fact independent) rv's X and Y are symmetric. Theorem 2.11 shows that the symmetry of $X + Y$ implies the symmetry of s.i. rv's X and Y if the latter are i.d. In the absence of s.i. one may have one of the following interesting cases.

Case (i). X and Y are symmetric and $X + Y$ is also symmetric.
Case (ii). X and Y are symmetric and i.d. but $X + Y$ is not symmetric.

The following Examples 2.2 and 2.3 will demonstrate these cases respectively.

Example 2.2

Let X and Y have the joint pdf

$$f(x,y)=\begin{cases}-\frac{1}{2xy^2}, & y<x<-1\\ \frac{1}{2xy^2}, & 1<x<y\\ 0, & \text{otherwise}\end{cases}$$

Then, clearly, the rv's X and Y and $X + Y$ are symmetric (indeed every linear combination of X and Y is symmetric.)

Example 2.3

Let U be a one-sided stable rv for $\alpha = 1$ (Feller, 1971, p. 542). Then cf of U, φ_U, is given by

$$\varphi_U(t)=\exp\left\{-|t|-i\frac{2}{\pi}t\ln(|t|)\right\}, \qquad t\in\mathbb{R}.$$

Let V_1, V_2, V_3 be i.i.d. with cf φ_U, and let $X = V_1 - V_2$, $Y = V_1 - V_3$. Then, $X + Y = 2V_1 - V_2 - V_3$ and

$$\varphi_X(t)=\varphi_Y(t)=e^{-2|t|}, \qquad t\in\mathbb{R},$$

$$\varphi_{X+Y}(t)=\exp\left\{-|4t|-i\left(\frac{4}{\pi}t\ln 2\right)t\right\},\ \ t\in\mathbb{R}.$$

We observe that X and Y are i.d. Cauchy rv's symmetric (about 0), but $X + Y$ is a Cauchy rv symmetric about $C = \frac{4\ln 2}{\pi}$.

The following theorem is Theorem 2 of Behboodian (1989) in which the assumption of independence is now replaced by that of sub-independence.

Theorem 2.12

Let X and Y be s.i. and $X + Y$ symmetric. If X is symmetric with cf $\varphi_X(t) \neq 0$ for all t, then Y must be symmetric.

Let $p_1 X_1, p_2 X_2, \ldots, p_{n-1} X_{n-1}, -X_n$ be s.i.i.d. rv's, where $p_i > 0$ for $i = 1, 2, \ldots, n-1$, and $\sum_{i=1}^{n-1} p_i = 1$ and let $Y = \sum_{i=1}^{n-1} p_i X_i - X_n$. It is clear that if X_1 is symmetric about c for some $c \in \mathbb{R}$, then Y is symmetric about 0. The question now is whether or not the converse statement is true. Hamedani and Volkmer (2003) showed that the converse is not true without additional assumptions. To see this, let φ be cf of X_1. Note that $\varphi\colon \mathbb{R} \to \mathbb{C}$ is continuous, $\varphi(0) = 1$ and $\varphi(-t) = \overline{\varphi(t)}$. X_1 is symmetric about c if and only if $e^{-ict}\ \varphi(t)$ is real-valued (see Feller, 1971). Hamedani and Volkmer (2003) gave the following characterization of symmetric rv's.

Theorem 2.13

Assume that Y (given above) is symmetric about 0 and $\phi(t)$ has no zeros and is right differentiable at 0. Then there exists c $\in \mathbb{R}$ such that X_1 is symmetric about c.

Remark 2.8

(a) It is clear that Theorem 2.13 holds for the special linear combination $Y = \sum_{i=1}^{n-1} X_i - nX_n$. (b) It is shown by an example (given below), in Hamedani and Volkmer (2003), that the conclusion of Theorem 2.13 is, in general, false if we drop the assumption that ϕ has no zeros.

Example 2.4

Define $u: \mathbb{R} \to \mathbb{C}$ by $u = \chi_{[0,1/2]} + i\,\chi_{[3/2,2]}$, where $\chi_{[a,\,b]}$ is the indicator function on $[a,\ b]$, and let

$$\varphi(t) = \int_{\mathbb{R}} u(s)\overline{u(s+t)}ds.$$

Since $\int_{\mathbb{R}}|u(s)|^2 ds = 1$, ϕ is a cf of a probability distribution (see Feller, 1971). The pdf f of this distribution is given by $f(x) = |\hat{u}(x)|^2$, where

$$\hat{u}(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{itx}u(t)dt.$$

For further details we refer the reader to their article.

2.6. Characterizations of Poisson and Cauchy Distributions

Here, we first present a sub-independent version of the famous Raikov's Theorem and then a simple characterization of the Cauchy distribution based on the notion of sub-independence.

Theorem 2.14 (Raikov)

If X and Y are non negative integer-valued rv's such $X + Y$ has a Poisson distribution and X and Y are s.i., then each of X and Y has a Poisson distribution.

Remark 2.9

The following two simple observations were made in Bansal et al. (1998): (i) If $W \sim C(0)$ (standard Cauchy), $X = \frac{1}{2}W$, and $Y = -\frac{1}{2W}$, then X and Y are s.i.i.d. with common distribution Cauchy with parameters $\theta = 0$ and $\gamma = \frac{1}{2}$. (ii) If the rv's W and $-\frac{1}{W}$ are s.i.i.d. and the rv $\frac{1}{2}\left(W - \frac{1}{W}\right) \sim C(0)$, then $W \sim C(0)$.

2.7. Central Limit Theorem for Sub-Independent Random Variables

As mentioned before, the well-known Khintchine's Law of Large Numbers and Lindeberg-Lévy's Central Limit Theorem as well as other important results can be stated in terms of s.i. rv's. Hamedani and Walter (1984a) reported several version of the central limit theorems for s.i.i.d. rv's. These results are stated in Propositions 2.7–2.9 below. For the sake of completeness, however, we first state the following two definitions.

Definition 2.1

Let $R_\lambda, \lambda \geq 0$ denote the set of all rv's X such that:

i. $E(|X|^\lambda) < \infty$,
ii. $E(X^k) = m_k$ for all $k = 1, 2, \dots, [\lambda]$, where m_k is the kth moment of Z, the standard normal rv.
Let M_λ denote the set of cdf's of $X \in R_\lambda$.

Definition 2.2

Let $d_\lambda, \lambda \geq 0$ be the function from $M_\lambda \times M_\lambda$ into $\mathbb{R}$ given by

$$d_\lambda(F, G) = \sup_{t\in\mathbb{R}}\left|E\left(\frac{e^{iXt} - e^{iYt}}{|t|^\lambda}\right)\right|,$$

where F and G are, respectively, the cdf's of two rv's X and Y.

Proposition 2.7

Let $(X_j)_{j\geq 1}$ be a sequence of s.i.i.d. rv's with mean 0, variance 1 and such that $E(|X|^\lambda) < \infty$ for some $\lambda > 2$. Then

$$2^{-\frac{n}{2}}\sum_{j=1}^{2^n} X_j \to Z\sim N(0,1) \text{ (has standard normal distribution)},$$

in distribution as $n \to \infty$ and their cdf's converge in the metric of M_λ. Moreover the rate of convergence is dominated by

$$d_\lambda\left(T_{\sqrt{2}}^{n}F, \Phi\right) < 2^{n(1-\lambda/2)}\left(E\left(|X|^\lambda\right) + E\left(|Z|^\lambda\right)\right),$$

where Φ is cdf of Z and $T_{\sqrt{2}}$ is a strictly contractive map on $(M_\lambda,\ d_\lambda)$.

Remark 2.10

Proposition 2.7 is valid only for the s.i.i.d. case and only for certain indices. However, the extension to the non-i.d. case and all indices is not difficult and is based on the following lemma.

Lemma 2.1

Let $X_1,\ X_2;\ Y_1,\ Y_2$ be pairs of s.i. rv's in R_λ and let $\alpha > 0$. Let $F_1,\ F_2;\ G_1,\ G_2$ be their cdf's. Then

$$d_\lambda(F, G) < \alpha^{-\lambda}\{d_\lambda(F_1, G_1) + d_\lambda(F_2, G_2)\},$$

where F is the cdf of $\frac{X_1+X_2}{\alpha}$ and G of $\frac{Y_1+Y_2}{\alpha}$.

Remark 2.11

It should be observed that Lemma 2.1 holds when $\frac{X_1+X_2}{\alpha}$ is not in R_λ necessarily.

Proposition 2.8

Let $(X_j)_{j\geq 1}$ be a sequence of s.i. rv's in R_λ for some $\lambda > 2$ whose distribution functions belong to a bounded set in M_λ. Then,

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n} X_j \to Z \ \textit{in} \text{ distribution as } n \to \infty.$$

Proposition 2.9

Let $(X_j)_{j\geq 1}$ be a sequence of s.i.i.d. rv's such that $E(|X|^\lambda) < \infty$ for some $0 < \lambda < 2$. If:

i. $1 \leq \lambda < 2$ *and the mean is 0, or*
ii. $0 < \lambda < 1$

and α satisfies $\alpha > 2^{1/\lambda}$, then

$$\alpha^{-n}\sum_{j=1}^{2^n} X_j \to Y \text{ in distribution as } n \to \infty,$$

where Y is the rv with the δ −distribution.

Remark 2.12

A similar idea was used by Trotter (1959) to give a proof of the central limit theorem. Although he gives a proof in which the use of cf is avoided, this will limit his result to the case of independent rv's. The approach in Hamedani and Walter (1984a) is considerably different and their result requires only the much weaker assumption of sub-independence.

2.8. A different But Equivalent Interpretation of Sub-Independence and Related Results

Ebrahimi et al. (2010) looked at the concept of sub-independence in different but equivalent definition which provides a better understanding of this concept. Here we copy a good portion of their article since it treats this notion in completely different direction than we have dealt with so far. They presented models for the joint distribution of uncorrelated variables that are not independent, but the distribution of their sum is given by the product of their marginal distributions. These models are referred to as the summable uncorrelated marginals distributions. They are developed utilizing the assumption of sub-independence, which has been employed in the present article in various directions, for the derivation of the distribution of the sum of random variables. One proposition, 2 lemmas, 3 definitions, and 3 examples which follow are due to Ebrahimi et al. (2010). The last example and theorem are new.

We will now revisit the definition of sub-independence of the rv's $X_1,\ X_2, \dots,\ X_n$. Let ($\mathbf{X} = (X_1,\ X_2, \dots,\ X_n)'$)' be a random vector with cdf F and cf $\Psi(\boldsymbol{t})$. Components of $\boldsymbol{X}$ are said to be s.i. if

(2.8.1)

$$\Psi(\mathbf{t}) = \prod_{i=1}^{n} \psi_i(t), \qquad \forall \mathbf{t} = (t, t, \dots, t)' \in \mathbb{R}^n,$$

where $\psi_i(t)$ is cf of X_i. We first consider the bivariate case $n = 2$ and let F be the cdf of $\boldsymbol{X} = (X_1,\ X_2)$, and $\mathbf{X}^* = (X_1^*, X_2^*)$ denote the random vector with cdf $F^*(x_1,\ x_2) = F_1(x_1)\, F_2(x_2)$, where F_i , $i = 1, 2$ is the marginal cdf of X_i.

Definition 2.3

F is said to be SUM (summable uncorrelated marginals) bivariate distribution if $X_1 + X_2 \stackrel{st}{=} X_1^* + X_2^*$, where $\stackrel{st}{=}$ denotes the stochastic equality. Random variables with a SUM joint distribution are referred to as SUM random variables.

It is clear that the SUM and sub-independence are equivalent, so the two terminologies can be used interchangeably. It is also clear that the class of SUM rv's are closed under scalar multiplication and addition under independence. That is, if $\boldsymbol{X} = (X_1,\ X_2)$ is a SUM random vector so is $a\,\boldsymbol{X}$ and if $\boldsymbol{Y} = (Y_1,\ Y_2)$ is another SUM random vector independent of $\boldsymbol{X}$, then $\boldsymbol{X} + \boldsymbol{Y}$ is also SUM random vector. However, the SUM property is directional in that X_1 and X_2 being SUM rv's does not imply that X_1 and aX_2 are SUM. Definition 2.3 can be generalized to any specific direction by $a_1X_1 + a_2X_2 \stackrel{st}{=} a_1X_1^* + a_2X_2^*$.

For continuous distributions, Kendall's tau τ and Spearman's rho ρ_s are given by

$$\tau = 4 \int \int_{\mathbb{R}^2} F(x_1, x_2) f(x_1, x_2) dx_1 dx_2 - 1$$

and

$$\rho_s = 12\int\int_{\mathbb{R}^2} F_1(x_1)F_2(x_2)f(x_1,x_2)dx_1dx_2 - 3.$$

These measures are invariant under strictly increasing transformations. For a SUM model both measures can be nonzero, one of them can be zero while the other one is not and both can be zero without the variables being independent.

We define a bivariate SUM copula to be a SUM distribution on the unit square $[0,1]^2$ with uniform marginals. We state the following two lemmas; the second one explains the construction of families of SUM models by linking the univariate pdf 's $f_i\,(x_i\,)$, $i\ =\ 1,2$.

Lemma 2.2

For any SUM copula, $\rho_s\ =\ 0$.

Lemma 2.3

Let $f_i\,(x_i)$, $i\ =\ 1,2$ *be pdf's and* $g(x_1,\ x_2)$ *a measurable function. Setx*

(2.8.2)

$$f_\beta(x_1,x_2) = f_1(x_1)f_2(x_1) + \beta_g(x_1,x_2), \qquad (x_1,x_2) \in \mathbb{R}^2.$$

Then for some $\beta\ \in\ \mathbb{R}$, $f_\beta(x_1,x_2)$ *is a SUM pdf with marginal pdf's* $f_i\,(x_i\,)$, $i\ =\ 1,2$ *provided that:*

a. $f_\beta(x_1,x_2) \geq 0,$
b. $\int_{\mathbb{R}} g(x_1,\ x_2)\ dx_1 = \int_{\mathbb{R}} g(x_1,\ x_2)\ dx_2 = 0$ *for all* $(x_1,\ x_2)\ \in\ \mathbb{R}^2$, *and*
c. $\int_{\mathbb{R}} g(c-t,t)dt = 0$ *for all* $c\ \in\ \mathbb{R}$.

The next example illustrates Lemmas 2.2 and 2.3.

Let $f_i(x_i)$, $i = 1, 2$ be two pdf's on [0, 1] and set

(2.8.3)

$$f_\beta(x_1,x_2) = f_1(x_1)f_2(x_2) + \beta\sin[2\pi(x_2-x_1)], \qquad (x_1,x_2) \in [0,1]^2,$$

such that for some $\beta\ \in\ \mathbb{R}$, $f_\beta(x_1,x_2)$ is a pdf on $[0,1]^2$. Two specific examples are as follows.

i. Let $f_i\,(x_i\,)\ =\ 1,\ i\ =\ 1,2$ be the pdf of uniform distribution on $[0,1]$ and $\beta = -\frac{1}{2}$. Then by Lemma 2.2, for (2.8.3), $\rho_s\ =\ 0$. It can be shown that $\tau\ \neq\ 0$.
ii. Let $f_1(x_1) = \frac{1}{2} + x_1$, $f_2(x_2) = 1$ and $\beta = -\frac{1}{2}$. It can be shown that, for (2.8.3), $\rho_s = -\frac{3}{4\pi^3}$ and $\tau = \frac{\pi-4}{8\pi^3}$.

For $g(x_1,\ x_2)\ =\ f_1(x_1)\ f_2(x_2)\ q(x_1,\ x_2)$, (2.8.2) will have the following form

(2.8.4)

$$f_\beta(x_1,x_2) = f_1(x_1)f_2(x_2)[1+\beta q(x_1,x_2)], (x_1,x_2) \in \mathbb{R}^2,$$

where $f_i\,(x_i\,)$, $i\ =\ 1,2$ are the marginal pdf's, $q(x_1,\ x_2)$ is a measurable bounded function on $\mathbb{R}^2$ with bound $|q(x_1,x_2)|\ \leq B$ and $\beta\ = B^{-1}$.

2.8.1. Bivariate SUM

The following result presents a method for constructing a bivariate SUM family with given marginal distributions and gives the mutual information measure, Kendall tau and Spearman's rho for the family.

Proposition 2.10

Let $f_i(x) = f(x)$, $i = 1, 2$ in (2.8.4) be a symmetric pdf and the linking function $q(x_1, x_2)$ be such that

(2.8.5)

$$-q(x_1, x_2) = q(x_2, x_1) = q(-x_1, x_2) = q(x_1, -x_2).$$

Then:

a. *the bivariate function (2.8.4) is the pdf of a family of SUM distributions with marginals $f_i(x) = f(x)$, $i = 1, 2$ and $(a_1 X_1, a_2 X_2)$, $a_i = \pm 1$, $i = 1, 2$ are SUM variables;*
b. *the mutual information for the family is given by*

$$M_\beta(X_1, X_2) = \sum_{n=1}^{\infty} \frac{\beta^{2n}}{(2n-1)2n} E_2\{E_1[q(X_1, X_2)]^{2n}\},$$

where E_i, $i = 1, 2$ denotes the expectation with respect to f_i;
c. $\tau = \rho_S = 0.$

Remark 2.13

(a) As applications of Proposition 2.10, Ebrahimi et al. (2010) presented two examples of bivariate pdf's, with appropriate functions $q(x_1, x_2)$, for which (X_1, X_2) has a SUM distribution and X_1 and X_2 are i.d. $N(0, 1)$. For one of the examples they find a closed form for $M_\beta(X_1, X_2)$ and for the other they provide an approximation. For both examples, of course, $\rho = \tau = \rho_S = 0$. We mention here examples in which (X_1, X_2) has a SUM distribution for appropriate functions $q(x_1, x_2)$ and X_1 and X_2 are i.d. with symmetric pdf's other than $N(0, 1)$.

i. Standard Cauchy: $f(x) = \frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$;
ii. Laplace Double Exponential: $f(x) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma}$, $x \in \mathbb{R}$;
iii. Hyperbolic Secant: $f(x) = \frac{1}{2_\gamma K_1(\alpha\gamma)} e^{-\alpha\sqrt{\gamma^2+(x-\mu)^2}}$, $x \in \mathbb{R}$, where K_1 is a modified Bessel function of the second kind;
iv. Logistic or Sech-Square(d): $f(x) = \frac{e^{-(x-\mu)/s}}{s\left(1+e^{-(x-\mu)/s}\right)^2} = \frac{1}{4s} Sech\left(\frac{x-\mu}{2s}\right)$, $\mu = mean$ and s is proportion to standard deviation;
v. Raised Cosine: $f(x) = \frac{1}{2s}\left[1 + \cos\left(\frac{\pi(x-\mu)}{s}\right)\right]$, $\mu - s \leq x \leq \mu + s$;
vi. Wigner Semicircle: $f(x) = \frac{2}{\pi r^2}\sqrt{r^2 - x^2}$, $-r < x < r$;
vii. $f(x) = \frac{1}{2\pi} \frac{\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2}$; $f(x) = \frac{2(2+x^2)\sin^2\left(\frac{x}{4}\right)+x\left(x+\sin\left(\frac{x}{2}\right)\right)}{2\pi x^2(1+x^2)}$; $f(x) = \frac{4\sin^2\left(\frac{x}{4}\right)}{\pi x^2}$; $f(x) = \frac{4\left(x-2\sin\left(\frac{x}{2}\right)\right)}{\pi x^3}$, $x \in \mathbb{R}$.

(b) The cf's corresponding to pdf's in (vii) are, respectively,

$$\varphi(t) = \begin{cases} 1-|t|, & \text{if } |t| \le 1 \\ 0, & \text{if } |t| > 1 \end{cases} \quad ; \qquad \varphi(t) = \begin{cases} 1-|t|, & \text{if } |t| \le \frac{1}{2} \\ \frac{1}{2} e^{-|t|+\frac{1}{2}}, & \text{if } |t| > \frac{1}{2} \end{cases} \quad ;$$

$$\varphi_1(t) = \begin{cases} 1-2|t|, & \text{if } |t| \le \frac{1}{2} \\ 0, & \text{if } |t| > \frac{1}{2} \end{cases} \quad ; \quad \varphi_2(t) = |\varphi_1(t)|^2, \quad t \in \mathbb{R} \quad .$$

(c) The graphs of the first two pdf's in (vii) are bell shaped and can be used to approximate normal pdf.
(d) Hamedani et al. (2013) presented various examples of bivariate mixture SUM distributions based on the pdf's given in (vii).

2.8.2. Multivariate SUM

We consider multivariate SUM random variables. Let F be the cdf of $\mathbf{X} = (X_1, X_2, \dots, X_n)'$ and $\mathbf{X}^* = (X_1^*, X_2^*, \dots, X_n^*)'$ denote the random vector with cdf $F^* = \prod_{i=1}^n F_i$, where F_i is the cdf of X_i .

Definition 2.4

F is said to be a SUMn (SUM distribution of order n) if $\sum_{i=1}^n X_i \overset{st}{=} \sum_{i=1}^n X_i^*$.

Definition 2.4 can be extended to the product of a linear combination of marginals, that is $\mathbf{a}'\mathbf{X} \overset{st}{=} \mathbf{a}'\mathbf{X}^*$, where $\mathbf{a}' = (a_1, a_2, \dots, a_n)$. A particular case of interest is when $a_i = 0, 1$, which leads to the following extension of Definition 2.4.

Definition 2.5

F is said to be a multivariate SUM distribution if it is SUMn and all p-dimensional marginal distributions, $p < n$ are SUMp. That is, $\mathbf{a}'\mathbf{X} \overset{st}{=} \mathbf{a}'\mathbf{X}^*$, for all $\mathbf{a}'s$ such that $a_k = 0, 1$ and $\sum_{k=1}^n a_k = p \le n$.

The following examples show variant of SUM distributions.

Example 2.6

Let $\mathbf{X} = (X_1, X_2, X_3)'$.

a. Consider the distribution with pdf

$$f_\beta(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}} \left(1 + \beta(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}}\right), \mathbf{x} \in \mathbb{R}^3,$$

where $\beta = B^{-1}$ and
(2.8.6)

$$\left|(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}}\right| \le B.$$

The corresponding cf is

$$\Psi_\beta(\mathbf{t}) = e^{-\frac{1}{2}\mathbf{t}'\mathbf{t}} - \frac{1}{2^{9/2}} \beta i (t_1 - t_2)(t_1 - t_3)(t_2 - t_3) e^{-\frac{1}{4}\mathbf{t}'\mathbf{t}}, \qquad \mathbf{t} \in \mathbb{R}^3,$$

where $\mathbf{t} = (t_1, t_2, t_3)'$. Clearly, $f_\beta(\mathbf{x})$ is SUM3. It can be shown that $f_\beta(x_i, x_j)$, $i \ne j = 1, 2, 3$ are SUM2 for all β satisfying (2.8.6). So, $f_\beta(\mathbf{x})$ is a trivariate SUM distribution. The univariate marginals are $N(0, 1)$, so the distributions of $\mathbf{a}'\mathbf{X}$ where $\sum_{k=1}^3 a_k = p \le 3$ are $N(0, p)$, $p = 2, 3$ given by the independent trivariate normal model.

b. consider the distribution with pdf

$$f_\beta(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}} \left(1 + \beta x_2 (x_1^2 - x_3^2) e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}}\right), \qquad \mathbf{x} \in \mathbb{R}^3,$$

where $\beta = B^{-1}$ and
(2.8.7)

$$\left| x_2 (x_1^2 - x_3^2) e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}} \right| \le B.$$

The corresponding cf is

$$\Psi_\beta(\mathbf{t}) = e^{-\frac{1}{2}\mathbf{t}'\mathbf{t}} - \frac{1}{2^{9/2}} \beta i t_2 (t_1^2 - t_3^2) e^{-\frac{1}{4}\mathbf{t}'\mathbf{t}}, \qquad \mathbf{t} \in \mathbb{R}^3.$$

Clearly, $f_\beta(\mathbf{x})$ is SUM3. It can be shown that for $\beta \neq 0$, $f_\beta(x_1, x_2)$, and $f_\beta(x_2, x_3)$ are not SUM2 and $f_\beta(x_1, x_3)$ is an independent BVN (bivariate normal) for all β satisfying (2.8.7). So, $f_\beta(\mathbf{x})$ is SUM3 but not trivariate SUM distribution. The univariate marginals are $N(0, 1)$, so the distribution of $X_1 + X_2 + X_3$ is $N(0, 3)$ given by the independent trivariate normal model.

Example 2.7

Let ($\mathbf{X} = (X_1, X_2, \dots, X_n)'$ have pdf

$$f_\beta(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}} \left(1 + \beta (x_1^2 - x_2^2) e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}} \prod_{k=1}^{n} x_k \right), \qquad \mathbf{x} \in \mathbb{R}^n,$$

so that $\beta = B^{-1}$ and

$$\left| (x_1^2 - x_2^2) e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}} \prod_{k=1}^{n} x_k \right| \le B.$$

The corresponding cf is

$$\Psi_\beta(\mathbf{t}) = e^{-\frac{1}{2}\mathbf{t}'\mathbf{t}} - \frac{\beta}{4} \left(\frac{i}{2\sqrt{2}}\right)^n (t_1^2 - t_2^2) e^{-\frac{1}{4}\mathbf{t}'\mathbf{t}} \prod_{k=1}^{n} t_k\,, \qquad \mathbf{t} \in \mathbb{R}^n,$$

$\mathbf{t}' = (t_1, t_2, \dots, t_n)$. Clearly, $f_\beta(\mathbf{x})$ is SUMn. It can be shown that all p-dimensional marginals, $p < n$, are independent normal. So, $f_\beta(\mathbf{x})$ is a multivariate SUM distribution. The univariate marginals are $N(0, 1)$, so the distributions of $\mathbf{a}'\mathbf{X}$ where $\sum_{k=1}^{n} a_k^2 = p \le n$ are $N(0, p)$, $p = 2, 3, \dots, n$ given by the independent n-variate normal model.

The following example is quite interesting in the sense that any subset of size $r < n$ is s.i. but not independent.

Example 2.8

Let $(X_1, X_2, \dots, X_n)$ have pdf given by

$$f_\beta(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}} \Bigg\{ 1 + \frac{\beta}{(2c)^{\frac{n}{2}+6}} (x_2^2 - x_1^2)[12c^2 - 2c(x_2^2 - x_1^2) + x_1^2 x_2^2] \times \left[1 + \sum_{k=3}^{n} \left(\frac{1}{4c^2}\right)^{k-2} \prod_{i=3}^{k} \left(2c - x_i^2\right) \right] e^{-\left(\frac{1}{4c}-\frac{1}{2}\right)\mathbf{x}'\mathbf{x}} \Bigg\}, \qquad \mathbf{x} \in \mathbb{R}^n,$$

where $0 < c < \frac{1}{2}$, $\beta = B^{-1}$ and

$$\left| \frac{1}{(2c)^{\frac{n}{2}+6}} \left(x_2^2 - x_1^2\right) \left[12c^2 - 2c\left(x_2^2 - x_1^2\right) + x_1^2 x_2^2\right] \times \left[1 + \sum_{k=3}^{n} \left(\frac{1}{4c^2}\right)^{k-2} \prod_{i=3}^{k} \left(2c - x_i^2\right) \right] e^{-\left(\frac{1}{4c}-\frac{1}{2}\right)\mathbf{x}'\mathbf{x}} \right| \leq B.$$

The cf for f_β is

$$\Psi_\beta(t_1, t_2, \dots, t_n) = e^{-\frac{1}{2}\sum_{j=1}^{n} t_j^2} + \beta e^{-c\sum_{j=1}^{n} t_j^2} \times \left(\sum_{k=2}^{n} \prod_{i=1}^{k} t_i^2 \right) (t_1^2 - t_2^2), (t_1, t_2, \dots, t_n) \in \mathbb{R}^n.$$

Then $X_j's$ are s.i.i.d. $N(0,1)$. The same is also true for random vector $(X_1, X_2, \dots, X_j)$, $j = 2, 3, \dots, n-1$. So, $X_1, X_2, \dots, X_n$ indeed form a sequence of s.i.i.d. rv's.

For $c = \frac{1}{4}$, we have pdf

$$f_\beta(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}} \Bigg\{ 1 + \beta 2^{\frac{n}{2}+4} (x_2^2 - x_1^2)[3 - 2(x_2^2 - x_1^2) + 4x_1^2 x_2^2] \times \left[1 + \sum_{k=3}^{n} \prod_{i=3}^{k} 2^{2k-5} \left(1 - 2x_i^2\right) \right] e^{-\left(\frac{1}{4c}-\frac{1}{2}\right)\mathbf{x}'\mathbf{x}} \Bigg\}, \qquad \mathbf{x} \in \mathbb{R}^n,$$

where $\beta = B_1^{-1}$ and

$$\left| 2^{\frac{n}{2}+4} \left(x_2^2 - x_1^2\right) \left[\left[3 - 2\left(x_2^2 - x_1^2\right) + 4x_1^2 x_2^2\right]\right] \times \left[1 + \sum_{k=3}^{n} 2^{2k-5} \prod_{i=3}^{k} \left(1 - 2x_i^2\right) \right] e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}} \right| \leq B_1.$$

The corresponding cf is

$$\Psi_\beta(t_1, t_2, \dots, t_n) = e^{-\frac{1}{2}\sum_{j=1}^{n} t_j^2} + \beta\, e^{-\frac{1}{2}\sum_{j=1}^{n} t_j^2} \times \left(\sum_{k=2}^{n} \prod_{i=1}^{k} t_i^2\right)(t_1^2 - t_2^2), (t_1, t_2, \dots, t_n) \in \mathbb{R}^n.$$

We end this subsection with a characterization of the multivariate SUM distribution.

Theorem 2.15

Let φ_j , $j = 1, 2, \dots, n$ be cf's and let

$$\Psi_\beta(t_1, t_2, \dots, t_n) = \prod_{j=1}^{n} \varphi_j(t_j) + \beta q(t_1, t_2, \dots, t_n),$$

where $q(\boldsymbol{t})$, $\boldsymbol{t} \in \mathbb{R}^n$ is non negative definite, continuous at the origin and $q(t, t, \dots, t) = 0$ for $t \in \mathbb{R}$. Then for some constant β, $\boldsymbol{\Psi}_\beta$ is cf of a SUM distribution if $|\boldsymbol{\Psi}_\beta(\boldsymbol{t})| \leq 1$ for all $\boldsymbol{t} \in \mathbb{R}^n$.

Proof

$\boldsymbol{\Psi}_\beta$ is non negative definite, continuous at the origin and $\boldsymbol{\Psi}_\beta(\boldsymbol{0}) = 1$. Then by Bochner's Theorem $\boldsymbol{\Psi}_\beta$ is a cf.

Remark 2.14

Construction of a desirable $\boldsymbol{\Psi}_\beta$ boils down to choosing appropriate function $q(t_1, t_2, \dots, t_n)$.

2.9. Equivalence of Sub-Independence and Independence in Special Cases

We raised the question earlier that under what conditions sub-independence implies independence. It is possible to have an answer for this question if the underlying joint distribution has a specific form, for example, jointly distributed and uncorrelated normal rv's are independent. Mohammadpour (2010) had an answer for our question which again is based on a specific underlying distribution as follows.

Theorem 2.16

Let $0 < \alpha < 1$ and $X_1, X_2, \dots, X_n$ be jointly α-stable rv's. Then $X_1, X_2, \dots, X_n$ are s.i. if and only if they are independent.

Proposition 2.11

Let $0 < \alpha < 1$. A stable random vector $\boldsymbol{X} = (X_1, X_2, \dots, X_n)'$ with −ln cf (2.4.3) is s.i. if and only if all of its components $X_j's$ are independent rv's.

Proposition 2.12

Let $A = (a_{jk})$ be a positive definite matrix such that $a_{jk} \leq 0$ for $j \neq k$, $j, k = 1, 2, \dots, n$. A sub-Gaussian random vector $\boldsymbol{X}$ with

$$-\ln \varphi_{\mathrm{X}}(\mathbf{t}) = (\mathbf{t}' A \mathbf{t})^{\alpha/2} - i\mathbf{t}'\boldsymbol{\mu}, \qquad 0 < \alpha \leq 2,$$

is s.i. if and only if $\alpha = 2$ and A is a diagonal matrix (i.e., components of $\boldsymbol{X}$ are independent and normally distributed).

Proposition 2.13

A Gaussian random vector $\boldsymbol{X} = (X_1, X_2, \dots, X_n)'$ is s.i. if and only if all its components $X_j's$ are independent rv's.

The final result relates the SUM distributions to the well-known notions of POD (Positive Orthant Dependence) and NOD (Negative Orthant Dependence) defined as follows. We like to mention here that Mohammadpour (2010) has implicitly used POD for stable distribution which was discussed earlier.

Definition 2.6

A multivariate distribution F is said to be POD (NOD) if

$$\bar{F}(x_1,\, x_2, \dots,\, x_n) \geq (\leq) \prod_{i=1}^{n} \bar{F}_i(x_i),$$

where $\bar{F}(x_1,\, x_2, \dots,\, x_n) = P(X_1 > x_1,\, X_2 > x_2, \dots,\, X_n > x_n)$ and $\bar{F}_i(x_i) = P(X_i > x_i)$.

It should be noted that POD (NOD) are the weakest among all the existing notions of dependence. The special case of $n = 2$ is known as positive (negative) quadrant dependence. It is known that under POD (NOD), if $\rho(X_i\,,\, X_j) = 0$, then X_i and X_j are pairwise independent, without implying any higher order dependency among $X_i's$. For details about POD (NOD) and other notions of dependence see Lehmann (1966) and Barlow and Proschan (1981). The following result due to Ebrahimi et al. (2010) shows that under POD (NOD), SUM\ model implies independence.

Lemma 2.4

Let $\boldsymbol{X} = (X_1,\, X_2, \dots,\, X_n)'$ be a non negative random vector with a POD (NOD) distribution F. Then F is a SUM distribution if and only if $F(x) = \prod_{i=1}^{n} F_i(x_i)$, where F_i is cdf of X_i.

2.10. Dissociation, Sub-Independence

De Paula (2008) presented a bivariate distribution for which

(2.10.1)

$$E(Y^n|X) = E(Y^n) \text{ and } E(X^n|Y) = E(X^n), \qquad n = 1,2, \dots,$$

i.e., X^m and Y^n are uncorrelated for all positive integers m and n, but X and Y are not independent. De Paula's goal was to show a measure of dissociation between two dependent rv's X and Y beyond the concept of uncorrelatedness of X and Y. Hamedani and Volkmer (2009a,b) showed that the rv's considered in De Paula (2008) are not s.i. Then, they presented a bivariate distribution for which (2.10.1) holds, X and Y are s.i. but not independent. This provides a stronger measure of dissociation between X and Y. Here is the example.

Example 2.9

Let

$$\theta(s) = \begin{cases} Ae^{1/(4s^2-1)} & \text{if } |s| < \dfrac{1}{2} \\ 0 & \text{otherwise} \end{cases},$$

where $A > 0$ is chosen so that

$$\int_{-\infty}^{\infty} \theta^2(s)ds = 1.$$

The inverse Fourier transformation g of θ is

$$g(x) = (2\pi)^{-1} A \int_{-1/2}^{1/2} e^{1/(4s^2-1)} e^{-isx} ds$$

$$= \pi^{-1} A \int_{0}^{1/2} e^{1/(4s^2-1)} \cos(sx)\, ds.$$

Define φ by

$$\varphi(s) = \int_{-\infty}^{\infty} \theta(u)\theta(s+u)du.$$

Then φ is the cf of a pdf $f(x)$ and $\varphi(s) = 0$ for $|s| \geq 1$. In fact, we have

$$f(x) = 2\pi g^2(x) = 2\pi^{-1} A^2 \left(\int_{0}^{1/2} e^{1/(4s^2-1)} \cos(sx)\, ds \right)^2.$$

Now define

$$h(x, y) = f(x)f(y)[1 + \cos(x)\cos(3y)], \qquad (x, y) \in \mathbb{R}^2.$$

Then $h(x, y)$ is a pdf with marginals $f(x)$, $f(y)$, and it can be shown that $Cov(X^m, Y^n) = 0$ for $m, n = 1, 2, \ldots$ and X and Y are s.i.

Remark 2.15

a. A general version of the above example with corresponding technical derivations can be found in Hamedani and Volkmer (2009a).
b. One can easily generalize the above example to a random vector $(X_1, X_2, \ldots, X_n)$.

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