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Robust PI controller design satisfying sensitivity and uncertainty specifications

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ABSTRACT

This paper presents a control design method for determining proportional-integral-type controllers satisfying specifications on gain margin, phase margin, and an upper bound on the (complementary) sensitivity for a finite set of plants. The approach can be applied to plants that are stable or unstable, plants given by a model or measured data, and plants of any order, including plants with delays. The algorithm is efficient and fast, and as such can be used in near real-time to determine controller parameters (for online modification of the plant model including its uncertainty and/or the specifications). The method gives an optimal controller for a practical definition of optimality. Furthermore, it enables the graphical portrayal of design tradeoffs in a single plot, highlighting the effects of the gain margin, complementary sensitivity bound, low frequency sensitivity and high frequency sensor noise amplification.

SECTION I. INTRODUCTION

Although many methods for tuning proportional–integral (PI) and proportional–integral–derivative (PID) controllers exist, extensive research in design techniques continues, driven by the strong use of such controllers in industry. Depending on the types of specifications that the design must satisfy, the tuning methods reported in the literature can be summarized as falling into one of two categories.

One class of methods considers gain and phase margin specifications. Ho et al. [1], [2] developed simple analytical formulae to tune PI and PID controllers for commonly used first-order and second-order plus dead-time plant models to meet gain and phase margin specifications. Ho et al. [3], [4] reported tuning formulae for the design of PID controllers that satisfy both robustness and performance requirements. Crowe and Johnson [5] presented an automatic PI control design algorithm to satisfy gain and phase margin based on a converging algorithm. Suchomoski [6] developed a tuning method for PI and PID controllers that can shape the nominal stability, transient performance, and control signal to meet gain and phase margins.

A second class of design methods focuses on sensitivity specifications, and is based on the premise that gain and phase margin specifications may fail to guarantee a reasonable bound on the sensitivity. Ogawa [7] used the QFT-framework to propose a PI design technique that satisfies a bound on the sensitivity for an uncertain plant. Poulin and Pomerleau [8] developed a PI design methodology for integrating processes that bounds the maximum peak resonance of the closed loop. The peak resonance constraint is equivalent to bounding the complementary sensitivity, which can be converted to bounding the sensitivity. Cavicchi [9] described a design method for bounding the sensitivity while achieving desired steady-state performance. The method can also be applied to measured data. However, plant uncertainty is not considered, and the procedure fits a simple compensation structure. Crowe and Johnson [10] reported a design approach to find a PI/PID controller that bounds the sensitivity while satisfying a phase margin condition. Kristiansson and Lennartson [11] emphasized the need to bound the sensitivity and complementary sensitivity. They suggested the use of an optimization routine to design PI and PID controllers with low-pass filters on the derivative gain to optimize for control efforts, disturbance rejection and bound on the sensitivity. They also provided tuning rules for nonoscillatory stable plants and plants with a single integrator. Astrom et al. [12] described a numerical method for designing PI controllers based on optimization of load disturbance rejection with constraints on sensitivity and weighting of set point response.

Other investigators have pursued research into tuning methods. For example, Yeung et al. [13] presented a nontrial and error graphical design technique for controller design of the lead-lag structure that enables simultaneous fulfillment of gain margin, phase margin and crossover frequency.

These papers and many others apply gain and phase margin constraints in finding PI and PID controller designs. Some add limitations on the (complementary) sensitivity. However, there are several differences between approaches reported in the literature and the idea proposed here. First, the approach here bounds the sensitivity of the closed-loop transfer function for all frequencies, not just at the crossover frequencies where the gain and phase margins are satisfied. Second, the approach developed here accounts for plant uncertainty, in that the controller design must satisfy the specifications for a set of plants. Third, the approach presented provides explicit equations to determine the set of all possible controllers. Fourth, with this method it is possible to extract the optimal control design solution for many practical optimization criteria. Fifth, the algorithm can be applied to many types of plants, including continuous and discrete plants, plants with pure delay, nonminimum phase plants, and stable and unstable plants. Sixth, since the algorithm uses explicit equations, and not optimization routines, it is very fast.

SECTION II. PROBLEM STATEMENT

Consider an open-loop transfer function, $L(s)$

$$L(s) = C(s)P(s) \quad (1)$$

where $P(s)$ is a member of a finite set of plants, $P_1(s), \dots, P_n(s)$, and $C(s)$ is a PI controller

$$C(s) = \frac{a(1+bs)}{s}. \quad (2)$$

The gain and phase margin conditions, the typical measures of robustness, are replaced by a condition on the closed-loop sensitivity inequality

$$\left| \frac{1}{1+kL(s)} \right| \leq M \text{ for } s = j\omega, \forall \omega \geq 0, k \in [1, K] \quad (3)$$

where the sensitivity bound $M > 1$ and the gain uncertainty of the plant, k , is in the interval $[1, K]$. It can be shown [14] that when $\arg L(j\omega) = -\pi$ rad, then (3) requires $|L(j\omega)| \leq (M-1)/M$ for $K = 1$ and, thus, the gain margin for a given K is at least

$$GM = 20\log_{10}(K) + 20\log_{10}\left(\frac{M}{M-1}\right). \quad (4)$$

Similarly, when $|L(j\omega)| = 1$, (3) requires $\arg L(j\omega) > -\pi + 2\arcsin[(2M)^{-1}]$ and, thus, the phase margin is at least

$$PM = 2\arcsin\left(\frac{1}{2M}\right). \quad (5)$$

Inequality (3) is a more encompassing measure of robustness than gain and phase margin. It places a bound on the sensitivity at all frequencies, not just at the two frequencies associated with the gain and phase margins.

The design problem of interest is to find all (a, b) pairs that satisfy (3) for all $P(s) \in [P_1(s), \dots, P_n(s)]$. For plants that include at least one integrator, the sensitivity is proportional to $1/a$ at low frequencies, and for any plant the sensor noise at the plant input is amplified by ab at high frequencies. As such, it is of particular interest to find the pair (a, b) for which a is maximum and its associated ab is smallest.

SECTION III. MAIN RESULTS

To determine the (a, b) values for which the closed-loop system is stable and (3) is satisfied, consider first the special case of no gain uncertainty, i.e., $K = 1$, and a single plant $P(s)$. Splitting $P(s)$ for $s = j\omega$ into its real and imaginary parts, $P(j\omega) = A(\omega) + jB(\omega)$, and substituting it and (2) into (3) gives

$$Da^2(1 + b^2\omega^2) + 2aA - 2ab\omega B + 1 - M^{-2} \geq 0 \forall \omega \geq 0 \quad (6)$$

where $D = A^2 + B^2$. For an (a, b) pair which is on the boundary region of the allowed (a, b) values, there exists ω such that (6) is an equality. Moreover, since at that particular ω , (6) is minimum, its derivative (with respect to ω) at the same ω is zero. Thus

$$[2E(1 + b^2\omega^2) + 2D\omega b^2]a + 2A - 2b(\omega B + B) = 0 \quad (7)$$

where $E = AA + BB$ and the dot indicates derivative with respect to ω . From (7)

$$a = \frac{-A + b\omega B + bB}{E + Eb^2\omega^2 + D\omega b^2}. \quad (8)$$

Substituting (8) into the equality of (6) gives a fourth-order equation for b

$$x_4b^4 + x_3b^3 + x_2b^2 + x_1b + x_0 = 0 \quad (9)$$

where $Q = 1 - M^2$ and

$$x_4 = (QE^2 - 2BBE + DB^2)\omega^4 + (-2B^2E + 2QED)\omega^3 + (-DB^2 + QD^2)\omega^2$$

$$x_3 = (2BAE + 2ABE - 2DAB)\omega^3 + (2ABE + 2ABD)\omega^2 + 2ABD\omega$$

$$x_2 = (DB^2 + DA^2 + 2QE^2 - 2AAE - 2BBE)\omega^2 + (2QED - 2AAD - 2B^2E + 2DBB)\omega + DB^2$$

$$x_1 = (2BAE + 2ABE - 2DAB)\omega - 2DAB + 2ABE$$

$$x_0 = -2AAE + DA^2 + QE^2$$

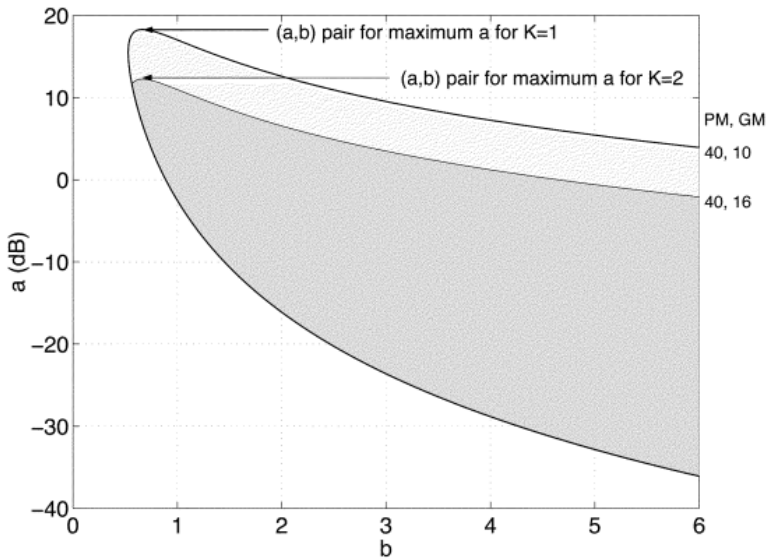


Fig. 1. Region of (a, b) values for $M = 1.46$, equivalent to 40° phase margin (PM) or greater and 10-dB-gain margin or greater (GM for $K = 1$) for Example 1. Lower shaded region is for $M = 1.46$ with additional 6-dB-plant gain uncertainty ($K = 2$) for a total of 16-dB or greater

The allowed (a, b) region for a given M value can be calculated as follows: For a given ω solve (9) for b . Noting that b has four solutions (for a given ω), select the positive real solution for which the resulting closed-loop system is stable and (3) is satisfied for $K = 1$. Then, use (8) to find its corresponding a . Searching over a range of frequencies ω enables the boundary of the (a, b) region to be identified.

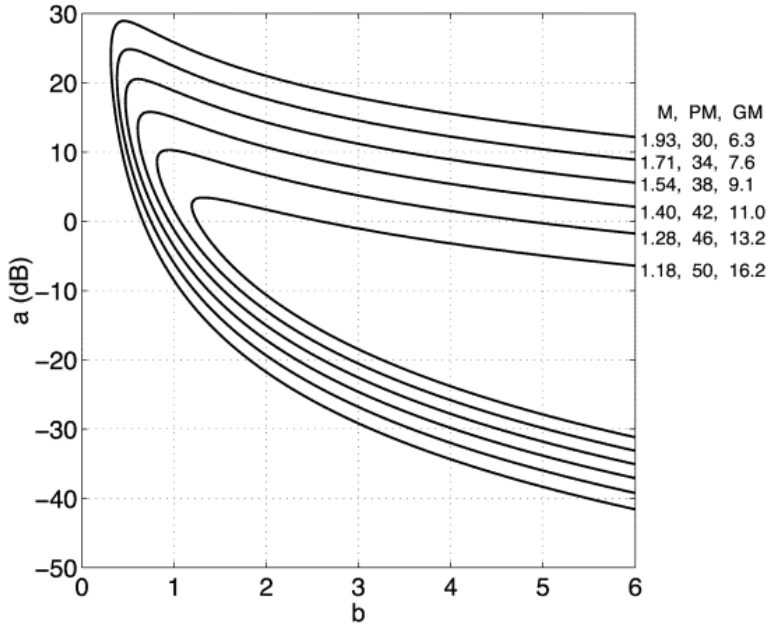


Fig. 2. Boundary curves of (a, b) region that satisfy $|(1/1 + L)| < M$ for Example 1. Marked on the right of each curve is its M value, minimal phase margin (PM) and minimal gain margin (GM in dB for $K = 1$) according to (4) and (5)

Remark 3.1

A PI controller exists if and only if there exists a frequency for which an (a, b) pair solving (9) can be found for which the resulting closed-loop system is stable and (3) for $K = 1$ is satisfied. If a PI controller does not exist and is required, try increasing M .

A. Example 1: Simplified dc Motor

Consider the plant

$$P(s) = \frac{1}{s(1+\frac{s}{10})} \quad (10)$$

which can represent a simplified model of an armature-controlled dc motor with the input being motor current and the output being speed. For this plant

$$A(\omega) = \frac{-10}{(100 + \omega^2)} \quad A(\omega) = \frac{20\omega}{(100 + \omega^2)^2}$$

$$B(\omega) = \frac{-100}{\omega(100 + \omega^2)} \quad B(\omega) = \frac{100(100 + 3\omega^2)}{\omega^2(100 + \omega^2)^2}$$

Fig. 1 depicts the (a, b) values for the particular case of $M = 1.46$, which is equivalent to a 40° phase margin or greater and a 10-dB-gain margin or greater. [The (a, b) values fall in both shaded regions.] Fig. 1 can also be used to find the (a, b) values which satisfy any gain margin constraint. For example, if 6-dB-gain margin uncertainty is desired (i.e., $K = 2$), then for any b , the allowed a values should be 6-dB less in order to cope with the increase in uncertainty. The (a, b) region will, therefore, be the lower shaded region depicted in Fig. 1 where the upper curve is shifted down by 6 dB. The maximum a for $K = 1$ occurs at $(a, b) = (18.2\text{dB}, 0.67)$ and maximum a for $K = 2$ occurs at $(a, b) = (12.2\text{dB}, 0.67)$, giving the controller designs corresponding to lowest sensitivity at low frequencies. Note that if gain uncertainty K is required then a solution is guaranteed only if there exists at least a single b corresponding to a range of a values in an interval $[a_1, a_2]$ such that $a_2/a_1 \geq K$.

The solution for several M values for plant (10) is depicted in Fig. 2. Each curve is the boundary of the allowed (a, b) values for a given M . The corresponding PM and GM values indicated are the minimum values along the boundary curve, i.e., the PM and GM are equal or greater along the curve.

B. Extension to Complementary Sensitivity

Replacing the sensitivity margin constraint (3) by the complementary sensitivity

$$\left| \frac{kL(j\omega)}{1+kL(j\omega)} \right| \leq M \forall \omega \geq 0, k \in [1, K] \quad (11)$$

it can be shown that $L = L_0$ satisfies (3) if and only if $L = (M^2/M^2 - 1)L_0$ satisfies (11). This leads to the following corollary: If (a, b) is a pair that solves the problem stated in Section II, then the pair

$$\left(\frac{M^2 - 1}{M^2} a, b \right)$$

solves the same problem where (3) is replaced by (11).

SECTION IV. OPTIMIZATION

The answer to the question “Which is the best (a, b) pair?” of course depends on the optimization criterion. Seron and Goodwin [15] note that “In general, the process noise spectrum is typically concentrated at low frequencies, while the measurement noise spectrum is typically more significant at high frequencies.” It follows that an optimal controller can be found by weighting the performance at low frequencies and noise at high frequencies. Since the high frequency noise is proportional to ab and the low frequency performance to a , the optimal solution must lie on the boundary of the (a, b) curve. Moreover, if there exists more than one boundary pair for the same a , the one with the lowest ab will be the best. The same condition appears in Kristiansson and Lennartson [11] who proposed several evaluation criteria, one being the ability of the system to handle low frequency load disturbance, represented here by parameter a .

Note that if the open-loop system does not include an integrator, the maximum gain may not correspond to a practical optimal choice.

SECTION V. EXTENSION TO UNCERTAIN PLANTS

Assume that the plant, $P(s)$, is known to be one of a finite set of plants, $P_1(s), \dots, P_n(s)$. The controller design challenge here is to find all (a, b) pairs that solve the problem stated in Section II where $P(s)$ can be any member of the set. This (a, b) region will be the intersection of all (a, b) regions of members of the set (if this intersection region is empty, then there exists no PI solution). As an example, consider the plant set

$$P(s) = \frac{\text{gain}}{s \left(1 + \frac{s}{\text{pole}}\right)} \text{ for gain} = [1,3]$$

$$\text{and pole} = [10,12,14,16,18,20]$$

where $M = 1.46$ as before. Fig. 3 shows the intersection as the shaded region. The pair corresponding to maximum a is $(a, b) = (8.6\text{dB}, 0.62)$.

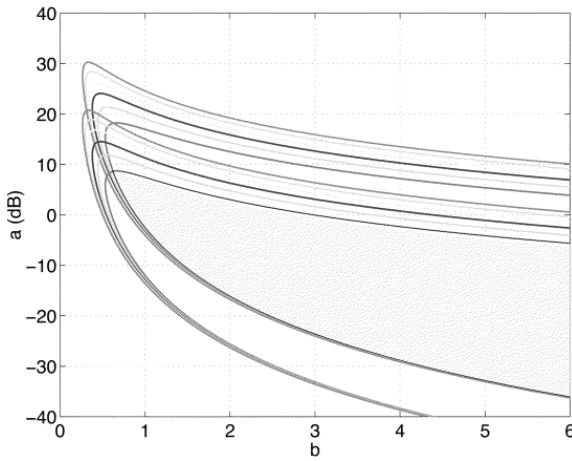


Fig. 3. Boundary curves of (a, b) region that satisfy $|(1/1 + L)| < M$ for $M = 1.46$ for a set of plants. The intersection of all regions is the allowed region. $M = 1.46$ is equivalent to 40° phase margin (PM) or greater and 10-dB-gain margin or greater

SECTION VI. DISCRETE PI CONTROLLERS

The problem can be recast in its discrete form, where the plant is $P(z)$ and the controller (2) is replaced by its discrete equivalent

$$C(z) = a_d(1 + b_d(1 - z^{-1})).$$

Using the bilinear transformation, $z = ((1 + j\Omega)/(1 - j\Omega))$ where $\Omega = j\omega T/2$, the plant can be written in the form $P((1 + j\Omega)/(1 - j\Omega))$, the controller in the form

$$C(\Omega) = \frac{a_d(1 + (2b_d + 1)j\Omega)}{1 + j\Omega}$$

and the open-loop transfer function in the form

$$L(\Omega) = a_d(1 + (2b_d + 1)j\Omega) \frac{P\left(\frac{1 + j\Omega}{1 - j\Omega}\right)}{1 + j\Omega}.$$

The latter three equations translate the discrete problem into the one previously defined for finding the (a_d, b_d) region. The procedure is as follows: Solve the problem defined in Section II where $P(z)$ at frequencies on the unit circle is replaced by $P((1 + j\Omega)/(1 - j\Omega))/(1 + j\Omega)$ to determine the (a, b) region. Then, (a_d, b_d) will be the region defined by $(a, (b - 1)/2)$.

SECTION VII. CONCLUSION

The note presents explicit equations for calculating PI controllers that simultaneously stabilize a given set of plants and satisfy design specifications, namely gain margin and phase margin constraints and a bound on the (complementary) sensitivity, for continuous as well as discrete-time systems. The algorithm fits any plant dimension including pure delay. Moreover, the algorithm answers the question if a solution whose bandwidth is in a given interval exists or not.

The two parameters of PI controllers satisfying the constraints correspond to a domain in a plane whose boundary is a curve given explicitly. For a practical optimization criterion presented here, the optimal controller lies on the curve. By inspection, the design plot enables identification of the PI controller for desired robustness conditions, and in particular, gives the PI controller for lowest sensitivity. Tradeoffs among high-frequency sensor noise, low frequency sensitivity, and gain and phase margin constraints are also directly available.

The algorithm can be executed very fast for highly uncertain plants, and as such the controller design can be updated in near real-time to reflect changes in plant uncertainty and/or closed-loop specifications.

Work on extending the PI algorithm proposed here to the important class of PID controllers and to controllers based on extended PID structures, such as PID controllers with filtered D-terms, is now in progress.

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