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THE ANTISYMMETRY BETWEENNESS AXIOM AND HAUSDORFF CONTINUA

PAUL BANKSTON

ABSTRACT. An interpretation of betweenness on a set satisfies the *antisymmetry* axiom at a point a if it is impossible for each of two distinct points to lie between the other and a. In this paper we study the role of antisymmetry as it applies to the K-*interpretation of betweenness* in a Hausdorff continuum X, where a point c lies *between* points a and b exactly when every subcontinuum of X containing both a and b contains c as well.

1. INTRODUCTION

An interpretation of betweenness on a set satisfies the *antisymmetry* axiom at a point a, or is *antisymmetric at* a, if it is impossible for each of two distinct points to lie between the other and a. Expressed as a first-order formula (see, e.g., [8]) involving just one ternary relation symbol and equality, this axiom is

$$\text{Antisymmetry at } a\text{: } \forall\, xy\,(([a,y,x] \wedge [a,x,y]) \rightarrow x = y).$$

And from this it is clear that "antisymmetry at a" is the usual order-theoretic notion of antisymmetry for the (generally reflexive and transitive) binary relation $\leq_a$, defined by saying $x \leq_a y$ exactly when x lies between a and y. Binary antisymmetry is the very condition that turns a

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pre-order into a partial order; we say that an interpretetion of betweenness is *antisymmetric* if it is antisymmetric at each of its points. This notion has been around a long time in studies of betweenness (called "Postulate C" in [9], "closure" in [18]), and is traditionally taken to be as fundamental an assumption about betweenness relations as is *symmetry*, the condition that lying between a and b is the same as lying between b and a.

Here we take a different view of antisymmetry, and treat it as a feature of betweenness that is as "honoured in the breach as in the observance." Indeed, we consider interpretations of betweenness for which this feature fails quite dramatically.

In [2, 3] (and further in [4]), we discuss three topological interpretations of betweenness, each reflecting an aspect of connectedness. The most restrictive of these was introduced by L. E. Ward [22] to study cut points in an abstract setting, and is what we call the Q-*interpretation*: in a topological space X, $[a, c, b]_{\mathsf{Q}}$ holds just in case either $c \in \{a, b\}$ or a and b lie in different quasicomponents of $X \setminus \{c\}$. That is, there are disjoint sets A and B, each clopen in $X \setminus \{c\}$, such that $a \in A$ and $b \in B$.

When we replace "quasicomponent" in the definition above with "component," we obtain the C-*interpretation* $[\cdot, \cdot, \cdot]_{\mathsf{C}}$ of betweenness. The C-interpretation is generally weaker than the Q-interpretation, as quasicomponents are unions of components; a space is called QC-*complete* if the two interpretations agree. It is easy to show that a connected T_1 space is QC-complete if it is *locally connected*, i.e., in possession of an open base consisting of connected sets, but local connectedness is not necessary for QC-completeness to occur (see Example 3.4 (ii) below). A cut point of a connected space is precisely one that lies between two points other than itself, in either the Q- or the C-interpretation.

Example 1.1. In the euclidean plane, we set $X = \{a, b\} \cup \bigcup_{n=1}^{\infty} A_n$, where $a = \langle \frac{1}{2}, 0\rangle$, $b = \langle 1, 0\rangle$, and $A_n = \{\langle x, \frac{x}{n}\rangle : 0 \leq x \leq 1\}$, $n = 1, 2, \ldots$ Then X is a connected metrizable space. If $c = \langle 0, 0\rangle$, then $[a, c, b]_{\mathsf{C}}$ holds because the components of $X \setminus \{c\}$ consist of $\{a\}$, $\{b\}$, and the half-closed segments $A_n \setminus \{c\}$, $n = 1, 2, \ldots$ However, if U is a clopen neighborhood of a in $X \setminus \{c\}$, then $A_n \setminus \{c\} \subseteq U$ for all but finitely many n. Hence we have $b \in U$, and infer that $[a, c, b]_{\mathsf{Q}}$ does not hold. Thus X is not QC-complete.

The following is a direct consequence of [2, Theorem 6.1.2]; we include a simple proof in order to highlight a classic result from the elementary theory of connected spaces.

Proposition 1.2. *The* Q- *and* C-*interpretations of betweenness in a connected space are antisymmetric.*

Proof. Suppose we have X a connected topological space, with $[a, c, b]_{\mathsf{C}}$ holding for some $a, b, c \in X$, $b \neq c$. It suffices to find a connected subset of X that contains a and c, but not b. Clearly we are done if $c = a$; so the alternative is that the three points are distinct, and there must be distinct components A and B of $X \setminus \{c\}$ such that $a \in A$ and $b \in B$. This tells us that $X \setminus B$ is a connected subset of $X \setminus \{b\}$ containing both a and c (see [17, Theorem IV.3.3], due to K. Kuratowski and B. Knaster), and hence that $[a, b, c]_{\mathsf{C}}$ cannot hold.

Because the Q-interpretation is more restrictive than the C-interpretation, it too must be antisymmetric. □

In this paper, a *continuum* is a connected compact Hausdorff space. Thus the terms "continuum" and "Hausdorff continuum" are synonymous; i.e., we do not assume our continua to be metrizable. A continuum (or any set) is *nondegenerate* if it contains at least two points; a subset of a topological space is a *subcontinuum* if it is a continuum in its subspace topology.

Note that if $c \notin \{a, b\}$, then $[a, c, b]_{\mathsf{C}}$ holds just in case no connected subset of $X \setminus \{c\}$ contains both a and b. If we replace "connected subset" in this condition with "subcontinuum," we obtain the K-*interpretation* of betweenness: for $c \notin \{a, b\}$, $[a, c, b]_{\mathsf{K}}$ holds just in case a and b lie in separate *continuum components* of $X \setminus \{c\}$. A point $c \in X$ lies between two points other than itself in the K-interpretation precisely when c is a weak cut point of X. (Aspects of the relations $\leq_a$ associated with the K-interpretation are also studied in [12]. There the pre-order $\leq_a$ is termed the "weak cut point order" based at a.)

The Q- and the C-interpretations of betweenness are antisymmetric; not so the K-interpretation. For if X is the $\sin(\frac{1}{x})$-*continuum* in the euclidean plane (see, e.g., [16] and Example 2.3 below), a is a point on the graph of $y = \sin(\frac{1}{x})$, $0 < x \leq 1$, and b and c are any two points on the vertical segment $\{0\} \times [-1, 1]$, then we have both $[a, b, c]_{\mathsf{K}}$ and $[a, c, b]_{\mathsf{K}}$ holding. Generally, it is easy to find continua that are not K-antisymmetric, and hence not CK-complete.

As an obvious shorthand, when we say that a continuum is *antisymmetric (at a point)*, we have the K-interpretation of betweenness firmly in mind. Thus the $\sin(\frac{1}{x})$-continuum is not antisymmetric, but it does have points of antisymmetry (see Example 2.3).

Remark 1.3. Antisymmetry in metrizable continua has also been studied, under the label "Property C," by B. E. Wilder [24].

2. Antisymmetric Road Systems

In [2, 3] we view betweenness as arising from the primitive structure given by a *road system*. This is a family $\mathcal{R}$ of nonempty subsets of a set X—the *roads* of the system—such that: (1) every singleton subset of X

is a road; and (2) every doubleton subset of X is contained in a road. Roads "connect" one point to another in a very minimal sense; the set of roads connecting $a, b \in X$ is denoted $\mathcal{R}(a,b) := \{R \in \mathcal{R} : a, b \in R\}$. The ternary relation $[\cdot,\cdot,\cdot]_{\mathcal{R}}$ *induced* by $\mathcal{R}$ is defined by saying that $[a,c,b]_{\mathcal{R}}$ holds just in case $c \in R$ for every $R \in \mathcal{R}(a,b)$. A ternary relation $[\cdot,\cdot,\cdot]$ on a set X is an *R-relation* if it equals $[\cdot,\cdot,\cdot]_{\mathcal{R}}$ for some road system $\mathcal{R}$ on X.

Remark 2.1. If $\mathcal{A}$ is any collection of nonempty subsets of X, we may define $[\cdot,\cdot,\cdot]_{\mathcal{A}}$ as above. If $\mathcal{A}$ also satisfies the condition that for each two points $a, b \in X$, there are sets $A, B \in \mathcal{A}$ with $a \in A \subseteq X \setminus \{b\}$ and $b \in B \subseteq X \setminus \{a\}$, then $\mathcal{R} = \mathcal{A} \cup \{X\} \cup \{\{a\} : a \in X\}$ is a road system with $[\cdot,\cdot,\cdot]_{\mathcal{R}} = [\cdot,\cdot,\cdot]_{\mathcal{A}}$.

When a and b are points of a road system $\langle X, \mathcal{R}\rangle$, the $\mathcal{R}$*-interval* $[a,b]_{\mathcal{R}}$ is defined to be the 1-slice $[a,\cdot,b]_{\mathcal{R}}$ (i.e., the intersection $\bigcap \mathcal{R}(a,b)$). The road system is *antisymmetric (at a point)* if the same can be said for its induced R-relation. Phrased in interval terms, antisymmetry at a says that $[a,b]_{\mathcal{R}} \neq [a,c]_{\mathcal{R}}$ whenever $b \neq c$.

If X is a connected topological space, it is easy to see that $[\cdot,\cdot,\cdot]_{\mathsf{C}} = [\cdot,\cdot,\cdot]_{\mathcal{C}}$, where $\mathcal{C}$ is the antisymmetric road system comprising the connected subsets of X. (In light of Remark 2.1, it is not really necessary to assume X is connected: we could otherwise turn $\mathcal{C}$ into a road system simply by declaring X to be a road.) It is equally easy to see that $[\cdot,\cdot,\cdot]_{\mathsf{K}} = [\cdot,\cdot,\cdot]_{\mathcal{K}}$, where $\mathcal{K}$ is the (not necessarily antisymmetric) road system comprising the subcontinua of X (again, with the possibility of X being thrown in). Both of these road systems satisfy the important property of *additivity*, which says that the union of two overlapping roads is a road.

An obvious question at this point is whether there is an additive antisymmetric road system $\mathcal{Q}$ inducing the Q-interpretation of betweenness on a connected space. While there is an affirmative answer to this, no inducing road system so far obtained seems to arise "naturally."

Theorem 2.2. [2, Corollary 6.2.2] *If X is a connected topological space, then the Q-interpretation of betweenness is induced by an additive antisymmetric road system $\mathcal{Q}$, which may be taken to contain $\mathcal{C}$.*

As mentioned above, antisymmetry in the betweenness context is closely related to the binary notion of antisymmetry that turns pre-orderings into partial orderings. That is, if $\langle X, \mathcal{R}\rangle$ is a road system which is antisymmetric at $a \in X$, and we define the binary relation $\leq_a$ to be the 2-slice $[a,\cdot,\cdot]_{\mathcal{R}}$ (i.e., $c \leq_a b$ just in case $[a,c,b]_{\mathcal{R}}$ holds), then $\leq_a$ is a partial ordering with bottom element a.

Example 2.3. Let X be the $\sin(\frac{1}{x})$-continuum in the euclidean plane; i.e., $X = A \cup S$, where $A = \{0\} \times [-1, 1]$ and $S = \{\langle x, \sin(\frac{1}{x})\rangle : 0 < x \leq 1\}$. Then $a \in X$ is a point of antisymmetry for X if and only if $a \in A$. If a is one of the non-cut points of the line segment A, say $a = \langle 0, -1\rangle$, then $\leq_a$ is described as follows: (1) for $b \in A$ and $c \in S$ we have $b \leq_a c$; (2) for $b, c \in A$, say $b = \langle 0, s\rangle$ and $c = \langle 0, t\rangle$, we have $b \leq_a c$ just when $s \leq t$; and (3) for $b, c \in S$, say $b = \langle s, \sin(\frac{1}{s})\rangle$ and $c = \langle t, \sin(\frac{1}{t})\rangle$, we have $b \leq_a c$ just when $s \leq t$. Thus $\leq_a$ is a total ordering. If it happens that a is a cut point of A, say $a = \langle 0, 0\rangle$, then the new description of $\leq_a$ differs from that above only in clause 2: for $b = \langle 0, s\rangle$ and $c = \langle 0, t\rangle$, $b \leq_a c$ just when either $0 \leq s \leq t$ or $t \leq s \leq 0$. This ordering is not total because the two non-cut points of A are $\leq_a$-incomparable.

A partial ordering is a *tree* ordering if: (1) each two elements have a common lower bound; and (2) no two incomparable elements have a common upper bound. In Example 2.3, with a a cut point of A, $\leq_a$ is not a tree ordering because any point of S is a common upper bound for the two $\leq_a$-incomparable non-cut points of A. However, if enough else is going on for a road system, the orderings $\leq_a$ do turn out to be trees (see Lemma 2.4 below) .

If $\langle X, \mathcal{R}\rangle$ is a road system, it is always the case that $[a, c]_{\mathcal{R}} \cup [c, b]_{\mathcal{R}} \subseteq [a, b]_{\mathcal{R}}$ whenever $c \in [a, b]_{\mathcal{R}}$. If the reverse inclusion also holds, we say the road system—or the induced R-relation—is *weakly disjunctive.* If $\mathcal{R}$ is additive, as is the case with all three of our topological betweenness interpretations, then the induced R-relation is actually *disjunctive*; i.e., it satisfies the stronger condition that $[a, b]_{\mathcal{R}} \subseteq [a, c]_{\mathcal{R}} \cup [c, b]_{\mathcal{R}}$ for all $a, b, c \in X$ (not just for $c \in [a, b]_{\mathcal{R}}$).

For any R-relation $\langle X, [\cdot, \cdot, \cdot]\rangle$ and $a, b \in X$, define the binary relation $\leq_{ab}$ on X to be the restriction of $\leq_a$ to the interval $[a, b]$. The following two results will be used extensively in the sequel.

Lemma 2.4. [2, Propositions 5.0.4 and 5.0.5] *If $\langle X, [\cdot, \cdot, \cdot]\rangle$ is an R-relation that is antisymmetric and weakly disjunctive, then each partial ordering $\leq_a$ is a tree ordering, and each partial ordering $\leq_{ab}$ is a total ordering. Moreover, $\leq_{ba}$ is the relation-inverse of $\leq_{ab}$.*

Lemma 2.5. [2, Theorem 5.0.6] *For a weakly disjunctive R-relation, the following conditions are equivalent:*

- (i) *Antisymmetry.*
- (ii) *Slenderness: the property that if $c \in [a, b]$, then $[a, c] \cap [c, b] = \{c\}$.*
- (iii) *Reciprocity: the property that if $c, d \in [a, b]$ and $c \in [a, d]$, then $d \in [c, b]$.*
- (iv) *Uniqueness of Centroids: the property that $[a, b] \cap [a, c] \cap [b, c]$ has at most one element, for each $a, b, c \in X$.*

3. Antisymmetry, Aposyndesis, and Decomposability

From here on, all topological spaces are assumed to be Hausdorff; as mentioned earlier, this separation axiom–but not metrizability–is built in to our definition of "continuum." A continuum is *aposyndetic* (see, e.g., [11]) if for each two of its points, there is a subcontinuum excluding one of the points and containing the other in its interior. Aposyndesis has the syntactic shape of a souped-up T_1 axiom, but is actually a weak form of local connectedness.

One consequence of aposyndesis concerns the betweenness relation $[\cdot,\cdot,\cdot]_{\mathsf{K}}$ itself, as a subset of the cartesian cube of a continuum. Define a continuum X to be K-*closed* if $[\cdot,\cdot,\cdot]_{\mathsf{K}}$ is closed in X^3.

Theorem 3.1. *All aposyndetic continua are* K-*closed.*

Proof. Suppose X is aposyndetic and that $[a,c,b]_{\mathsf{K}}$ does not hold. Then there is a subcontinuum $M \in \mathcal{K}(a,b)$ with $c \notin M$. Using aposyndesis, for each $x \in \{a,b\}$, we have an open set U_x and subcontinuum M_x such that $x \in U_x \subseteq M_x \subseteq X \setminus \{c\}$. Let U_c be an open neighborhood of c missing the subcontinuum $M_a \cup M \cup M_b$. Then $U_a \times U_c \times U_b$ is an open neighborhood of $\langle a,c,b\rangle$ in X^3 that does not intersect $[\cdot,\cdot,\cdot]_{\mathsf{K}}$. Hence X is K-closed. □

In an aposyndetic continuum, not only is $[\cdot,\cdot,\cdot]_{\mathsf{K}}$ a compact relation, but so are all of its slices (including $\leq_a = [a,\cdot,\cdot]_{\mathsf{K}}$). Of course the 1-slice $[a,\cdot,b]_{\mathsf{K}}$ is always compact, but that is the only nontrivial slice of $[\cdot,\cdot,\cdot]_{\mathsf{K}}$ guaranteed to be so (see Example 3.4 (ii) below).

A second consequence of aposyndesis is that there is a collapsing of betweenness interpretations.

Theorem 3.2. *Aposyndetic continua are* CK-*complete, and therefore antisymmetric. Locally connected continua are* QK-*complete.*

Proof. Assume X is aposyndetic, with $a,b \in X$. If $c \notin [a,b]_{\mathsf{C}}$, then there is a witness $A \in \mathcal{C}(a,b)$ with $c \notin A$. For each $x \in A$, use aposyndesis to find open set U_x and subcontinuum M_x, with $x \in U_x \subseteq M_x \subseteq X \setminus \{c\}$. Then $\mathcal{U} = \{U_x : x \in A\}$ is a cover of the connected set A by open sets; hence, for some $n = 1,2,\ldots$, there is an n-tuple $\langle U_{x_1},\ldots,U_{x_n}\rangle$ from $\mathcal{U}$, with $a \in U_{x_1}$, $b \in U_{x_n}$, and $U_{x_i} \cap U_{x_{i+1}} \neq \emptyset$ for each $1 \leq i \leq n-1$. Thus $M = M_{x_1} \cup \cdots \cup M_{x_n} \in \mathcal{K}(a,b)$, and $c \notin M$. This says $c \notin [a,b]_{\mathsf{K}}$, and we conclude that X is CK-complete.

That X is antisymmetric now follows from Proposition 1.2. If X is locally connected, then, as mentioned earlier, X is QC-complete as well as CK-complete. Hence X is QK-complete. □

Remark 3.3. That antisymmetry in metrizable continua is a consequence of aposyndesis was previously shown in [24].

The following examples show that aposyndesis is a strong assumption in Theorems 3.1 and 3.2.

Examples 3.4.

(i) Let X be the *topologist's oscilloscope* in the euclidean plane; i.e.,

$$X = V_0 \cup V_1 \cup H_0 \cup H_1 \cup S,$$

where, for $i = 0, 1$, $V_i = \{i\} \times [-1, 1]$ and $H_i = [0, 1] \times \{(-1)^i\}$, and $S = \{\langle x, \frac{1}{2}\sin(\frac{\pi}{x})\rangle : 0 < x \leq 1\}$. Then X is both QK-complete and K-closed, but not aposyndetic.

Failure of aposyndesis is clear. As for the other assertions, note that $[a, c, b]_{\mathsf{K}}$ holds in X if and only if either $c = a$ or $c = b$. Thus K-intervals are trivial, and we have QK-completeness immediately. Also we see that $[\cdot, \cdot, \cdot]_{\mathsf{K}} = (\Delta_X \times X) \cup (X \times \Delta_X)$, where $\Delta_X = \{\langle x, x\rangle : x \in X\}$; and so K-closedness is also immediate.

(ii) Let X be the *comb space* in the euclidean plane; i.e.,

$$X = ([0, 1] \times \{0\}) \cup (\{0\} \times [0, 1]) \cup \bigcup_{n=1}^{\infty} (\{\tfrac{1}{n}\} \times [0, 1]).$$

Then X is antisymmetric without being either CK-complete or K-closed. In particular (Theorem 3.2), X is not aposyndetic. (X is, however, QC-complete.)

Antisymmetry is easy to see. As for failure of CK-completeness, let $a = \langle 0, 0\rangle$ and $b = \langle 0, 1\rangle$. Then $[a, b]_{\mathsf{K}} = \{0\} \times [0, 1]$, while $[a, b]_{\mathsf{C}} = \{a, b\}$. QC-completeness is easy to check; as for failure of K-closedness, note that if we take c to be $\langle 0, \frac{1}{2}\rangle$, then the 1-slice $[b, c, \cdot]_{\mathsf{K}}$ is $X \setminus (\{0\} \times (\frac{1}{2}, 1])$, which is clearly not closed in X.

Remarks 3.5.

(i) Theorem 3.2 shows that CK-completeness interpolates between aposyndesis and antisymmetry; Examples 3.4 (i,ii) show that the three notions are distinct.

(ii) The topologist's oscilloscope (Example 3.4 (i)) shows that aposyndesis does not follow from K-closedness alone. It does follow, however, if we also assume hereditary unicoherence (see Theorem 4.2 below).

By a *decomposition* of a continuum X we mean a pair $\langle M, N\rangle$, where M and N are proper subcontinua of X and $X = M \cup N$. X is *decomposable* if it has a decomposition, and *indecomposable* otherwise.

For any continuum X and $A \subseteq X$, recall that X is *irredicible about* A if the only subcontinuum of X containing A is X itself. X is *irreducible* if X is irreducible about a two-point set; i.e., if $[a, b]_{\mathsf{K}} = X$ for some $a, b \in X$,

$a \neq b$. The point a is a *point of irreducibility* for X if $[a,b]_{\mathsf{K}} = X$ for some $b \in X \setminus \{a\}$.

The *composant* κ_a of a in X is the union of all proper subcontinua of X that contain a. Hence a is a point of irreducibility for X if and only if $\kappa_a \neq X$. A composant of continuum X is clearly connected; less obvious is the fact that it is also dense in X. (This follows easily from one of the so-called "boundary bumping" theorems, namely [16, Theorem 5.4]: if U is a nonempty proper open subset of continuum X and K is a component of the closure $\overline{U}$ of U, then K intersects $X \setminus U$.)

If X is decomposable, then [16, Theorem 11.13] either: (1) X is irreducible and has precisely three composants (including itself); or (2) X is not irreducible and has just itself as composant. In any event, decomposable continua have either one or three composants, with no two of them disjoint. On the other hand, if X is indecomposable, then no two of its composants can overlap. Moreover, the number of composants of a nondegenerate indecomposable continuum that is metrizable is $\mathfrak{c}$, the cardinality of the real line [14, Theorem 1]. Metrizability is crucial for this result, as it is possible for a nondegenerate indecomposable continuum to have either one or two composants [7, Theorem 1 (& Corollary)]. Nevertheless, it is still the case that every nondegenerate indecomposable continuum contains an indecomposable subcontinuum with $\mathfrak{c}$ composants [6, Corollary 5].

In the sequel, the default notion of betweenness in a continuum is the K-interpretation, and we thus drop the letter "K" from most prefixes and subscripts. If M is a subcontinuum of X and $a, b \in M$, then the interval $[a,b]^M = [a,b]^M_{\mathsf{K}}$ *relative to* M is defined to be $\bigcap\{K \in \mathcal{K}(a,b) : K \subseteq M\}$. Clearly $[a,b]^M \subseteq [a,b]^N$ whenever $a, b \in N \subseteq M$.

A continuum X is *hereditarily antisymmetric* (resp., *hereditarily decomposable*) if each of its nondegenerate subcontinua is antisymmetric (resp., decomposable).

Theorem 3.6. *Every hereditarily antisymmetric continuum is hereditarily decomposable.*

Proof. Let X be a continuum that is not hereditarily decomposable. Then, by definition, X must contain a nondegenerate indecomposabe subcontinuum M, which Bellamy's theorem [6] tells us may be assumed to contain two disjoint composants A and B. Since composants are dense, they're nondegenerate; hence we may pick $a \in A$ and $b, c \in B$, with $b \neq c$. Then $[a,b]^M = [a,c]^M = M$, so M is not antisymmetric. This shows that X is not hereditarily antisymmetric. $\square$

Remarks 3.7.

(i) The converse of Theorem 3.6 is false because the $\sin(\frac{1}{x})$-continuum of Example 2.3 is hereditarily decomposable without being (hereditarily) antisymmetric.

(ii) In [24], Wilder views the property of antisymmetry for metrizable continua as interpolating between aposyndesis and decomposability, much as Jones [11] views aposyndesis as interpolating between local connectedness and decomposability. While aposyndesis implies decomposability for general continua, we do not know whether antisymmetry does as well. Any indecomposable antisymmetric continuum, however, would necessarily have just one composant.

4. The Gap Free Axioms

Consider the following two first-order conditions that may be imposed on a ternary structure.

Gap Freeness: $\forall ab \exists x (a \neq b \to ([a,x,b] \wedge x \neq a \wedge x \neq b))$; and
Strong Gap Freeness: $\forall ab \exists x (a \neq b \to ([a,x,b] \wedge \neg[x,a,b] \wedge \neg[a,b,x]))$.

Gap freeness says that no interval has exactly two points, and is a straightforward generalization of density as understood in the order-theoretic context. Strong gap freeness clearly implies gap freeness for any R-relation; and if antisymmetry holds, the converse is also true. In the setting of continua, Q-gap freeness, i.e., gap freeness for $[\cdot,\cdot,\cdot]_{\mathsf{Q}}$, is the defining condition for a continuum to be a *dendron*, and is equivalent [23] to the *connected intersection property*: the intersection if any two connected subsets is connected.

In [3] we consider the problem of obtaining a similar result for the K-interpretation, and so far there is a complete answer only in the case of strong gap freeness.

First note that when the connected intersection property is formally weakened to allow only intersections of subcontinua, we arrive at the well-studied notion of *hereditary unicoherence*, a property equivalent [3, Proposition 2.1] to the condition that all intervals are connected. Hereditary unicoherence clearly then implies gap freeness; however the converse does not hold: by simply taking two pseudo-arcs and sewing them together along two disjoint nondegenerate subcontinua [3, Theorem 2.6], we obtain a *crooked annulus*, a continuum which is gap free, with plenty of disconnected intervals. We still do not have a nontrivial topological characterization of gap freeness; nor do we know of a first-order betweenness statement that captures hereditary unicoherence. For strong gap freeness, however, there is a satisfying characterization.

Theorem 4.1. [3, Theorem 4.4 and Corollary 4.5] *Let X be a continuum; the following statements are equivalent:*

(i) *X is strongly gap free.*
(ii) *Every nondegenerate interval in X is a decomposable continuum.*
(iii) *X is both hereditarily unicoherent and hereditarily decomposable.*

While K-closedness alone is not enough to ensure aposyndesis in a continuum (see Example 3.4 (i)), the addition of hereditary unicoherence does the trick.

Theorem 4.2. *Every hereditarily unicoherent K-closed continuum is aposyndetic.*

Proof. Assume X is hereditarily unicoherent, as well as K-closed, with a and b distinct points of X. Then $[a,b,a]$ does not hold; and by K-closedness, there are open sets U_a and U_b, with $a \in U_a$ and $b \in U_b$, such that if $\langle x,z,y\rangle \in U_a \times U_b \times U_a$, then $[x,z,y]$ does not hold either. In particular, for each $\langle x,z\rangle \in U_a \times U_b$, there is a subcontinuum of X that contains both a and x, but not z. Thus, for each $x \in U_a$ we have $[a,x] \cap U_b = \emptyset$, and so the closed subset

$$M = \overline{\bigcup_{x \in U_a} [a,x]}$$

of X contains U_a and misses U_b. By hereditary unicoherence, each $[a,x]$ is a subcontinuum of X [3, Proposition 2.1]. Hence M is a subcontinuum of X that contains a in its interior and excludes b; thereby establishing aposyndesis for X. □

Next in this section, we prove an analogue of Theorem 4.1 in which strong gap freeness in clause (i) is replaced by the conjunction of gap freeness and antisymmetry.

Recall (see, e.g., [16, Theorem 6.6]) that every nondegenerate continuum has at least two non-cut points; a continuum with exactly two is called an *arc*. (Sometimes called a *Hausdorff arc* or a *generalized arc*. It is a famous result of continuum theory that any two metrizable arcs are homeomorphic.) The next result is well known [16, Theorem 6.16], and crucial to our immediate endeavor.

Lemma 4.3. *Let X be a topological space, with $a, b \in X$ distinct. The following statements are equivalent:*

(i) *X is an arc, with a and b its two non-cut points.*
(ii) *The topology on X is induced by a bounded complete dense total ordering that has a and b for its two end points.*

If X is an antisymmetric continuum and $a \in X$, recall from Lemma 2.4 that each binary relation $\leq_{ab}$ is a total ordering on $[a, b]$, and is inverse to the ordering $\leq_{ba}$.

Lemma 4.4. *Let X be an antisymmetric continuum, with $a, b \in X$. Then the order topology on $[a, b]$ induced by $\leq_{ab}$ coincides with the subspace topology on $[a, b]$.*

Proof. Fix $a, b \in X$. For $x \leq_{ab} y$ in $[a, b]$, let $[x, y]_{ab}$ be the order interval $\{z \in [a, b] : x \leq_{ab} z \leq_{ab} y\}$. Then the order intervals $[a, y]_{ab}$ and $[x, b]_{ab}$, $x, y \in [a, b]$, subbasically generate the closed sets in the order topology on $[a, b]$. So fix $x \leq_{ab} y$ in $[a, b]$. Then for any $z \in [a, b]$, we have: $z \in [a, y]_{ab}$ if and only if $a \leq_{ab} z \leq_{ab} y$, if and only if $z \leq_a y$, if and only if $z \in [a, y]$. Also, $z \in [x, b]_{ab}$ if and only if $x \leq_{ab} z \leq_{ab} b$, if and only if $b \leq_{ba} z \leq_{ba} x$, if and only if $z \leq_b x$, if and only if $z \in [b, x] = [x, b]$. Intervals, being intersections of subcontinua, are closed in the subspace topology. Therefore the order-closed subsets of $[a, b]$ are subspace-closed, implying that the order topology on $[a, b]$ is compact. The order topology on $[a, b]$ is also Hausdorff. Since there cannot be two distinct compact Hausdorff topologies with one finer than the other, we conclude that the order topology and the subspace topology on $[a, b]$ coincide. $\square$

Theorem 4.5. *Let X be a continuum; the following statements are equivalent:*

(i) *X is antisymmetric and gap free.*
(ii) *Every nondegenerate interval $[a, b]$ in X is an arc, with non-cut points a and b.*
(iii) *Every interval in X is a locally connected continuum.*
(iv) *Every interval in X is an aposyndetic continuum.*
(v) *Every interval in X is an antisymmetric continuum.*

Proof. The implications (ii) $\implies$ (iii) $\implies$ (iv) are immediate, and the implication (iv) $\implies$ (v) follows from Theorem 3.2; so first assume (v) holds, and try to prove (i). If a and b are distinct in X, then $[a, b]$ is a nondegenerate continuum and hence must contain a third point. This gives us gap freeness. Suppose $a, b, c \in X$, $c \in [a, b]$, and $c \neq b$. Then $[a, b]$ is an antisymmetric continuum; hence $b \notin [a, c]$, and we infer that X is antisymmetric. This proves (i).

Now assume (i) holds, and try to prove (ii). If $a, b \in X$ are distinct, then, by Lemma 4.4, the total order $\leq_{ab}$ induces the subspace topology on $[a, b]$. Since intervals are compact, the ordering is complete; and, by gap freeness, the ordering is dense as well. Applying Lemma 4.3, $[a, b]$ is an arc with non-cut points a and b, and we have (ii) holding. $\square$

As another corollary of the two preceding lemmas, we have the following.

Theorem 4.6. *A continuum is an arc if and only if it is antisymmetric and irreducible.*

Proof. Arcs are antisymmetric and irreducible. For the converse, suppose X is antisymmetric and irreducible about distinct points a and b. Then, since $[a,b]=X$, the orderings $\leq_a$ and $\leq_{ab}$ are identical. By Lemma 4.4, X as a topological space is totally ordered by $\leq_{ab}$. Since X is a continuum, we may apply Lemma 4.3 to infer that X is an arc (with non-cut points a and b). $\square$

Remark 4.7. The version of Theorem 4.6 for the metrizable case has already been proved in [24].

A topological space is *arcwise connected* if each two of its points are the non-cut points of an arc in the space. Following the coinage in [15], call a continuum an *arboroid* if it is both hereditarily unicoherent and arcwise connected; call it a *λ-arboroid* if it is both hereditarily unicoherent and hereditarily decomposable. (Then a metrizable arboroid (resp., metrizable λ-arboroid) is just a dendroid (resp., λ-dendroid) in the usual sense; and if we add local connectedness in either case, we obtain a dendrite (see, e.g., [16]).) An immediate consequence of Theorems 4.1 and 4.5 is the following new characterization of arboroids and λ-arboroids.

Corollary 4.8.

(i) *A continuum is an arboroid if and only if it is antisymmetric and gap free.*
(ii) *A continuum is a λ-arboroid if and only if it is strongly gap free.*

Remarks 4.9.

(i) From Corollary 4.8, it is immediate that arboroids are hereditarily decomposable. This was first posed as a question by L. E. Ward [21] and answered by D. Bellamy [6, Corollary 11].
(ii) Dendrons, the continua that are Q-gap free, are known [20, Lemma 4] to be locally connected. Hence we may use Theorem 3.2 and Corollary 4.8 to see that dendrons are indeed arboroids.

More importantly, Corollary 4.8 allows us to view the continuum-theoretic notions of dendron, arboroid, and λ-arboroid as different versions of gap freeness. This suggests a notion of "arboriality" for R-relations in general, and is the subject of ongoing work (see [4]).

Recall that if X is an aposyndetic continuum, then X is K-closed (Theorem 3.1), and hence each 1-slice $[a,c,\cdot]$ is closed in X. Relative to the tree ordering $\leq_a$ (see Lemma 2.4 and Theorem 3.2), this set is the principal $\leq_a$-filter generated by c, and is itself a tree ordering with bottom element c. As with any partial ordering, a *branch* is a maximal totally ordered subset.

Theorem 4.10. *Let X be an aposyndetic continuum, with $a, c \in X$. If B is a branch of $[a, c, \cdot]$, then $B = [c, d]$ for some (unique) $d \in X$.*

Proof. We know (Theorem 3.2) that X is antisymmetric, and hence that $[a, c, \cdot]$ is a tree with respect to $\leq_a$. Let B be any branch of $[a, c, \cdot]$, with $b \in B$. Then the subset $[c, b] \cup [a, b, \cdot]$ of $[a, c, \cdot]$ is the result of "pruning $[a, c, \cdot]$ below b." We first claim that

$$B = \bigcap_{b \in B} ([c, b] \cup [a, b, \cdot]).$$

Indeed, fix $b \in B$, and let $x \in [a, c, \cdot]$ be arbitrary. If $x \in B$, then, since B is totally $\leq_a$-ordered, either $x \leq_a b$ or $b \leq_a x$. In the first case $x \in [a, b]$. But also we have $c \leq_a x$, so $c \in [a, x]$. Thus, by reciprocity (Lemma 2.5) we know $x \in [c, b]$. If $b \leq_a x$, then $x \in [a, b, \cdot]$, by definition. Thus $B \subseteq \bigcap_{b \in B}([c, b] \cup [a, b, \cdot])$. On the other hand, assume $x \in [a, c, \cdot] \setminus B$. Since any branch in a tree is an order ideal, there is no $b \in B$ such that $x \leq_a b$. Also, if $b \leq_a x$ for every $b \in B$, then $B \cup \{x\}$ is a totally ordered subset of $[a, c, \cdot]$, properly containing B; so again we contradict the maximality of B. Hence there is some $b \in B$ to which x is $\leq_a$-incomparable; and for this choice of b, we have $x \notin [c, b] \cup [a, b, \cdot]$. This proves the claimed equality.

Now, because X is aposyndetic, Theorem 3.1 shows that all 1-slices are closed in X. Thus any branch B in $[a, c, \cdot]$ is closed in X, by the equality above. This tells us that all branches of subtrees of the form $[a, c, \cdot]$ are compact subsets of X, and we may now mimic the proof of Lemma 4.4 to infer that the subbasic order-closed sets, being of the form $[c, b]$ and $B \cap [a, b, \cdot]$, $b \in B$, are subspace closed as well.

Thus B, with its subspace topology, is a compact totally ordered space; hence it has a greatest element d. This greatest element is unique, by antisymmetry; hence we conclude that B is the interval $[c, d]$. $\square$

5. Antisymmetry and Totality

An obvious restatement of the first-order condition given above to define antisymmetry for a ternary relation is

Antisymmetry at a: $\forall xy\, (x \neq y \to (\neg[a, y, x] \vee \neg[a, x, y]))$.

In formal contrast to this, we define *totality* at a point as follows.

Totality at a: $\forall xy\, (x \neq y \to ([a, y, x] \vee [a, x, y]))$.

Note that, in any "reasonable" interpretation of betweenness, such as an R-relation, the antecedent formula in the definition of totality is superfluous. Also clear is the fact that an R-relation is both antisymmetric and total at point a precisely when the pre-ordering $\leq_a$ is a total ordering.

Example 5.1. If X is an arc and we define betweenness using either the Q-, the C-, or the K-interpretation, then there are exactly two points at which X is both antisymmetric and total, namely the non-cut points of X.

Proposition 5.2. *An R-relation can have at most two points at which it is both antisymmetric and total.*

Proof. Suppose $\langle X, [\cdot,\cdot,\cdot]\rangle$ is an R-relation with three points $a, b, c \in X$ at which it is both antisymmetric and total. By totality at a, we have either $[a, c, b]$ or $[a, b, c]$ holding; say it is $[a, c, b]$. Then, by antisymmetry at a, we have $\neg[a, b, c]$.

By antisymmetry at b, and because $[b, c, a]$ holds, we also have $\neg[b, a, c]$. So $\neg[c, b, a]$ and $\neg[c, a, b]$ both hold, contradicting the assumption of totality at c. □

Example 5.3. Recalling the $\sin(\frac{1}{x})$-continuum in Example 2.3 and the K-interpretation of betweenness: the points of totality are the two non-cut points of A (also points of antisymmetry), as well as the unique non-cut point of S (not a point of antisymmetry). The $\sin(\frac{1}{x})$-continuum has no points of totality in either the Q- or the C-interpretation (see Theorem 5.4 below).

Theorem 5.4. *Let X be a nondegenerate continuum; the following statements are equivalent for the Q- or the C-interpretation of betweenness:*

(i) *X has at least one point of totality.*
(ii) *X has exactly two points of totality.*
(iii) *X is an arc.*

Proof. The implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are clear; so we assume (i) and prove (iii). Assume X is nondegenerate, and suppose $a \in X$ is a point of totality in either the Q- or the C-interpretation of betweenness. Then, if a point c lies properly between two other points, it must be the case that c is a cut point of X. So let $x, y \in X \setminus \{a\}$. Then either $[a, x, y]_{\mathsf{C}}$ or $[a, y, x]_{\mathsf{C}}$ holds; in either case, antisymmetry prevents $[x, a, y]_{\mathsf{C}}$ from holding. Thus a is a non-cut point of X.

X has at least one other non-cut point; say it is b. If x is any third point, we then have either $[a, b, x]_{\mathsf{C}}$ or $[a, x, b]_{\mathsf{C}}$. The first alternative forces b to be a cut point, so the second alternative must hold. Thus x is a cut point of X, telling us that a and b are the only non-cut points of X. Hence X is an arc. □

Note that Theorem 5.4 no longer holds for the K-interpretation of betweenness because of the $\sin(\frac{1}{x})$-continuum (see Example 5.3). If we tack on the assumption of aposyndesis (or even of CK-completeness, see Theorem 3.2), then Theorem 5.4 applies. We do not know whether antisymmetry is enough to ensure that a nondegenerate continuum with a point of totality is an arc, but we can get a positive answer if we also assume gap freeness.

Theorem 5.5. *If X is a nondegenerate antisymmetric continuum that is gap free and has a point of totality, then X is an arc.*

Proof. Assume X is a nondegenerate continuum that is both antisymmetric and gap free. Then, by Corollary 4.8, X is an arboroid, and is hence [15, Theorem 2] *nested.* This means that if $\mathcal{A}$ is a collection of arcs of X which is totally ordered by inclusion, then $\bigcup\mathcal{A}$ is contained in an arc of X. If $a \in X$ is a point of totality, then $\mathcal{A} = \{[a,b] : b \in X\}$ is totally ordered by inclusion, and each member of $\mathcal{A}$ is an arc. Since $\bigcup\mathcal{A}$ is all of X, we infer that X is an arc. □

An R-relation is called *total* if it satisfies totality at each of its points; a continuum is *total* if its K-interpretation of betweenness is total. A continuum is *hereditarily indecomposable* if no subcontinuum is indecomposable; i.e., if any two of its subcontinua are either disjoint or $\subseteq$-comparable. A hereditarily indecomposable continuum is clearly hereditarily unicoherent, and hence all of its intervals are connected. The crooked annulus mentioned in Section 4 is the union of two hereditarily indecomposable proper subcontinua, and also has some disconnected intervals.

Proposition 5.6. *A continuum is total if and only if it is hereditarily indecomposable.*

Proof. Let X be a total continuum, with M and N distinct subcontinua that overlap; say $a \in M \cap N$ and $b \in M \setminus N$. For any $x \in N$, totality gives us either $b \in [a,x]$ or $x \in [a,b]$. The first alternative is impossible, as it forces $b \in N$. Hence it must be the case that $x \in [a,b] \subseteq M$. This shows $N \subseteq M$; hence that X is hereditarily indecomposable.

Suppose X is hereditarily indecomposable, with $a,b,c \in X$ arbitrary. Since X is hereditarily unicoherent, all intervals are subcontinua; hence either $[a,b] \subseteq [a,c]$ or *vice versa.* This implies that X is total. □

Remarks 5.7.

(i) Because hereditary indecomposability implies hereditary unicoherence, Proposition 5.6 tells us that the first-order condition of totality implies the first-order condition of gap freeness for the K-interpretation of betweenness. This implication at the level of betweenness interpretations is not valid for all road systems, however: let the set X include the two points a and b, and let $\mathcal{R}$ consist of the singletons of X, the doubleton $\{a, b\}$, and X itself. Then $\langle X, \mathcal{R}\rangle$ is easily seen to satisfy totality, but not gap freeness.

(ii) Note that, since the C-interpretation is always antisymmetric (Proposition 1.2), Proposition 5.2 implies that totality for that interpretation is impossible in any connected topological space with more than two points. Thus, in the continuum context, we have the analogy: "K-total is to C-total, as hereditarily indecomposable is to degenerate."

(iii) Theorems 5.5 and 4.6 say that a nondegenerate antisymmetric continuum with a point of totality in the K-interpretation is an arc if it is either gap free or irreducible. A tempting conjecture is that points of totality are also points of irreducibility in general; but if that is the case, then Proposition 5.6 tells us that all nondegenerate hereditarily indecomposable continua have at least two composants, and thus answers a long-standing open problem (see [19] and [13, Problem 36]).

For R-relations, global antisymmetry allows at most two points of totality; and a natural question is to what extent global totality limits points of antisymmetry. For the Q- and C-interpretations of betweenness, global totality implies degeneracy (see Remark 5.7 (ii)), so this leaves the K-interpretation.

Theorem 5.8. *Let X be a nondegenerate total continuum. Then X has no points of antisymmetry.*

Proof. Suppose X is nondegenerate and total (in the K-interpretation), and let $a \in X$ be arbitrary. By another boundary bumping theorem [16, Corollary 5.5], there is a nondegenerate subcontinuum $M \subseteq X \setminus \{a\}$. Let $b, c \in M$ be distinct. Then, since X is hereditarily indecomposable (Proposition 5.6), and $a \notin M$, we know that the subcontinuum $[b, c]$ is contained in the subcontinuum $[a, b]$. In particular, we have $c \in [a, b]$. Similarly, $b \in [a, c]$, implying that a is not a point of antisymmetry. ☐

6. The Equivalence Relations $\equiv_a$

For any point a in a continuum X, we define the equivalence relation $\equiv_a$ by the condition $b \equiv_a c$ if $[a, b] = [a, c]$. Denote by $[b]_a$ the

$\equiv_a$-block (equivalence class) containing b. Then clearly we always have $[b]_a \subseteq \overline{[b]_a} \subseteq [a,b]$, $[a]_a = \{a\}$, and $[b]_a$ is degenerate for all $b \in X$ just in case a is a point of antisymmetry for X. In this section we are interested in topological properties of the $\equiv_a$-blocks, both absolute (e.g., nondegenerate, compact, connected) and relative to X (e.g., dense, nowhere dense, having nonempty interior).

The following fact about composants is well known. While it is stated in [16] for metrizable continua, its proof still works in the more general (Hausdorff) setting.

Lemma 6.1. [16, Theorem 11.4] *The complement of any composant of a continuum is connected (possibly empty).*

Proposition 6.2. *For any point a in continuum X, the composant κ_a is a union of $\equiv_a$-blocks. Moreover, if $\kappa_a \neq X$, then $X \setminus \kappa_a$ is a single $\equiv_a$-block, which is also connected.*

Proof. If $b \equiv_a c$ and $b \in \kappa_a$, then $[a,b] \neq X$ and $[a,b] = [a,c]$. Hence $c \in \kappa_a$ too. If $b, c \in X \setminus \kappa_a$, then $[a,b] = [a,c] = X$; so $b \equiv_a c$. If $b \notin \kappa_a$, then $[b]_a = X \setminus \kappa_a$ is connected, by Lemma 6.1. □

Recall that a subset of a topological space is *nowhere dense* if its closure has empty interior.

Example 6.3. In the $\sin(\frac{1}{x})$-continuum X (see Example 2.3), the $\equiv_a$-blocks are degenerate when $a \in A$. When $a \in S$, A itself is the only nondegenerate $\equiv_a$-block. No matter where a is chosen, however, the $\equiv_a$-blocks are nowhere dense subcontinua of X.

Theorem 6.4. *Let X be a nondegenerate continuum, with $a \in X$.*

(i) *Each $\equiv_a$-block has empty interior in X.*
(ii) *The only way for a $\equiv_a$-block to be dense in X is for it to equal $X \setminus \kappa_a$, in which case it is also connected. In particular, no more than one $\equiv_a$-block can be dense in X.*
(iii) *If X is decomposable, then no $\equiv_a$-block is dense in X.*
(iv) *If X is indecomposable, then all $\equiv_a$-blocks contained in κ_a are nowhere dense in X. If X is also irreducible, then $X \setminus \kappa_a$ is the unique $\equiv_a$-block that is dense in X.*
(v) *If X is hereditarily unicoherent, then each $\equiv_a$-block is connected.*
(vi) *If X is hereditarily unicoherent and hereditarily decomposable, then each $\equiv_a$-block is a nowhere dense subcontinuum of X.*

Proof. Ad (i): Singletons are nowhere dense, so we may assume $b \neq a$. Then (by standard continuum theory) we may find a subcontinuum $M \in \mathcal{K}(a,b)$ which is irreducible about $\{a,b\}$. Let A be the composant of a

in M. Then, because $[a,b]^M = M$, we know, by Proposition 6.2, that $[b]_a^M := \{x \in M : [a,x]^M = [a,b]^M\} = M \setminus A$. Since A is dense in M, $[b]_a^M$ can have no interior relative to M, let alone relative to X. Now, $[b]_a \subseteq [a,b] \subseteq [a,b]^M$. If $x \in [b]_a$ then $x \in [a,b] \subseteq [a,b]^M$. But also $b \in [a,x] \subseteq [a,x]^M$; so $[a,x]^M = [a,b]^M$, and we infer that $[b]_a \subseteq [b]_a^M$. Hence we know $[b]_a$ has empty interior in X.

Ad (ii): If $b \in \kappa_a$, then $[b]_a \subseteq [a,b] \neq X$, and $X \setminus [a,b]$ is a nonempty open set missing $[b]_a$. So for $[b]_a$ to be dense in X, it must be the case that $[b]_a = X \setminus \kappa_a$, a connected set, by Proposition 6.2.

Ad (iii): Let $\langle M, N\rangle$ be a decomposition of X. If $a \in M \cap N$, then $\kappa_a = X$; hence, by (ii), no $\equiv_a$-block is dense in X. If a is, say, in $M \setminus N$, then $M \subseteq \kappa_a$, and $X \setminus N$ is a nonempty open set disjoint from $X \setminus \kappa_a$. Again, by (ii), no $\equiv_a$-block can be dense in X.

Ad (iv): Let X be indecomposable, with $b \in \kappa_a$. Then there is a proper subcontinuum $M \in \mathcal{K}(a,b)$. Proper subcontinua of indecomposable continua have empty interior, and $\overline{[b]_a} \subseteq [a,b] \subseteq M$; hence $[b]_a$ is nowhere dense in X.

If X is also irreducible, then there is at least one composant of X disjoint from κ_a, and it must be contained in $[b]_a$ for $b \in X \setminus \kappa_a$. Since composants are dense, we know $[b]_a = X \setminus \kappa_a$ is the unique $\equiv_a$-block that is dense in X.

Ad (v): From the argument in (i) above, we have $[b]_a \subseteq [b]_a^M = M \setminus A$. By Lemma 6.1, we know $[b]_a^M$ is connected as well as having empty interior. Since X is hereditarily unicoherent, we may take M to be $[a,b]$ itself, in which case $[a,b]^M = [a,b]$ and $[b]_a^M = [b]_a$. Thus $[b]_a$ is connected.

Ad (vi): Assume X is both hereditarily unicoherent and hereditarily decomposable. By (i) and (v) we know that $\equiv_a$-blocks have empty interior in X and are connected, so what is left is to show they are also closed. Suppose, for the sake of obtaining a contradiction, that $[b]_a$ is not closed, and so fix a point $x \in \overline{[b]_a} \setminus [b]_a$. $[b]_a$ is connected and nondegenerate; so $\overline{[b]_a}$ is a nondegenerate subcontinuum of $[a,b]$, and it therefore has a decomposition $\langle H, K\rangle$. And since $[b]_a$ is dense in its closure, we may find points $y \in [b]_a \setminus K$ and $z \in [b]_a \setminus H$. Assume $x \in H$. Since $H \subseteq [a,b]$ and $x \notin [b]_a$, we know $[a,x]$ is a subcontinuum of $[a,b]$ that misses $[b]_a$. Since $x \in H$, we know $M = [a,x] \cup H$ is a subcontinuum of $[a,b]$. Since $z \in [b]_a \setminus H$ and $[a,x] \neq [a,b]$, we know that M does not contain z, and is hence a proper subcontinuum of $[a,b]$. But $a \in M$, and so $M \cap [b]_a$ must be empty. However, we have $y \in H \cap [b]_a \subseteq M \cap [b]_a$, and our contradiction. $\square$

Corollary 6.5. *If X contains a nondegenerate subcontinuum M that is both hereditarily unicoherent and hereditarily decomposable (i.e., a λ-arboroid), then the number of $\equiv_a$-blocks is uncountable for any $a \in M$.*

Proof. With $\equiv_a^M$ denoting $\equiv_a$ relative to M, we see that the collection of $\equiv_a^M$-blocks covers M; and Theorem 6.4 (vi) shows that each of them is nowhere dense in M. Now apply the Baire category theorem to infer that the number of $\equiv_a^M$-blocks is uncountable. If $b \in M$, then $[b]_a^M \supseteq [b]_a$. Hence there are uncountably many $\equiv_a$-blocks contained in M. $\square$

Question 6.6. Can there ever be just countably many $\equiv_a$-blocks? This would be another strong way of asserting the failure of antisymmetry at a.

Define a point a in continuum X to be *fuzzy* if antisymmetry fails at a to the modest extent that $[b]_a$ is nondegenerate for all $b \neq a$. X is *fuzzy* if each of its points is fuzzy. Fuzziness soundly implies the lack of points of antisymmetry; the following implies Theorem 5.8, but has a completely different proof.

Theorem 6.7. *Hereditarily indecomposable continua are fuzzy; in fact, nondegenerate equivalence classes are connected, and hence of cardinality $\geq \mathfrak{c}$.*

Proof. Let X be hereditarily indecomposable, with a and b distinct points of X. Then $M = [a, b]$ is a nondegenerate hereditarily indecomposable continuum that is irreducible about $\{a, b\}$. Thus the composants of M partition it into at least two dense sets. Let a be the composant of M containing a. Then $[b]_a = [b]_a^M = M \setminus A$ (see the proof of Theorem 6.4 (v)), and so $[b]_a$ contains a composant of M. This makes $[b]_a$ dense in $[a, b]$, so it is nondegenerate. Being the complement of a composant also makes it connected, by Proposition 6.2. Any subspace of a compact Hausdorff space is Tychonoff; hence any nondegenerate connected one has cardinality $\geq \mathfrak{c}$. $\square$

Fuzziness, like totality, is a first-order betweenness condition that is necessary for hereditary indecomposability to hold; and it is a natural question whether fuzziness, like totality, is also sufficient. An immediate consequence of Theorem 6.7 and the following is that the answer is no.

A continuum Z is a *wedge sum* of continua X and Y if there is a decomposition $\langle M, N \rangle$ of Z such that M is homeomorphic to X, N is homeomorphic to Y, and $M \cap N$ is a singleton.

Theorem 6.8. *A wedge sum of two fuzzy continua is fuzzy.*

Proof. Suppose $Z = M \cup N$, where M and N are proper fuzzy subcontinua, and $M \cap N = \{c\}$. If a and b are points in M, then $[a,b] \subseteq [a,b]^M$. We show $[a,b] = [a,b]^M$; and for this it suffices to prove that if H is any subcontinuum of Z containing a and b, then there is a subcontinuum K of M such that $a, b \in K \subseteq H$. Indeed, suppose H is such a subcontinuum, which we may assume intersects $N \setminus M$. Then c must lie in H and be one of its cut points. Thus $H \setminus N$ is clopen in $H \setminus \{c\}$; and, by [17, Theorem 3.4], it follows that $K = (H \setminus N) \cup \{c\}$ is a subcontinuum of M. Clearly we have $a, b \in K \subseteq H$, as desired.

So if $a \in Z$ is fixed, say $a \in M$, and if $b \in M \setminus \{a\}$, then $[a,b]^M = [a,b']^M$ for some $b' \in M$ distinct from b, since M is fuzzy. Since $[a,b] = [a,b]^M$, we infer that $[b]_a$ is nondegenerate. If $b \in N \setminus M$, then $c \in [a,b]$ because c is a cut point of Z. Hence $[a,b] = [a,c] \cup [c,b]$, by weak disjunctivity. Since $c \neq b$ and N is fuzzy, there is some $b' \in N \setminus \{b\}$ with $[c,b]^N = [c,b']^N$. Thus $[a,b'] = [a,c] \cup [c,b'] = [a,c] \cup [c,b']^N = [a,c] \cup [c,b]^N = [a,c] \cup [c,b] = [a,b]$; hence $[b]_a$ is nondegenerate in this case too. $\square$

7. Distal Continua

Let X be a continuum, with $a \in X$. The pre-order $\leq_a$ suggests that we may consider a as a "vantage point" by defining $d \in X$ to be *a-distal* if, for any $b \in X$, $d \leq_a b$ implies $b \leq_a d$. If d is a-distal, then d is "as far away from a as you can go." If a is a point of antisymmetry, then, the a-distal points are the maximal elements of the partial order $\leq_a$. The set of a-distal points is denoted δ_a, and the continuum is called *a-distal* if each $x \in X$ is between a and some $d \in \delta_a$. Finally, X is *distal* if X is a-distal for each $a \in X$.

The following facts are immediate from the definitions.

Proposition 7.1. *Let X be a nondegenerate continuum, with $a \in X$.*

(i) *δ_a is a union of $\equiv_a$-blocks.*
(ii) *$a \notin \delta_a$.*
(iii) *If $\kappa_a \neq X$, then $\delta_a = X \setminus \kappa_a$.*

Any $\equiv_a$-block contained in δ_a is called an *a-direction.* If α is a cardinal number, we say $a \in X$ is *α-directional* if α is the number of $\equiv_a$-blocks contained in δ_a. Since $X \setminus \kappa_a$ is a single $\equiv_a$-block when $\kappa_a \neq X$, we know a is one-directional in this case.

Question 7.2. Is δ_a ever empty?

Here are some examples of distal continua.

Examples 7.3.

(i) If X is an arc with non-cut points a and b, then $\delta_a = \{b\}$ and $\delta_b = \{a\}$; so non-cut points are one-directional. If c is a cut point, then $\delta_c = \{a, b\}$; so cut points are two-directional.

(ii) Let X be the $\sin(\frac{1}{x})$-continuum of Example 2.3. If $a \in A$ and b is the non-cut point of S, then $\delta_a = \{b\}$, and $\delta_b = A$, a single $\equiv_b$-block. Thus both b and points of A are one-directional. If c is a cut point of S, then $\delta_c = A \cup \{b\}$, a union of two $\equiv_c$-blocks, and hence two-directional.

(iii) Let X be the comb space of Example 3.4, with $b = \langle 0, 1\rangle$ and $b_n = \langle \frac{1}{n}, 1\rangle$, $n \geq 1$. If $a \in X$, then $\delta_a = (\{b\} \cup \{b_1, b_2, b_3, \ldots\} \setminus \{a\}$, and every point is $\aleph_0$-directional. We note that δ_a is closed in X if and only if $a \neq b$: indeed $b \in \overline{\delta_b} \setminus \delta_b$.

(iv) Let X be the unit circle in the euclidean plane. Then, for any $a \in X$, we have $\delta_a = X \setminus \{a\}$, which is not closed, even in the presence of local connectedness. Every point of the unit circle is $\mathfrak{c}$-directional.

Theorem 7.4. *Let X be a continuum, $a \in X$.*

(i) *If X is aposyndetic, then each member of δ_a is a non-cut point of X, and X is a-distal. In particular, aposyndetic continua are distal.*

(ii) *If X is a-distal, then X is irreducible about $\{a\} \cup \delta_a$.*

(iii) *If $\kappa_a \neq X$, then $\delta_a = X \setminus \kappa_a$, a single (connected) $\equiv_a$-block. Hence X is a-distal, and a is one-directional.*

(iv) *If X is indecomposable and irreducible, then X is distal and each of its points is one-directional.*

Proof. Ad (i): Suppose $d \in \delta_a$ is a cut point of X. Then, by Theorem 3.2, there exist points $x, y \in X$, with $d \in [x, y] \setminus \{x, y\}$. By disjunctivity, we have either $d \in [a, x]$ or $d \in [a, y]$. Since $d \in \delta_a$, we know either $x \in [a, d]$ or $y \in [a, d]$; hence either $x = d$ or $y = d$, by antisymmetry. This contradiction tells us d must be a non-cut point.

To show that X is a-distal, let $b \in X$ be arbitrary. We need to show $b \leq_a d$ for some $d \in \delta_a$. Since $[a, b]$ is a totally $\leq_a$-ordered set, a simple nod to Zorn's lemma shows that $[a, b] \subseteq B$ for some $\leq_a$-branch B. Hence, by Theorem 4.10, $B = [a, d]$ for some $d \in X$. Clearly $d \in \delta_a$ since B is an $\leq_a$-branch, and $b \leq_a d$ since $b \in B$.

Ad (ii): Assume X is a-distal, with K a subcontinuum of X containing $\{a\} \cup \delta_a$. With $b \in X$ arbitrary, find $d \in \delta_a$ such that $b \in [a, d]$. Since both a and d are in K, so is b. Hence $K = X$.

Ad (iii): This is immediate, from Proposition 6.2.

Ad (iv): This follows immediately from (iii) above, and the fact that every composant of X is a proper subset. □

8. Centroids

If $[\cdot,\cdot,\cdot]$ is an interpretation of betweenness on a set X, and $a, b, c \in X$, define $[abc]$ to be the intersection $[a,b]\cap[a,c]\cap[b,c]$. Elements of $[abc]$ are the *centroids* of the triple $\langle a, b, c\rangle$; the betweenness structure $\langle X, [\cdot,\cdot,\cdot]\rangle$ is *(uniquely) centroidal* if each triple has a (unique) centroid.

If $\langle X, \mathcal{R}\rangle$ is uniquely centroidal, we denote by $\gamma : X^3 \to X$ the associated centroid operation, and often abbreviate $\gamma(a, b, c)$ simply as abc. By Lemma 2.5, we know that antisymmetric weakly disjunctive R-relations are uniquely centroidal if they are centroidal at all, and the question arises whether, under such circumstances, the centroid operation is a "median," in the sense of [5, 10] and elsewhere.

From the standpoint of universal algebra, a *median* on a set X is a ternary operation $\mu : X^3 \to X$ that is symmetric (i.e., completely commutative) and satisfies the following universal equalities:

Absorption: $\forall\, xyz\, (\mu(x, y, y) = y)$; and

Weak Associativity: $\forall\, wxyz\, (\mu(\mu(w, x, y), x, z) = \mu(w, x, \mu(y, x, z)))$.

A *median algebra* is a set together with a distinguished median. Such structures most naturally arise in the study of distributive lattices, where $\mu(x, y, z)$ is defined to be $(x \vee y) \wedge (x \vee y) \wedge (y \vee z)$. (Indeed, abstract median algebras may be represented [5] as median-subalgebras of powers of the two-element lattice.) In the setting of R-relations, we also find medians in the form of centroids.

Lemma 8.1. *Let $\langle X, [\cdot,\cdot,\cdot]\rangle$ be an antisymmetric weakly disjunctive centroidal R-relation, with $a, b, c \in X$. Then $abc = a$ if and only if $a \in [b, c]$.*

Proof. By definition, $abc \in [b, c]$, so the "only if" direction is trivial. Assume $a \in [b, c]$. Then, by Lemma 2.5 (slenderness), $[b, a] \cap [a, c] = \{a\}$. Hence $\{abc\} = [abc] = [b, a] \cap [a, c] \cap [b, c] = \{a\} \cap [b, c] = \{a\}$, and we have $abc = a$. □

Theorem 8.2. *In an antisymmetric weakly disjunctive centroidal R-relation, the centroid operation is a median.*

Proof. Let $\langle X, [\cdot,\cdot,\cdot]\rangle$ be an antisymmetric weakly disjunctive centroidal R-relation. The definition of "centroid set" immediately gives us (setwise) symmetry (i.e., $[abc] = [acb] = [bac] = [bca] = [cab] = [cba]$) and absorption (i.e., $[abb] = \{b\}$), so we concentrate on weak associativity: given $a, b, c, d \in X$, we wish to show that $(abc)bd$ and $ab(cbd)$ are the same point.

Both abc and cbd lie in $[b, c]$; by weak disjunctivity, either $abc \in [b, cbd]$ or $abc \in [cbd, c]$. Suppose the first case holds. Then we have $[b, cbd] \subseteq [b, d]$; hence $abc \in [b, d]$, and thus $(abc)bd = abc$, by Lemma 8.1. On the other hand, $[ab(cbd)] = [a, b] \cap [a, cbd] \cap [b, cbd] \subseteq [a, b] \cap [a, cbd] \cap [b, c]$, since $cbd \in [b, c]$. By Lemma 2.5 (reciprocity), we have $cbd \in [abc, c]$ because $abc \in [b, cbd]$. Hence $cbd \in [a, c]$, and we have $[a, cbd] \subseteq [a, c]$. Thus we infer $[ab(cbd)] \subseteq [abc]$. Since both sets are singletons, we conclude that $(abc)bd$ and $ab(cbd)$ both equal abc.

Next, suppose the second case holds, that $abc \in [cbd, c]$. Then, by reciprocity, $cbd \in [b, abc] \subseteq [a, b]$; so $ab(cbd) = cbd$ (again by Lemma 2.5). Also we have $[(abc)bd] = [abc, b] \cap [abc, d] \cap [b, d] \subseteq [c, b] \cap [abc, d] \cap [b, d]$. But $abc \in [cbd, c]$, by assumption; so $abc \in [c, d]$, and thus $[abc, d] \subseteq [c, d]$. Hence $[(abc)bd] \subseteq [cbd]$, and we conclude that $(abc)bd$ and $ab(cbd)$ both equal cbd. □

Remark 8.3. Full associativity, the statement that $(vwx)yz = v(wxy)z = vw(xyz)$ universally holds, is generally false for centroids. For assume we have a linear ordering, where $a < b < c < d < e$ are five distinct points. Then $(abc)de = bde = d$, $a(bcd)e = ace = c$, and $ab(cde) = abd = b$.

In the setting of continua, we have a satisfying condition that suffices for centroid existence.

Lemma 8.4. [3, Proposition 3.1] *Let X be a continuum, with $a, b \in X$. If $[a, b]$ is connected, then $[abc] \neq \emptyset$ for any $c \in X$. In particular, if X is hereditarily unicoherent, then X is centroidal (and all of its centroid sets are subcontinua).*

Remark 8.5. Paraphrasing a well-known result (see [16, Corollary 11.20]), a metrizable continuum is indecomposable if and only if it equals one of its own centroid sets.

When we combine Lemmas 8.4 and 2.5 with Theorem 8.2, we immediately obtain

Corollary 8.6. *Let X be an antisymmetric hereditarily unicoherent continuum. Then X is uniquely centroidal, and the centroid operation γ is a median.*

Since a continuum's being centroidal is so manifestly a consequence of hereditary unicoherence, it is natural to ask whether the converse is true. The answer is generally no, as any crooked annulus will attest (see [3, Theorem 3.2]). So a weaker assertion, one for which a crooked annulus no longer offers a counterexample, is that centroidality implies gap freeness. We do not know the answer to this, even under the assumption of antisymmetry. But we do get a yes answer if we invoke aposyndesis.

Theorem 8.7. *In aposyndetic continua, being hereditarily unicoherent (or gap free) is equivalent to being centroidal.*

Proof. Assume X is an aposyndetic continuum. Then X is antisymmetric, by Theorem 3.2. We have already seen that hereditary unicoherence generally implies being centroidal (Lemma 8.4), and that gap freeness is sufficient for hereditary unicoherence in antisymmetric continua (Theorem 4.5), so it remains to show that being centroidal implies being gap free.

Assume X is centroidal but not gap free. Then there are two points $a \neq b$ in X such that $[a, b] = \{a, b\}$. For each $c \in X$, we have abc uniquely defined (Lemma 2.5), and hence either $abc = a$ or $abc = b$. Thus the sets $C_a = \{x \in X : abx = a\}$ and $C_b = \{x \in X : abx = b\}$ are disjoint, they cover X, and are both nonempty (since $a \in C_a$ and $b \in C_b$). Since X is connected, then, it cannot be the case that both C_a and C_b are closed in X. But C_a and C_b are the 1-slices $[b, a, \cdot]$ and $[a, b, \cdot]$, respectively (Lemma 8.1), and are indeed closed by Theorem 3.1. $\square$

Question 8.8. In light of Corollary 8.6, is the centroid operation γ for an antisymmetric hereditarily unicoherent continuum continuous in all (any) of its variables? In general, what do the inverse images of a point or closed set look like?

For each a, b in an antisymmetric hereditarily unicoherent continuum X, define $\gamma_{ab} : X \to X$ to be the function $x \mapsto abx$. Clearly γ_{ab} maps X onto $[a, b]$, and $\gamma_{ab}(c) = c$ if and only if $c \in [a, b]$. This is the defining condition for a continuous mapping from a space to a subspace to be a *retraction*, but continuity in this instance is not assured.

Example 8.9. Referring to the comb space of Example 3.4, we have an antisymmetric hereditarily unicoherent continuum. Let $a = \langle 0, 0\rangle$ and $b = \langle 0, 1\rangle$, with $b_n = \langle \frac{1}{n}, 1\rangle$, $n \geq 1$. Then $b = \lim_{n\to\infty} b_n$. However, $\gamma_{ab}(b) = abb = b$, while, for each n, we have $\gamma_{ab}(b_n) = abb_n = a$. Thus γ_{ab} is not continuous at b.

Theorem 8.10. *Let X be an aposyndetic centroidal continuum. For each $a, b, c \in X$, with $c \in [a, b]$, the inverse image $\gamma_{ab}^{-1}(c)$ is a subcontinuum of X.*

Proof. Suppose X is a continuum that is both aposyndetic and centroidal, with $a, b, c \in X$ such that $c \in [a, b]$. For any $x \in X$, we have $abx = c$ just in case $c \in [a, x] \cap [b, x]$; in the notation of 1-slices, this gives us $\gamma_{ab}^{-1}(c) = [a, c, \cdot] \cap [b, c, \cdot]$. (In the proof of Theorem 8.7, the sets C_a and C_b are $\gamma_{ab}^{-1}(a)$ and $\gamma_{ab}^{-1}(b)$, respectively.) This set is closed for aposyndetic X, by Theorem 3.1, and so is compact.

We next show $\gamma_{ab}^{-1}(c)$ is connected. First observe that if $x \in [a, c, \cdot]$, then $[c, x] \subseteq [a, c, \cdot]$. For if $y \in [a, x]$, then, since $c \in [a, x]$, we have $c \in [a, y]$, by Lemma 2.5 (reciprocity). This says $y \in [a, c, \cdot]$.

To finish the argument, suppose x and y are in $\gamma_{ab}^{-1}(c) = [a, c, \cdot] \cap [b, c, \cdot]$. Then, by the argument above, both $[c, x]$ and $[c, y]$ are contained in $\gamma_{ab}^{-1}(c)$. By hereditary unicoherence (Theorem 8.7), we infer that $[c, x] \cup [c, y]$ is a connected subset of $\gamma_{ab}^{-1}(c)$ that contains both x and y. This ensures that $\gamma_{ab}^{-1}(c)$ itself is connected. □

We do not know whether the centroid operation is continuous for aposyndetic centroidal continua. However, if we replace aposyndesis with local connectedness, we get an affirmative answer. Recall that a continuous mapping between continua is *monotone* if inverse images of subcontinua are subcontinua.

Theorem 8.11. *Let X be a locally connected centroidal continuum. Then the centroid operation $\gamma : X^3 \to X$ is continuous; and, for each $a, b \in X$, the mapping $\gamma_{ab} : X \to [a, b]$ is a monotone retraction.*

Proof. Assume X is a locally connected centroidal continuum. We aim to show that whenever D is closed in X, its inverse image $\gamma^{-1}(D)$ is closed in X^3.

We first observe that if $D \subseteq X$ is closed and $d \in X \setminus D$, we may use local connectedness to cover D with finitely many subcontinua, none containing d. Hence the collection $\mathcal{F}$, consisting of the closed subsets of X with finitely many components, forms a closed-set base. We lose no generality, then, in showing $\gamma^{-1}(D)$ is closed in X^3 for $D \in \mathcal{F}$; and indeed we may assume D itself is connected.

So assume $D \subseteq X$ is a subcontinuum and that $\langle a, b, c\rangle \in X^3$ is such that $abc \notin D$. Then we need open sets U_a, U_b, U_c, containing a, b, and c respectively, such that $a'b'c' \notin D$ for any $\langle a', b', c'\rangle \in U_a \times U_b \times U_c$.

Suppose D intersects both $[a, abc]$ and $[abc, b]$, say $u \in D \cap [a, abc]$ and $v \in D \cap [abc, b]$. By weak disjunctivity, we have $[a, b] = [a, u] \cup [u, v] \cup [v, b]$; hence abc must lie in one of these subintervals. If $abc \in [a, u]$, then $abc = u$ because $u \in [a, abc]$ and antisymmetry holds. Similarly, $abc = v$ if $abc \in [v, b]$. In any event, we have $abc \in [u, v]$. But since D is a subcontinuum, we have $[u, v] \subseteq D$, contradicting the assumption that $abc \notin D$.

Hence we infer that D must miss at least two out of the three intervals $[a, abc], [b, abc]$, and $[c, abc]$, and therefore that D misses at least one of the intervals $[a, b], [a, c]$, and $[b, c]$. Say it is the case that $D \cap [a, b] = \emptyset$. Then, using local connectedness, we may find connected open sets U_a and U_b, neighborhoods of a and b, respectively, such that $D \cap (\overline{U_a} \cup \overline{U_b}) = \emptyset$. Letting

U_c be any open neighborhood of c, we have that if $\langle a', b', c'\rangle \in U_a \times U_b \times U_c$, then $a'b'c' \in [a', b']$. Since $\overline{U_a} \cup [a, b] \cup \overline{U_b}$ is a subcontinuum containing a' and b', it must contain $[a', b']$; hence $a'b'c'$ cannot lie in D. This shows $\gamma^{-1}(D)$ is closed in X^3.

$\gamma_{ab} : X \to [a, b]$ is a retraction because it is continuous (and $abc = c$ if and only if $c \in [a, b]$). It is monotone, by Theorem 8.10, because local connectness implies aposyndesis and continuous surjections between continua are monotone whenever inverse images of singletons are connected. $\square$

Remark 8.12. The fact that dendrons "admit a natural continuous median" has long been known, but in a rather disguised context (see, e.g., [1]). The dendrons, being the Q-gap free continua, are precisely the locally connected centroidal continua (see Theorem 3.2 and [20, Lemma 4]).

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