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Geometry Based Synthesis of Planar Compliances with Redundant Mechanisms Having Five Compliant Components

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Abstract

In this paper, a geometric approach to the passive realization of any planar compliance with a redundant compliant mechanism is presented. The mechanisms considered are either simple serial mechanisms consisting of five elastic joints or simple parallel mechanisms consisting of five springs. For each type of mechanism, realization conditions to achieve a given compliance are derived. The physical significance of each condition is identified and graphically interpreted. Geometry based synthesis procedures to achieve any given compliance are developed for both types of mechanisms. Since each realization condition imposes restrictions solely on the mechanism geometry, the procedures allow one to choose the geometric properties of each component (from a set of admissible options) independently from the selection of the elastic properties of each component.

Keywords

Compliance synthesis, Passive realization of compliance, Redundant compliant mechanisms

1. Introduction

In [robotic](https://www.sciencedirect.com/topics/engineering/robotics) manipulation, compliance is a common means of providing force regulation and stable positioning relative to constraints [\[1\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0001) [\[2\].](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0002) For small, quasi-static [elastic deformations](https://www.sciencedirect.com/topics/engineering/elastic-deformation) from equilibrium, a linear relationship exists between a twist (the displacement and rotation in Cartesian space) and a wrench (force and moment in Cartesian space). This relationship can be represented by a symmetric positive semidefinite (PSD) matrix, the [compliance matrix](https://www.sciencedirect.com/topics/engineering/compliance-matrix) C , or the [stiffness matrix](https://www.sciencedirect.com/topics/engineering/stiffness-matrix) K , the inverse of C .

A rigid body suspended by a [compliant mechanism](https://www.sciencedirect.com/topics/engineering/compliant-mechanism) can achieve a general elastic behavior. A compliant mechanism consists of elastic components connected in different ways. [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0001) 1 illustrates two types of simple compliant mechanisms. In a serial mechanism [\(Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0001) 1a), each joint is loaded with a joint compliance; in a parallel mechanism [\(Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0001) 1b), each spring is connected independently to the suspended body. For both types of mechanisms, each joint compliance/stiffness can be obtained with a conventional torsional/translational spring or can be controlled in real-time using variable stiffness [actuation](https://www.sciencedirect.com/topics/engineering/actuation) (VSA) [\[3\].](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0003) Thus, mechanisms considered are compliant mechanisms with lumped compliances. The elastic behavior of a compliant mechanism depends on its configuration and the value of compliance/stiffness of each joint/spring. As such, Cartesian compliance synthesis (or realization) requires identification of both the mechanism configuration and the compliance/stiffness of each joint/spring. In many robotic tasks, the configuration of the manipulator is an important concern due to physical constraints. Identification of the compliance [realization](https://www.sciencedirect.com/topics/engineering/realization) conditions (both on the configuration and on the elastic component properties) is the primary motivation for this work. In addition, a better understanding and interpretation of these conditions for relatively simple compliant mechanisms yields a new means of compliant mechanism synthesis and provides insight into the design of hybrid serial/parallel compliant mechanisms.

Fig. 1. Simple redundant [compliant mechanisms](https://www.sciencedirect.com/topics/engineering/compliant-mechanism) with five components. (a) A 5-joint serial mechanism with joint compliances $c_i \geq 0$. The location of each joint J_i is presented by a position vector r_i . (b) A 5-spring parallel mechanism with spring rates $k_i\geq 0$. The axis of each spring is represented by wrench w_i and r_i is the perpendicular vector from the frame to w_i .

In the design of a compliant mechanism, the space of realizable compliances with the mechanism is an important consideration. Since variable stiffness actuators enable [joint stiffness](https://www.sciencedirect.com/topics/engineering/joint-stiffness) to vary in real-time, the realizable space of compliances is significantly increased. However, as shown in $[4]$, $[5]$, $[6]$, an arbitrary compliance cannot be achieved by varying the joint compliance/stiffness alone. The realizable space of compliant behaviors is highly restricted by the mobility of the mechanism. In order to further increase the mechanism mobility and enlarge the realizable space, mechanisms with kinematic redundancy can be used. Due to the increase in degrees of freedom, a redundant *serial* manipulator can reach an even larger space of compliances because the configuration of the mechanism can vary without affecting the pose of the [end](https://www.sciencedirect.com/topics/engineering/end-effector)[effector.](https://www.sciencedirect.com/topics/engineering/end-effector)

1.1. Related work

Many researchers have studied general spatial compliant behaviors. [Screw theory](https://www.sciencedirect.com/topics/engineering/screw-theory) [\[7\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0007) [\[8\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0008) [\[9\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0009) [\[10\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0010) [\[11\]](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0011) and Lie groups [\[12\]](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0012) have been widely used in spatial compliance analysis.

In previous work in *spatial* compliance realization, the conditions on stiffness/compliance that can be passively realized by *simple* parallel/serial compliant mechanisms have been identified [\[13\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0013) [\[14\].](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0014) Synthesis procedures for any stiffness/compliance that is realizable with a simple compliant mechanism have been developed and refined [\[13\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0013) [\[14\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0014) [\[15\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0015) [\[16\].](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0016) The realization of an arbitrary spatial compliance/stiffness using more complicated compliant components (helical joints in a serial mechanism or screw springs in a parallel mechanism) has also been addressed [\[17\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0017) [\[18\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0018) [\[19\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0019) [\[20\].](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0020) In these approaches, the compliant behavior realization depends on a compliance/stiffness matrix decomposition into rank-1 components. In each process, mechanism geometry was not considered or constrained.

In more recent research, *spatial* compliance realization procedures that take into account some mechanism geometric properties have been developed [\[21\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0021) [\[22\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0022) [\[23\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0023) [\[24\].](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0024) In our most recent work in spatial compliance realization [\[25\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0025) a geometry-based approach to realizing an arbitrary compliance was presented. In the synthesis procedure, each elastic component is selected from a restricted space based on its location, direction, and/or pitch.

In recent work on the realization of *planar* compliance, achieving a specified planar translational compliance (point compliance in Euclidean space *E*(2)) using a 3*R* serial mechanism having given geometry has been addressed $[26]$. In the approach, an optimization was used. In $[27]$, $[28]$, methods to achieve an [isotropic](https://www.sciencedirect.com/topics/engineering/isotropic) compliance in 2- and 3-dimensional Euclidean spaces with a simple serial mechanism were presented. In [\[4\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0004) conditions on mechanism geometry to attain every point compliance in *E*(2) were identified, and synthesis procedures to achieve any given compliance in $E(2)$ using a 3 [revolute-joint](https://www.sciencedirect.com/topics/engineering/revolute-joint) mechanism with given [link lengths](https://www.sciencedirect.com/topics/engineering/link-length) were developed. In [\[5\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0005) the results for 3*R* mechanisms [\[4\]](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0004) were extended to general 3-joint mechanism (containing revolute and prismatic joints) having given (constrained) geometry.

In [\[29\]](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0029), a general 3 × 3 planar compliance (in *SE*(2)) synthesis procedure for the design of a 4-spring parallel mechanism having a specific [symmetric structure](https://www.sciencedirect.com/topics/engineering/symmetric-structure) was developed. In [\[6\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0006) a geometric synthesis procedure for a general planar compliance with a 3-component (non-redundant) compliant mechanism was presented. Most recently, compliance realization with a *redundant* mechanism consisting of 4 compliant components was addressed [\[30\].](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0030) Realization conditions on the mechanism configuration to achieve a given compliance was identified. Synthesis procedures based on the mechanism geometry were developed.

1.2. Contributions of the paper

The limitations of prior work are best expressed in terms of serial mechanisms. The main limitation of previous work for 3*J* and 4*J* mechanisms is that the realizable space of compliant behaviors achieved in each mechanism is very limited. It is known that the space of all passive planar compliances is a 6-dimensional cone (the 3×3 PSD cone). The space of compliances that can be achieved at a configuration of a *n*-joint mechanism is a *n*dimensional *polyhedral* subcone. Although an arbitrary compliance can be realized with a 3*J* or 4*J* mechanism by properly choosing the joint locations, the set of acceptable joint locations is highly constrained.

When a 3*R* mechanism is considered for the realization of a compliance, the realization conditions impose three equality constraints on the mechanism configuration $[6]$, i.e., a wrench passing through any two joint locations must yield a twist centered at the location of the third joint [\[6\].](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0006) Because a 3*R* mechanism does not have redundancy, there is a unique mechanism configuration when the locations of the mechanism base and endeffector are specified. Since the realization conditions on 3*J* mechanism are extremely restrictive [\[6\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0006) it is highly

unlikely (if not impossible) that a desired compliance can be achieved at a specified [endpoint](https://www.sciencedirect.com/topics/engineering/endpoint) pose, even with infinite variability in joint elastic behavior.

When a *redundant* mechanism consisting of 4 compliant components is considered (which increases the dimension of the realizable compliance space at a configuration by one), the realization conditions are still highly restrictive. Geometrically, the realization conditions require that a wrench passing through two joints yields a twist centered on a line segment connecting the other two joints. If the base location of the mechanism and pose of end-effector are specified, the remaining two joints must satisfy two equality conditions and four inequality conditions simultaneously to achieve the desired compliance. Since the space constrained by the realization conditions is so small, an arbitrarily specified compliance is still unlikely to be achieved by the mechanism.

It can been seen that the limitations of 3*J* and 4*J* mechanisms for compliance realization are related to their limited number of controllable compliant components. Thus, an investigation into compliance realization using mechanisms with additional compliant components is needed. Because the space of planar compliance is 6 dimensional, increasing the number of components to 5 will increase the dimension of the space of realizable compliance behaviors.

This paper addresses planar compliance synthesis with a *redundant* mechanism having five compliant components. The approaches developed for mechanisms of three and four compliant components [\[6\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0006) [\[30\]](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0030) cannot be used directly for this case for the following reasons: (1) the realizable compliance space of a 5-component mechanism is not simply the union of compliance [subspaces](https://www.sciencedirect.com/topics/engineering/subspace) realized with all combinations of three and four components in the mechanism (the contributions of all components must be considered *simultaneously*); (2) although the constraints on compliance are less restrictive due to the reduced number of equality conditions, the number of inequality conditions for passive realization is increased; and (3) the geometric significance of each realization condition is completely different due to the increase of compliant components.

The main contributions of the paper are:

- identification of necessary and sufficient conditions for 5-component mechanisms to realize an arbitrary compliance;
- identification of the geometric significance of each realization condition and interpretation of these conditions in terms of mechanism geometry;
- development of geometry-based synthesis procedures for both 5-joint serial mechanisms and 5-spring parallel mechanisms for the realization of an arbitrary compliance.

1.3. Overview

This paper presents a geometric approach to the design of a redundant compliant mechanism having five compliant components that passively realizes an arbitrary planar compliance. Realization conditions on mechanism configurations for an arbitrarily given compliance are identified. Geometry based synthesis procedures are developed for both 5-joint serial and 5-spring parallel mechanisms. With these procedures, the realization of any given compliant behavior can be accomplished by choosing each elastic component based on its geometry from an allowable space. [Section](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0005) 2 provides some background needed for the realization of a compliance by a simple serial/parallel mechanism. In [Section](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0009) 3, realization conditions on the mechanism configuration to achieve a given compliance are identified. In [Section](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0013) 4, the geometric significance of each realization condition is presented. In [Section](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0022) 5, geometry based synthesis procedures are developed for both types of mechanisms. In [Section](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0026) 6, [numerical examples](https://www.sciencedirect.com/topics/engineering/numerical-example) are presented to demonstrate the synthesis process. A brief conclusion and summary are provided in [Section](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0029) 7.

2. Technical background

In this section, some background needed for the [realization](https://www.sciencedirect.com/topics/engineering/realization) of a planar compliant behavior with a simple serial/parallel mechanism is presented. For each type of mechanism, only simple compliant components (no helical joints or screw springs) are needed.

2.1. Screw representation of a mechanism configuration

Consider a serial mechanism having *n* joints J_i $(i = 1, 2, ..., n)$ as shown in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0001) 1a. Each joint can be described by a planar unit twist (the *joint twist*). In Plücker [axis coordinates,](https://www.sciencedirect.com/topics/engineering/axis-coordinate) the joint twists associated with a revolute [joint](https://www.sciencedirect.com/topics/engineering/revolute-joint) J_{τ} and [prismatic joint](https://www.sciencedirect.com/topics/engineering/prismatic-joint) J_{ρ} are:

$$
(1) \t\mathbf{t}_{\tau} = \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix}, \mathbf{t}_{\rho} = \begin{bmatrix} \widehat{\mathbf{n}} \\ 0 \end{bmatrix},
$$

where $\bf u=r\times\hat k$,, $\bf r$ is the position vector of J_τ , $\hat{\bf k}$ is the <u>[unit vector orthogonal](https://www.sciencedirect.com/topics/engineering/orthogonal-unit-vector)</u> to the mechanism plane, and $\bf\hat n$ is the (unit) vector along the axis of J_{ρ} .

If a twist \mathbf{t}_{τ} is specified, the center of the twist, T_{τ} , can be calculated using

$$
(2) r = Su
$$

where $\mathbf{S} \in \mathbb{R}^{2 \times 2}$ is the skew-symmetric matrix:

$$
(3) S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
$$

Thus, once a joint twist t_i for a revolute joint is identified, the location of the joint $J_i=T_i$ is uniquely determined as illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0002) 2a. Since a twist t_ρ associated with a prismatic joint has infinite pitch (free vector), it only determines the direction of the prismatic axis. The joint location is arbitrary in the plane.

 (b) Fig. 2. Screw representation of the location of a mechanism component. (a) The location of joint J_i in a serial mechanism can be represented by a unit twist ${\sf t}_i.$ (b) The axis of a spring in a parallel mechanism is represented by a unit wrench ${\sf w}_i.$

For the parallel mechanism shown in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0001) 1b, each spring can be described by a planar unit wrench (the *spring wrench*). In Plücker ray coordinates, the spring wrenches corresponding to a line (translational) spring and a torsional spring are:

(4) $\mathbf{w}_{\rho} = \begin{bmatrix} \widehat{\mathbf{n}} \\ d \end{bmatrix}$ $\begin{bmatrix} \widehat{\mathbf{n}} \\ d \end{bmatrix}$, $\mathbf{w}_{\tau} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

where $\hat{\bf n}$ is a (unit) vector along the axis of spring, $d = ({\bf r} \times \hat{\bf n}) \cdot \hat{\bf k}$, and ${\bf r}$ is the <u>[orthogonal vector](https://www.sciencedirect.com/topics/engineering/orthogonal-vector)</u> from frame *O* to the spring axis as shown in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0002) 2b.

If a wrench w_o is specified, the location of its axis (the line of action l) is determined by the perpendicular vector r, which can be determined using

$$
(5) r = -dS\hat{\mathbf{n}},
$$

where matrix S is defined in $Eq. (3)$.

Thus, once a spring wrench w_o associated with a line spring is identified, the axis of the spring *l* is uniquely determined. Since the spring wrench w_{τ} in $\underline{{\sf Eq.}}$ $\underline{{\sf Eq.}}$ $\underline{{\sf Eq.}}$ (4) has infinite pitch, a torsional spring can be located anywhere in the plane.

2.2. Screw product of a twist and wrench

Since the center of a twist t , T , is independent of its density (magnitude), a scalar multiplication of a twist does not change its location. Similarly, a scalar multiplication of a wrench does not change its line of action l . For planar cases, a point T is uniquely identified by a unit twist \hat{t} . A line l is uniquely identified by a unit wrench \hat{w} .

The screw product (reciprocal product) of a twist t and a wrench w, if expressed in Plücker axis and ray coordinates respectively, is defined as

$$
\mathbf{w}^T \mathbf{t} = \mathbf{t}^T \mathbf{w}
$$

which indicates the work done by the wrench along the twist.

If t is a finite pitch unit twist expressed in $Eq. (1)$ $Eq. (1)$ and w is a finite pitch unit wrench expressed in Eq. (4), then the reciprocal product of t and w indicates the distance from the twist center of t to the axis of w , i.e., if a point T is represented by a unit twist t and a line l is represented by a unit wrench w, then the distance from T to l is calculated by

(6)
$$
d = \mathbf{w}^T \mathbf{t} = \mathbf{t}^T \mathbf{w}.
$$

A twist and a wrench **w** are called reciprocal [\[7\]](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0007) if the screw product of the two is zero:

(7)
$$
\mathbf{w}^T \mathbf{t} = \mathbf{t}^T \mathbf{w} = 0,
$$

which indicates that the wrench w does no work along twist t.

For planar cases, the reciprocal relationship of a twist t and wrench w can be represented geometrically if both have finite pitches. If T is the center of t and l is the axis of w, then t and w are reciprocal if and only if T is on l. Note that the reciprocal product of two screws is [coordinate frame](https://www.sciencedirect.com/topics/engineering/coordinate-frame) independent.

2.3. Compliance achieved by a passive elastic mechanism

Consider a *n*-joint serial mechanism. Each joint has joint twists t_i and joint compliance $c_i \geq 0$ ($i = 1, ..., n$). The Cartesian compliance C at the configuration is $[18]$:

$$
(8) \ \mathbf{C} = c_1 \mathbf{t}_1 \mathbf{t}_1^T + c_2 \mathbf{t}_2 \mathbf{t}_2^T + \dots + c_n \mathbf{t}_n \mathbf{t}_n^T.
$$

This equation can be express in the form:

$$
(9) \ \mathbf{C} = \mathbf{TC}_J \mathbf{T}^T,
$$

where $T=[t_1,t_2,...,t_n]$ and C_j is the (diagonal) joint <u>compliance matrix</u> $C_j=$ diag $[c_1,c_2,...,c_n]$. Since each column of T presents a unit twist contributed by a joint, T is the [Jacobian](https://www.sciencedirect.com/topics/engineering/jacobian) of the serial mechanism. Thus, the realization of Cartesian compliance matrix C requires that an appropriate set of joint twists ${\bf t}_i$ (or the mechanism Jacobian) and corresponding joint compliances c_i are identified. In general, due to the non-uniqueness of the decomposition in $Eq. (8)$, there is an infinite number of mechanism configurations that can realize a given C.

Dual to serial mechanism realization, for a parallel mechanism having [spring](https://www.sciencedirect.com/topics/engineering/spring-stiffness) wrenches w_i and spring [stiffnesses](https://www.sciencedirect.com/topics/engineering/spring-stiffness) $k_i \geq 0$ $(i = 1, ..., n)$, the Cartesian stiffness K [\[17\]](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0017) is:

$$
(10)\mathbf{K} = k_1 \mathbf{w}_1 \mathbf{w}_1^T + \dots + k_n \mathbf{w}_n \mathbf{w}_n^T.
$$

Similar to the serial case, Eq. (10) can be expressed as

$$
(11)\mathbf{K} = \mathbf{W}\mathbf{K}_J\mathbf{W}^T,
$$

where $W = [w_1, w_2, ..., w_n]$ and K_j is the diagonal joint <u>[stiffness matrix](https://www.sciencedirect.com/topics/engineering/stiffness-matrix)</u> $K_j =$ diag $[k_1, k_2, ..., k_n]$. Since each column of W presents a unit wrench imposed by each spring, W^T is the Jacobian of the parallel mechanism. Thus, the realization of Cartesian stiffness K requires that an appropriate set of spring wrenches w_i (or the mechanism Jacobian) and the corresponding spring stiffnesses $k_i \geq 0$ are identified.

Note that [Eq.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0008) (8) for the Cartesian compliance of a serial mechanism and Eq. [\(10\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0010) for the Cartesian stiffness of a parallel mechanism each applies for a general *n*-component mechanism (for both redundant and non-redundant mechanisms). It can be seen that if used for the realization of a full-rank 3 × 3 compliance or stiffness matrix, a mechanism must have at least 3 components ($n \geq 3$). Due to the rank deficiency of a prismatic joint twist and a torsional spring wrench, a serial mechanism containing prismatic joints alone can only attain a rank-2 matrix, and a parallel mechanism containing only torsional springs can only attain a rank-1 matrix. Hence, if a serial mechanism is used to realize a full-rank compliance matrix, it must have at least one revolute joint; and if a parallel mechanism is used to realize an arbitrary stiffness, at least two line springs must be used. Realization of a given compliance requires conditions on the locations of revolute joints (represented by joint twists) of a serial mechanism, and conditions on the spring axes (represented by spring wrenches) of a parallel mechanism. Since each joint twist t_i and each spring wrench w_i has geometric meaning, using screw representation, mathematical conditions for compliance realization can be interpreted in terms of mechanism geometry and expressed graphically in the plane.

3. Realization conditions

In this section, [realization](https://www.sciencedirect.com/topics/engineering/realization) conditions for a 5-component [compliant mechanism](https://www.sciencedirect.com/topics/engineering/compliant-mechanism) to achieve a given compliance are identified. Serial mechanisms are considered first. Then, by duality, the results are modified and extended to parallel mechanisms.

3.1. Realization conditions for 5-joint serial mechanisms

Consider a 5-joint serial mechanism. Each joint J_i is represented by joint twist ${\bf t}_i$ $(i=1,2,...,5)$. A line that passes through two joints J_i and J_i (represented by a wrench \bf{w}_{ij}) satisfies the reciprocal conditions for both ${\bf t}_i$ and ${\bf t}_j$:

(12) $\mathbf{w}_{ij}^T \mathbf{t}_i = 0$ and $\mathbf{w}_{ij}^T \mathbf{t}_j = 0$.

Since a scalar multiplication of a wrench does not change its axis, w_{ij} can be calculated by

 (13) $\mathbf{w}_{ij} = \alpha (\mathbf{t}_i \times \mathbf{t}_i), \forall i \neq j,$

where *α* is an arbitrary scalar.

Suppose that a [symmetric matrix](https://www.sciencedirect.com/topics/engineering/symmetric-matrix) C is expressed as:

$$
(14)\mathbf{C} = c_1 \mathbf{t}_1 \mathbf{t}_1^T + c_2 \mathbf{t}_2 \mathbf{t}_2^T + c_3 \mathbf{t}_3 \mathbf{t}_3^T + c_4 \mathbf{t}_4 \mathbf{t}_4^T + c_5 \mathbf{t}_5 \mathbf{t}_5^T,
$$

and suppose that wrench w_{12} passes through joints J_1 and J_2 ; and wrench w_{34} passes through joints J_3 and J_4 . Then, using Eq. [\(12\),](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0012)

$$
\begin{array}{rcl}\n\mathbf{C}\mathbf{w}_{12} & = & \left(c_1\mathbf{t}_1\mathbf{t}_1^T + c_2\mathbf{t}_2\mathbf{t}_2^T + c_3\mathbf{t}_3\mathbf{t}_3^T + c_4\mathbf{t}_4\mathbf{t}_4^T + c_5\mathbf{t}_5\mathbf{t}_5^T\right)\mathbf{w}_{12} \\
& = & c_3(\mathbf{t}_3^T\mathbf{w}_{12})\mathbf{t}_3 + c_4(\mathbf{t}_4^T\mathbf{w}_{12})\mathbf{t}_4 + c_5(\mathbf{t}_5^T\mathbf{w}_{12})\mathbf{t}_5.\n\end{array}
$$

Applying Eq. [\(12\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0012) again to the above equation yields:

 $(15)w_{34}^T C w_{12} = c_5(\mathbf{t}_5^T w_{12})(\mathbf{t}_5^T w_{34}).$

Applying the same process with w_{13} and w_{24} yields:

$$
(16) \mathbf{w}_{24}^T \mathbf{C} \mathbf{w}_{13} = c_5 (\mathbf{t}_5^T \mathbf{w}_{24}) (\mathbf{t}_5^T \mathbf{w}_{13}).
$$

Solving both [Eqs. \(15\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0015) and (16) for c_5 yields:

$$
(17)\frac{w_{24}^T c w_{13}}{(t_5^T w_{24})(t_5^T w_{13})} = \frac{w_{34}^T c w_{12}}{(t_5^T w_{12})(t_5^T w_{34})}.
$$

In general, if the decomposition (14) applies to C, then, for any permutation (i, j, p, q, r) from {1, 2, 3, 4, 5}:

$$
(18)\frac{\mathbf{w}_{ij}^T\mathbf{C}\mathbf{w}_{pq}}{(\mathbf{t}_r^T\mathbf{w}_{ij})(\mathbf{t}_r^T\mathbf{w}_{pq})} = \frac{\mathbf{w}_{ip}^T\mathbf{C}\mathbf{w}_{jq}}{(\mathbf{t}_r^T\mathbf{w}_{ip})(\mathbf{t}_r^T\mathbf{w}_{jq})}
$$

where w_{ij} is a wrench passing through the centers of ${\bf t}_i$ and ${\bf t}_j$ $(J_i$ and $J_j)$ defined in [Eqs.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0012) (12) or [\(13\).](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0013)

Thus, Eq. [\(18\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0018) is a *necessary* condition for C to be achieved at a configuration of a 5-joint serial mechanism. Using a process similar to that presented in $[30]$ for 4-component mechanisms, it can be shown that $Eq. (18)$ $Eq. (18)$ is also a *sufficient* condition. Thus, if Eq. [\(18\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0018) is satisfied for one permutation, then it must be satisfied for all permutations; and if matrix C can be decomposed into the form of [\(14\),](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0014) then for every permutation (i, j, p, q, r) from {1, 2, 3, 4, 5},

$$
(19)c_r = \frac{\mathbf{w}_{ij}^T \mathbf{C} \mathbf{w}_{pq}}{(\mathbf{w}_{ij}^T \mathbf{t}_r)(\mathbf{w}_{pq}^T \mathbf{t}_r)} = \frac{\mathbf{w}_{ip}^T \mathbf{C} \mathbf{w}_{jq}}{(\mathbf{w}_{ip}^T \mathbf{t}_r)(\mathbf{w}_{jq}^T \mathbf{t}_r)}.
$$

Therefore, satisfaction of condition (18) for any one permutation (i, j, p, q, r) of $\{1, 2, 3, 4, 5\}$ is a necessary and sufficient condition for C to be decomposed into the form of (14) . Note that in the decomposition of [Eq. \(14\),](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0014) no restriction on each coefficient c_i is yet imposed. This means that if condition (18) is violated for one permutation, the mechanism cannot achieve the given C at the configuration even if each joint compliance can vary indefinitely in $(-\infty, +\infty)$.

For a PSD [compliance matrix](https://www.sciencedirect.com/topics/engineering/compliance-matrix) C to be realized passively (using conventional springs or variable stiffness actuation), each joint compliance must be nonnegative, which requires $c_r \geq 0$ in Eq. [\(19\).](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0019) It is readily shown that, if for any five permutations with r being each of the joints ($r = 1,2,3,4,5$),

$$
\frac{\mathbf{w}_{ij}^T \mathbf{C} \mathbf{w}_{pq}}{(\mathbf{w}_{ij}^T \mathbf{t}_r)(\mathbf{w}_{pq}^T \mathbf{t}_r)} \geq 0,
$$

then, the inequality conditions must be satisfied for all permutations of $\{1, 2, 3, 4, 5\}$. Thus, the realization conditions can be summarized as:

Proposition 1

Suppose t_i (i = 1,2, ...,5) are the joint twists of a 5-joint serial mechanism. A given compliance *C* can be achieved *passively at the mechanism configuration if and only if the following conditions hold:*

(i) For any one permutation (i, j, p, q, r) of $\{1, 2, 3, 4, 5\}$,

$$
(20)\frac{\mathbf{w}_{ij}^T \mathbf{C} \mathbf{w}_{pq}}{(\mathbf{t}_r^T \mathbf{w}_{ij})(\mathbf{t}_r^T \mathbf{w}_{pq})} = \frac{\mathbf{w}_{ip}^T \mathbf{C} \mathbf{w}_{jq}}{(\mathbf{t}_r^T \mathbf{w}_{ip})(\mathbf{t}_r^T \mathbf{w}_{jq})}.
$$

(ii) For any five permutations with $r = 1,2,3,4,5$,

$$
(21)\frac{\mathbf{w}_{ij}^T \mathbf{C} \mathbf{w}_{pq}}{(\mathbf{w}_{ij}^T \mathbf{t}_r)(\mathbf{w}_{pq}^T \mathbf{t}_r)} \geq 0.
$$

Therefore, the realization of a compliance at a given configuration requires the satisfaction of one equality condition in the form of (20) and five inequality conditions in the form of (21) .

3.2. Realization conditions for 5-spring parallel mechanisms

By duality, a set of necessary and sufficient conditions on a 5-spring parallel mechanism to realize a given stiffness can be obtained.

Suppose that w_i ($(i = 1, 2, ..., 5)$) are the spring wrenches of a parallel mechanism. If two wrenches w_i and w_j intersect at T_{ij} , then the two wrenches must be reciprocal to a twist t_{ij} centered at T_{ij} . Using the reciprocal condition, t_{ij} can be calculated using

 (22) t_{ij} = β (**w**_i × **w**_i)_n

where β is an arbitrary scalar. With appropriate modification, the results of [Proposition 1](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0001) obtained for a 5-joint serial mechanism apply to a 5-spring parallel mechanism for the realization of an arbitrary stiffness.

Proposition 2

Suppose \mathbf{w}_i $(i=1,...,5)$ *) are spring wrenches of a 5-spring parallel mechanism. A given stiffness* \boldsymbol{K} *can be achieved passively by the mechanism if and only if the following conditions hold:*

(i) For an arbitrary permutation (i, j, p, q, r) of $\{1, 2, 3, 4, 5\}$,

$$
(23)\frac{\mathbf{t}_{ij}^T \mathbf{K} \mathbf{t}_{pq}}{(\mathbf{t}_{ij}^T \mathbf{w}_r)(\mathbf{t}_{pq}^T \mathbf{w}_r)} = \frac{\mathbf{t}_{ip}^T \mathbf{K} \mathbf{t}_{jq}}{(\mathbf{t}_{ip}^T \mathbf{w}_r)(\mathbf{t}_{jq}^T \mathbf{w}_r)}.
$$

(ii) For any five permutations of (i, j, p, q, r) of $\{1, 2, 3, 4, 5\}$ with $r = 1, 2, 3, 4, 5$,

$$
(24)\frac{\mathbf{t}_{ij}^T \mathbf{K} \mathbf{t}_{pq}}{(\mathbf{t}_{ij}^T \mathbf{w}_r)(\mathbf{t}_{pq}^T \mathbf{w}_r)} \geq 0.
$$

Dual to the serial case, if Eq. [\(23\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0023) holds for any one permutation, then it must hold for all permutations. If inequality [\(24\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0024) holds for any five permutations (with $s = 1,2,3,4,5$), then the inequality must hold for all permutations. Thus, to realize a given stiffness using a 5-spring parallel mechanism, one equality condition in the form of (23) and five inequality conditions in the form of (24) must be satisfied.

3.3. The uniqueness of the realization

If at a given configuration, a 5*J* serial mechanism realizes a compliance, the realization is typically unique. This is due to the fact that, if among the five joint twists no three are [linearly dependent](https://www.sciencedirect.com/topics/engineering/linearly-dependent) (the generic case), then each joint compliance c_r calculated by Eq. [\(19\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0019) is unique regardless of the permutation used in the calculation. Geometrically, the independence of three joint twists indicates that the locations of the three joints corresponding to these twists are not located on a single line. Since the mechanism [Jacobian](https://www.sciencedirect.com/topics/engineering/jacobian) is full-rank for this generic case, the realization is unique for any non-singular configuration. It is easy to see that, at a singular configuration, the <u>[denominators](https://www.sciencedirect.com/topics/engineering/denominator)</u> of some c_i in Eq. [\(24\)a](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0024)re zero. The joint compliance of joint J_r can be uniquely determined with (i, j, p, q) being any permutation of {1, 2, 3, 4, 5} excluding r, i.e., $(i, j, p, q) = \{1, 2, 3, 4, 5\} \setminus r$:

$$
(25)c_r = \frac{\mathbf{w}_{ij}^T \mathbf{C} \mathbf{w}_{pq}}{(\mathbf{w}_{ij}^T \mathbf{t}_r)(\mathbf{w}_{pq}^T \mathbf{t}_r)}.
$$

Although different permutations can be used in calculating c_r using Eq. [\(25\),](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0025) condition (ii) of [Proposition](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0001) [1](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0001) ensures that the joint compliance c_r obtained is the same for the given set of joint twists at the configuration.

Similarly, if a 5-spring parallel mechanism realizes a stiffness, and if among the five spring wrenches, no three are linearly dependent, then each [spring stiffness](https://www.sciencedirect.com/topics/engineering/spring-stiffness) k_r in the realization must be unique. Geometrically, the independence of three spring wrenches indicates that the three wrenches do not intersect at a single point or are not parallel to each other. For this case, the mechanism Jacobian W^T in Eq. [\(11\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0011) is full-rank (has rank 3). Thus, for a non-singular configuration, each spring stiffness can be uniquely obtained with an arbitrary permutation $(i, j, p, q) = \{1, 2, 3, 4, 5\} \setminus r$:

$$
(26) k_r = \frac{\mathbf{t}_{ij}^T \mathbf{K} \mathbf{t}_{pq}}{(\mathbf{t}_{ij}^T \mathbf{w}_r)(\mathbf{t}_{pq}^T \mathbf{w}_r)}.
$$

4. Geometric significance of the realization conditions

The set of [realization](https://www.sciencedirect.com/topics/engineering/realization) conditions developed in Section 3 can be viewed in terms of the mechanism geometry. Since the realization conditions involve the product of two wrenches/twists about a given compliance/stiffness in [Eq. \(20\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0020) or [\(23\),](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0023) the physical significance of just the two products is identified first. Then, the geometric interpretations of all of the realization conditions are provided for the two types of simple 5-component mechanisms.

4.1. Screw product about compliance

As stated in [Section](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0007) 2.2, if a twist t and wrench w are both unit screws, then the screw product of t and w indicates the distance from the twist instantaneous center T to the wrench axis l . Below, we identify the physical significance of the product of two unit wrenches about a given compliance C.

Consider two unit wrenches w_1 and w_2 . We show that, for a given [compliance matrix](https://www.sciencedirect.com/topics/engineering/compliance-matrix) C, the quantity

$$
(26)h = \mathbf{w}_2^T \mathbf{C} \mathbf{w}_1
$$

is related to the location of the compliance center.

Note that *h* defined in $\underline{Eq. (27)}$ $\underline{Eq. (27)}$ $\underline{Eq. (27)}$ is invariant under [coordinate transformation](https://www.sciencedirect.com/topics/engineering/coordinate-transformation) for w_1 , w_2 and C. To clearly show the physical significance of *h*, a **[coordinate frame](https://www.sciencedirect.com/topics/engineering/coordinate-frame)** located at the compliance center C_c is used initially. In this coordinate frame, the compliance matrix C has the block diagonal form:

$$
(27)\mathbf{C}_n = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0}^T & c_{\phi} \end{bmatrix},
$$

where **D** is a 2 × 2 symmetric PSD matrix, 0 is the zero 2-vector, and $c_{\phi} > 0$ is the principal [rotational](https://www.sciencedirect.com/topics/engineering/rotational) compliance of C. Note that with the compliance matrix expressed in this form, the value of c_{ϕ} is unique.

In the compliance center based frame, an arbitrary unit wrench w_1 has the form:

$$
(28)\mathbf{w}_1 = \begin{bmatrix} \mathbf{n}_1 \\ d_1^c \end{bmatrix},
$$

where d_1^c is the distance from the axis of w_1 to the frame origin, i.e., to the center of compliance \mathcal{C}_c . The twist corresponding to w_1 is:

$$
(29)\mathbf{t}_1 = \mathbf{C}_n \mathbf{w}_1 = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0}^T & c_{\phi} \end{bmatrix} \begin{bmatrix} \mathbf{n}_1 \\ d_1^c \end{bmatrix} = \begin{bmatrix} \mathbf{D} \mathbf{n}_1 \\ c_{\phi} d_1^c \end{bmatrix}.
$$

The unit twist associated with t_1 is:

$$
(30)\hat{\mathbf{t}}_1 = \frac{1}{c_{\phi}d_1^c} \mathbf{t}_1,
$$

which has the same center as t_1 . Premultiplying t_1 by unit wrench w_2 yields:

$$
(31)\mathbf{w}_2^T\mathbf{t}_1 = \mathbf{w}_2^T\mathbf{C}_n\mathbf{w}_1 = (c_{\phi}d_1^c)(\mathbf{w}_2^T\hat{\mathbf{t}}_1).
$$

Since $\hat{\bf t}_1$ is a unit twist and ${\bf w}_2$ is a unit wrench, $({\bf w}_2^T\hat{\bf t}_1)$ indicates the distance from the center of ${\bf t}_1$, T_1 , to the wrench axis of w_2 . Thus,

$$
(32)\mathbf{w}_2^T \mathbf{C}_n \mathbf{w}_1 = c_{\phi}(\mathbf{w}_1^T \mathbf{t}_c)(\mathbf{w}_2^T \hat{\mathbf{t}}_1) = c_{\phi} d_1^c d_2^1,
$$

where \bf{t}_c is the unit twist located at the compliance center \cal{C}_c , d_1^c indicates the distance from \cal{C}_c to the wrench axis of w_1 , and d_2^1 indicates the distance from T_1 to the wrench axis of w_2 as illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0003) 3a. Similarly,

$$
(33) \mathbf{w}_1^T \mathbf{C}_n \mathbf{w}_2 = c_{\phi} d_2^c d_{1}^2,
$$

where d_2^c indicates the distance from \mathcal{C}_c to the wrench axis of \mathbf{w}_2 , d_2^1 indicates the distance from T_2 to the wrench axis of w_1 (illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0003) 3b). Since the products in *Eqs.* (33) and [\(34\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0034) are frame independent, and since C is symmetric, in an arbitrary frame,

$$
(34)\mathbf{w}_1^T \mathbf{C} \mathbf{w}_2 = \mathbf{w}_2^T \mathbf{C} \mathbf{w}_1 = c_{\phi} d_1^c d_2^1 = c_{\phi} d_2^c d_1^2.
$$

Thus, the product *h* defined in **Eq.** (27) is related to the rotational principal compliance of C and the positions of w_i relative to the compliance center.

Fig. 3. Geometric significance of the product of two wrenches about C.

In a dual development, the product $t_2 K t_1$ is invariant under coordinate transformation for t_1 , t_2 and K. When expressed in a frame located at its stiffness center C_k , a stiffness K has the following block diagonal form:

$$
(35)\mathbf{K}_n = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}^T & k_{\phi} \end{bmatrix},
$$

where $k_{\phi} > 0$ is the principal <u>[rotational stiffness](https://www.sciencedirect.com/topics/engineering/rotational-stiffness)</u> of **K** and is uniquely determined by the stiffness behavior.

Suppose that t_1 and t_2 are two unit twists, and that, in the coordinate frame located at C_k , the two twists have the form:

$$
(36)\mathbf{t}_i = \begin{bmatrix} \mathbf{u}_i \\ 1 \end{bmatrix}, i = 1, 2.
$$

Then, the wrench associated with t_1 acting on K_n is:

$$
(37)\mathbf{w}_1 = \mathbf{K}_n \mathbf{t}_1 = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}^T & k_{\phi} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{u}_1 \\ k_{\phi} \end{bmatrix}.
$$

The unit wrench associated with w_1 is:

$$
(38)\widehat{\mathbf{W}}_1 = \frac{1}{\|\mathbf{A}\mathbf{u}_1\|} \mathbf{W}_1,
$$

The third component of \hat{w}_1 is the distance from the wrench axis to the frame origin, which is at the stiffness center C_k , i.e.,

$$
(39)d_1^c=\frac{k_\phi}{\|\mathbf{A}\mathbf{u}_1\|}.
$$

Thus,

$$
(40)\mathbf{t}_2^T\mathbf{K}_n\mathbf{t}_1 = \mathbf{t}_2^T\mathbf{w}_1 = \parallel \mathbf{A}\mathbf{u}_1 \parallel (\mathbf{t}_2^T\widehat{\mathbf{w}}_1).
$$

Using Eq. [\(40\),](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0040)

$$
\mathbf{t}_2^T \mathbf{K}_n \mathbf{t}_1 = k_\phi \frac{(\mathbf{t}_2^T \widehat{\mathbf{w}}_1)}{d_1^c}.
$$

Since t_2 and $\widehat w_1$ are both unit screws, $t_2^T\widehat w_1$ indicates the distance from T_2 (the twist center of t_2) to the wrench axis of w_1 . Since the product in $\underline{Eq. (41)}$ $\underline{Eq. (41)}$ $\underline{Eq. (41)}$ is frame independent, in an arbitrary coordinate frame,

$$
(41)\mathbf{t}_2^T \mathbf{K} \mathbf{t}_1 = k_\phi \frac{\mathbf{t}_2^T \mathbf{w}_1}{\mathbf{w}_1^T \mathbf{t}_c},
$$

where t_c is the unit twist located at the center of stiffness. Reversing the order of multiplication yields:

$$
(42)\mathbf{t}_1^T \mathbf{K} \mathbf{t}_2 = k_\phi \frac{\mathbf{t}_1^T \mathbf{w}_2}{\mathbf{w}_2^T \mathbf{t}_c}.
$$

Since **K** is symmetric, [Eqs. \(42\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0042) and [\(43\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0043) are equal and can be expressed in terms of [geometric parameters:](https://www.sciencedirect.com/topics/engineering/geometric-parameter)

(43)
$$
\mathbf{t}_1^T \mathbf{K} \mathbf{t}_2 = \mathbf{t}_2^T \mathbf{K} \mathbf{t}_1 = k_\phi \frac{d_1^2}{d_1^c} = k_\phi \frac{d_2^2}{d_2^c}
$$
,

where d_1^2 indicates the distance from T_2 to the wrench axis of ${\bf w}_1$, d_1^c indicates the distance from \mathcal{C}_k to the wrench axis of w_1 as illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0004) 4a; and where d_2^1 is the distance from T_1 to the wrench axis of w_2 , d_2^c indicates the distance from \mathcal{C}_k to the wrench axis of w_2 as illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0004) 4b.

Fig. 4. Geometric significance of the product of two twists about K.

4.2. Geometric significance of the realization conditions for serial mechanisms For a 5-joint mechanism with joint twists ${\bf t}_i$ $(i=1,2,...,5)$,, let ${\bf w}_{ij}$ be the unit wrench reciprocal to twists ${\sf t}_i$ and ${\sf t}_j$, then the axis of ${\sf w}_{ij}$ passes through the two joints, J_i and $J_j.$ Denote:

 $t_{ij} = C w_{ij}.$

Then, w_{ij} and t_{ij} are solely determined by the mechanism geometry and the desired compliance C. The realization conditions require that one equality condition in the form of (20) is satisfied and five inequality conditions in the form of (21) are satisfied. Below, the geometric significance of each of these conditions is identified.

4.2.1. Equality condition for serial mechanisms

Using the results presented in [Section](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0014) 4.1, the geometric significance of equality condition [\(20\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0020) can be obtained. Here, we consider the case $[i, j, p, q, r] = [1, 2, 3, 4, 5]$. The equality realization condition [\(20\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0020) for this case requires that

$$
(44)\frac{w_{12}^T \text{C} w_{34}}{(w_{12}^T \text{t}_5)(w_{34}^T \text{t}_5)} = \frac{w_{23}^T \text{C} w_{14}}{(w_{23}^T \text{t}_5)(w_{14}^T \text{t}_5)}.
$$

Using [Eq. \(35\),](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0035) this can be expressed in geometric terms as:

$$
c_{\phi} \frac{d_{12}^c d_{34}^{12}}{d_{12}^5 d_{34}^5} = c_{\phi} \frac{d_{23}^c d_{14}^{23}}{d_{23}^5 d_{14}^5}.
$$

Since c_{ϕ} is constant for a given compliance, the condition is reduced to only geometric terms:

$$
(45) \frac{d_{12}^c d_{34}^{12}}{d_{12}^5 d_{34}^5} = \frac{d_{23}^c d_{14}^{23}}{d_{23}^5 d_{14}^5},
$$

where d_{12}^c is the distance from the compliance center \mathcal{C}_c to line J_1J_2 , d_{34}^{12} is the distance from the center of t_{12} to line J_3J_4 , d_{12}^5 is the distance from J_5 to line J_1J_2 , and d_{34}^5 is the distance from J_5 to line J_3J_4 (each illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0005) 5a); and where d_{23}^c is the distance from the compliance center \mathcal{C}_c to line J_2J_3 , d_{14}^{23} is the distance from the center of t_{23} to line J_1J_4 , d_{23}^5 is the distance from J_5 to line J_2J_3 , and d_{14}^5 is the distance from J_5 to line J_1J_4 (each illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0005) 5b). The equality condition (46) requires that the ratios between the distances in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0005) 5a and [5b](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0005) must be the same. Thus, the equality realization condition is expressed in terms of the mechanism geometry and compliance properties.

Fig. 5. Geometric interpretation on the equality condition for serial mechanisms. The distance ratios defined in Eq. [\(46\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0046) for (a) and (b) must be the same.

4.2.2. Inequality conditions for serial mechanisms

Suppose l_{ij} is the line of action of w_{ij} that passes through joints J_i and J_j . First, consider the case in which l_{ij} does not intersect the triangle with the other 3 joints $(J_p, J_q$ and $J_r)$ as vertices. For the locations of 5 joints shown in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0006) 6, suppose, with no loss of generality, that l_{12} , the line passing through joints J_1 and J_2 , does not intersect triangle $J_3J_4J_5$. We show that, for this case, if C is achieved passively at the configuration, the twist $t_{12} = Cw_{12}$ must be centered inside triangle $J_3J_4J_5$.

Fig. 6. Geometric interpretation on the inequality conditions. (a) Twist center T_{12} is located in the triangle formed by joints J_3 , J_4 and J_5 . (b) Twist center T_{13} is located outside the triangle formed by J_3 , J_4 and J_5 and in the shaded area.

Because [Eq. \(14\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0014) holds for C with each $c_i \ge 0$, substituting [Eqs. \(12\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0012) and [\(14\)i](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0014)nto $t_{12} = Cw_{12}$ yields:

$$
(46)\mathbf{t}_{12} = c_3(\mathbf{t}_3^T \mathbf{w}_{12}) \mathbf{t}_3 + c_4(\mathbf{t}_4^T \mathbf{w}_{12}) \mathbf{t}_4 + c_5(\mathbf{t}_5^T \mathbf{w}_{12}) \mathbf{t}_5.
$$

Since the line of action of w_{12} does not intersect triangle $J_3J_4J_5$, the three vertices are on the same side of l_{12} . Thus, (($\bf{t}_3^T\bf{w}_{12}$),), ($\bf{t}_4^T\bf{w}_{12}$)and $\bf(t_5^T\bf{w}_{12})$ must have the same sign. Because $c_i\,\geq\,0$ for passive realization, the coefficients $c_i(t_i^Tw_{12})>0$, or $c_i(t_i^Tw_{12})< 0$ $(i=3,4,5)$. Because twist t_{12} in Eq. [\(47\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0047) is a positive (or negative) combination of the other three unit twists, T_{12} must be located inside triangle $J_3J_4J_5$.

Conversely, if T_{12} is located inside triangle $J_3J_4J_5$, then the three coefficients of ${\bf t}_i$ $(i=$ 3,4,5)in [\(47\),](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0047) $(t_3^Tw_{12})$,, $(t_4^Tw_{12})$ and $(t_5^Tw_{12})$,, must have the same sign, which indicates c_3 , c_4 and c_5 have the same sign. We show that c_3 , c_4 and c_5 cannot be negative.

Because **C** is a symmetric PSD matrix,

$$
\mathbf{w}_{12}^T \mathbf{C} \mathbf{w}_{12} = c_3 (\mathbf{t}_3^T \mathbf{w}_{12})^2 + c_4 (\mathbf{t}_4^T \mathbf{w}_{12})^2 + c_5 (\mathbf{t}_5^T \mathbf{w}_{12})^2 \ge 0.
$$

Therefore each *cⁱ* must be non-negative. Thus, for this case, the inequality condition is equivalently represented by the location of T_{12} , which is demonstrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0006) 6a.

Next, consider the case in which the axis of w_{ij} intersects triangle $J_pJ_rJ_s$. Here assume, without loss of generality, that the axis of w_{13} intersects triangle $J_2J_4J_5$ (as shown in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0006) 6b). If the compliance C is passively achieved at the configuration shown, each coefficient $c_i \ge 0$ in [Eq. \(14\).](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0014) Then, similar to Eq. [\(47\),](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0047) we have:

$$
(47)\mathbf{t}_{13} = c_2(\mathbf{t}_2^T \mathbf{w}_{13}) \mathbf{t}_2 + c_4(\mathbf{t}_4^T \mathbf{w}_{13}) \mathbf{t}_4 + c_5(\mathbf{t}_5^T \mathbf{w}_{13}) \mathbf{t}_5.
$$

Since J_2 , J_4 and J_5 are not all on the same side of w_{13} , $t_2^Tw_{13}$,, $t_4^Tw_{13}$ and $t_5^Tw_{13}$ do not all have the same sign. Here, suppose that only $\mathbf{t}_4^T\mathbf{w}_{13}$ and $\mathbf{t}_5^T\mathbf{w}_{13}$ have the same sign. Thus, the center of \mathbf{t}_{13} , T_{13} , must be outside the triangle formed by $J_2J_4J_5$ and bounded by the two lines J_2J_4 and J_2J_5 (in the shaded area in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0006) 6b). On the other hand, if point T_{13} is inside the shaded area, the coefficients of t_4 and t_5 in (48), $c_4(t_4^Tw_{13})$ and $c_5(t_5^Tw_{13})$,, must have the opposite sign of the coefficient of t_2 , $c_2(\mathsf{t}_2^T\mathsf{w}_{13})$. Since $(\mathsf{t}_2^T\mathsf{w}_{13})$ has a sign opposite to that of $(t_4^Tw_{13})$ and $(t_5^Tw_{13})$, c_2 , c_4 and c_5 have the same sign. Because C is a symmetric PSD matrix, by the same process used for the first case, c_2 , c_4 and c_5 must be non-negative.

For a serial mechanism having five joints among which no three are collinear, the realization conditions can be equivalently interpreted geometrically as:

Proposition 3

A given compliance matrix can be passively realized at a configuration of a 5-joint serial mechanism if and only if the following conditions hold:

(i) The distance ratios for different sets of joints described in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0005) 5a and [5](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0005)*b are the same (satisfy Eq. [\(46\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0046)*);

(ii) A wrench whose axis passes through any 2 joints, (a) if not intersecting the triangle with the vertices being the other three joints, when multiplied by , yields a twist centered in the shaded triangle described in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0006) 6a; (b) if intersecting the triangle, when multiplied by C, yields a twist centered in the shaded area described in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0006) 6b.

Note that condition (i) in **[Proposition 3](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0003)** is equivalent to the equality condition (20) which ensures the decomposition of the desired compliance C into the form of (14) . Condition (ii) for any two joints J_i and J_i ensures that the joint compliances for the other three joints are non-negative, and ensures that three inequality conditions in $Eq. (21)$ are satisfied. Thus, satisfaction of condition (ii) for two sets of different joints (e.g., $\{J_1, J_2\}$ and $\{J_3, J_4\}$, or $\{J_1, J_3\}$ and $\{J_2, J_5\}$) guarantees all coefficients *c*_i's in [Eq. \(14\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0014) are non-negative, which ensures the passive realization of C with the 5*J* serial mechanism.

4.3. Geometric significance of the realization conditions for parallel mechanisms By duality, the results presented in [Section](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0015) 4.2 for 5-joint serial mechanisms can be modified and applied to 5 spring parallel mechanisms.

Suppose that w_i $(i = 1, ..., 5)$ are spring wrenches of a 5-spring parallel mechanism and suppose that wrenches w_i and w_j intersect at T_{ij} . Then the unit twist centered at T_{ij} , t_{ij} , must be reciprocal to both w_i and w_j . Denote:

 $w_{ij} = Kt_{ij}$.

Then, t_{ij} and w_{ij} are solely determined by the five spring wrenches and the desired stiffness K. The realization conditions require that one equality condition in the form of (23) is satisfied and five inequality conditions in the form of [\(24\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0024) are satisfied. Below, the geometric significance of each of these conditions is identified.

4.3.1. Equality condition for parallel mechanisms

Using the results presented in [Section](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0014) 4.1, the equality condition (23) can be written as:

$$
\frac{t_{34}^T w_{12}}{(t_c^T w_{12})(t_{12}^T w_5)(t_{34}^T w_5)} = \frac{t_{14}^T w_{23}}{(t_c^T w_{23})(t_{23}^T w_5)(t_{14}^T w_5)}..
$$

Since the screw product of a unit twist and a unit wrench indicates the distance from the twist center to the wrench axis, normalizing the wrenches and twists yields

$$
(48) \frac{d_{12}^{34}}{d_{12}^c d_5^{12} d_5^{34}} = \frac{d_{23}^{14}}{d_{23}^c d_5^{23} d_5^{14}} \, ,
$$

where d_{pq}^{ij} is the distance from the twist center of t_{ij} to the wrench axis of w_{pq} . The geometric meaning of each distance d_{pq}^{ij} in <u>Eq. [\(49\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0049)</u> is illustrated in [Figs. 7a](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0007) and [7b](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0007). The ratios of the distances in Eq. (49) in the two figures must be equal. Thus, the equality realization condition is expressed in terms of the mechanism geometry and [stiffness properties.](https://www.sciencedirect.com/topics/engineering/stiffness-property)

Fig. 7. Geometric interpretation on the equality conditions for parallel mechanisms. The distance ratios defined in Eq. (49) for (a) and (b) must be the same.

4.3.2. Inequality conditions for parallel mechanisms

Suppose a 5-spring parallel mechanism has spring wrenches w_i $(i = 1, 2, ..., 5)$ among which no three are concurrent or parallel to each other. Dual to [Eq. \(14\),](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0014) a stiffness **K** can be passively achieved by the mechanism if and only if K can be expressed as:

$$
(49)\mathbf{K} = k_1 \mathbf{w}_1 \mathbf{w}_1^T + k_2 \mathbf{w}_2 \mathbf{w}_2^T + k_3 \mathbf{w}_3 \mathbf{w}_3^T + k_4 \mathbf{w}_4 \mathbf{w}_4^T + k_5 \mathbf{w}_5 \mathbf{w}_5^T
$$

with each $k_i \geq 0$.

Suppose that t_{ij} is the unit twist centered at T_{ij} , the intersection of wrenches w_i and w_j , and that w_{ij} is the wrench resulting from t_{ii} imposed on K, i.e.,

 $w_{ij} = Kt_{ij}$.

Using a procedure equivalent to that for serial mechanisms, suppose, without loss of generality, (i,j) = $(1,2)$,, and first consider the case in which T_{12} is located in the triangle formed by the other three wrenches w_3 , w_4 , w_5 (as shown in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0008) 8a). For this case, it is shown that the line of action of w_{12} does not intersect the triangle.

Fig. 8. Geometric interpretation of the inequality conditions. A twist centered at T_{12} , (a) if located in the triangle formed by wrench axes w_3 , w_4 and w_5 , when multiplied by K, yields a wrench w_{12} that does not intersect the triangle; (b) if located outside the triangle, when multiplied by K, yields a wrench w_{12} that intersects the triangle without crossing the shaded area.

In fact,

$$
(50)\mathbf{w}_{12} = \mathbf{K} \mathbf{t}_{12} = k_3 (\mathbf{w}_3^T \mathbf{t}_{12}) \mathbf{w}_3 + k_4 (\mathbf{w}_4^T \mathbf{t}_{12}) \mathbf{w}_4 + k_5 (\mathbf{w}_5^T \mathbf{t}_{12}) \mathbf{w}_5.
$$

Then,

$$
(51)\mathbf{t}_{34}^T \mathbf{K} \mathbf{t}_{12} = k_5 (\mathbf{w}_5^T \mathbf{t}_{12}) (\mathbf{w}_5^T \mathbf{t}_{34}).
$$

Since T_{12} is in the triangle, T_{12} and T_{34} are on the same side of w_5 . Thus, $w_5^Tt_{12}$ and $w_5^Tt_{34}$ have the same sign, which together with Eq. [\(52\)i](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0052)ndicates $\bf{t}_{34}^T\bf{K}t_{12}>0$. Similarly, $\bf{t}_{35}^T\bf{K}t_{12}>0$, $\bf{t}_{45}^T\bf{K}t_{12}>0$. Summarizing,

$$
(52)\mathbf{t}_{34}^T \mathbf{K} \mathbf{t}_{12} > 0, \mathbf{t}_{35}^T \mathbf{K} \mathbf{t}_{12} > 0, \mathbf{t}_{45}^T \mathbf{K} \mathbf{t}_{12} > 0.
$$

If w_{12} intersects the triangle, it must intersect one of three line segments: $T_{12}T_{34}$, $T_{12}T_{35}$, or $T_{12}T_{45}$. Suppose that w_{12} intersects segment $T_{12}T_{34}$ at T (as shown by the dashed line in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0008) 8a), and t is the unit twist centered at T. Then, since T is located between T_{12} and T_{34} , twist t can be expressed as a positive combination of the two unit twists t_{12} and t_{34} :

 $t = \alpha t_{12} + \beta t_{34}$ where $\alpha > 0$ and $\beta > 0$. Then, $\mathbf{t}^T \mathbf{w}_{12} = (\alpha \mathbf{t}_{12} + \beta \mathbf{t}_{34})^T \mathbf{K} \mathbf{t}_{12} = \alpha \mathbf{t}_{12}^T \mathbf{K} \mathbf{t}_{12} + \beta \mathbf{t}_{34}^T \mathbf{K} \mathbf{t}_{12} = 0.$ Since $\mathbf{t}_{12}^T \mathbf{K} \mathbf{t}_{12} \geq 0, \alpha > 0$ and $\beta > 0$, (53) t $_{34}^{T}$ Kt₁₂ < 0,

which conflicts with the first inequality in (53) . Therefore, w_{12} cannot intersect segment $T_{12}T_{34}$. By the same reasoning, w_{12} cannot intersect segment $T_{12}T_{35}$ or segment $T_{12}T_{45}$. Thus, w_{12} cannot intersect triangle $J_3J_4J_5$.

Conversely, if the line of action of w_{12} does not intersect the triangle, then, the three vertices T_{34} , T_{35} and T_{45} of the triangle must be on the same side of w_{12} , and thus $\rm t_{34}^T w_{12}$, $\rm t_{35}^T w_{12}$ and $\rm t_{45}^T w_{12}$ must have the same sign. Below, we show these three quantities must be positive. To prove this, we only need to show that $\mathrm{t}_{34}^T\mathrm{w}_{12} > 0$,.

Consider the line *l* passing through T_{12} and T_{34} which intersects w_{12} at T' (as shown [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0008) 8a). Since T' is outside the triangle, it must be outside the segment $T_{12}T_{34}$. Then, the unit twist t' centered at T' can be expressed as

$$
\mathbf{t}' = \alpha' \mathbf{t}_{12} + \beta' \mathbf{t}_{34},
$$

where the two scalars α' and β' have opposite signs, i.e., $\alpha'\beta' < 0$. Since t' and w_{12} are reciprocal,

$$
\mathbf{t}'^{T}\mathbf{w}_{12} = (\alpha' \mathbf{t}_{12} + \beta' \mathbf{t}_{34})^{T} \mathbf{K} \mathbf{t}_{12} = \alpha' \mathbf{t}_{12}^{T} \mathbf{K} \mathbf{t}_{12} + \beta' \mathbf{t}_{34}^{T} \mathbf{K} \mathbf{t}_{12} = 0.
$$

Thus, $t_{12}^T Kt_{12}$ and $t_{34}^T Kt_{12}$ must have the same sign. Because $t_{12}^T Kt_{12} > 0$, then $t_{34}^T Kt_{12} > 0$,, satisfying the first inequality of (53) . With similar procedures, the three inequalities in (53) are each satisfied, which implies that k_3 , k_4 and k_5 in $\underline{Eq. (51)}$ $\underline{Eq. (51)}$ $\underline{Eq. (51)}$ are non-negative. Summarizing, coefficients k_3 , k_4 and k_5 in $\underline{Eq. (51)}$ are nonnegative, if and only if w_{12} does not intersect the shaded triangle formed by wrenches w_3 , w_4 , and w_5 as illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0008) 8a.

Second, consider the case in which T_{12} is located outside the triangle formed by wrenches w_3 , w_4 , and w_5 . Using a process similar to that used for the serial case, it can be proved that, for this case, the line of action of w_{12} must intersect the triangle without crossing the shaded area described in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0008) 8b.

For a parallel mechanism having five springs among which no three are concurrent or parallel to each other, the realization conditions can be equivalently interpreted geometrically as:

Proposition 4

A [stiffness matrix](https://www.sciencedirect.com/topics/engineering/stiffness-matrix) can be passively realized with a 5-spring parallel mechanism if and only if the following conditions hold:

(i) The distance ratios for different sets of wrenches described in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0007) 7a and [7b](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0007) are the same (satisfy Eq. [\(49\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0049));

(ii)A twist centered at the intersection of 2 spring axes, (a) if located within the triangle enclosed by the other 3 springs, when multiplied by K, yields a wrench that does not intersect the triangle; (b) if located outside the triangle, when multiplied by K, yields a wrench that intersects the triangle without crossing the shaded area described in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0008) 8b.

Dual to the serial case, condition (i) in [Proposition 4](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0004) ensures the decomposition of K into the form of Eq. [\(50\).](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0050) Condition (ii) of [Proposition 4](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0004) for any two springs ensures that the [spring stiffnesses](https://www.sciencedirect.com/topics/engineering/spring-stiffness) for the other three springs are non-negative. Thus, satisfaction of condition (ii) for two sets of different springs (e.g., $\{w_1, w_2\}$ and $\{w_3, w_4\}$, or ${w_1, w_3}$ and ${w_2, w_4}$ guarantees the passive realization of K with the 5-spring parallel mechanism.

4.4. The compliance center and the mechanism configuration

For any given planar compliant behavior, there is a particular point at which both the compliance matrix and stiffness matrix can be described in a diagonal form. This unique point is defined to be the center of compliance/stiffness. It was shown [\[6\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0006) [\[30\]](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0030) that, if a compliant behavior is passively achieved with a mechanism having 3 or 4 elastic components, then the center must be located within an area enclosed by the locations of these components. These results can be extended to mechanisms having five elastic components.

First consider a serial 5J mechanism. Let l_{ij} be the line passing through J_i and J_j , and w_{ij} be the unit wrench associated with l_{ij} . Suppose, with no loss of generality, that l_{12} does not intersect the triangle with vertices at the other joints J_3 , J_4 and J_5 . Then by realization inequality condition (i) of [Proposition 3,](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0003) T_{12} , the center of twist $T_{12} = Cw_{12}$, must be inside triangle $J_3J_4J_5$. Since the compliance center and T_{12} must be on the same side of l_{12} [\[30\],](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#bib0030) C_c must be on the same side of l_{12} as the triangle (the shaded area as illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0009) 9a). Applying the same process to all lines l_{ij} that have the other three joints on the same side, the center of compliance must be within the convex hull formed by the five joints as demonstrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0009) 9b.

Fig. 9. Locus of compliance centers C_c . (a) When considering only two joints, J_1 and J_2 , C_c is on the same side of line l_{12} as triangle $J_3J_4J_5$. (b) When considering all joints, C_c must be within the convex hull formed by the five joints. (c) For a 5-spring parallel mechanism, the stiffness center C_k must be the within the union of triangles formed by any three spring axes.

Similarly, it can be shown that, if a stiffness is passively achieved by a 5-spring parallel mechanism, the stiffness center must be inside the union of triangles formed by any 3 spring axes as illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0009) 9c. Note that for a 5-spring parallel mechanism, the locus area of stiffness centers C_k is not convex.

5. Compliance synthesis

In this section, synthesis procedures for both types of mechanisms are developed using the conditions on the mechanism geometry developed in [Section](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0013) 4. In each procedure, the center of the elastic behavior is used.

5.1. Synthesis with a 5-joint serial mechanism

Suppose C is a 3×3 [compliance matrix](https://www.sciencedirect.com/topics/engineering/compliance-matrix) having the form:

$$
\mathbf{C} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^T & c_{33} \end{bmatrix},
$$

the compliance center \mathcal{C}_c is determined by:

$$
(54)\mathbf{r}_c = \frac{1}{c_{33}}\mathbf{Sb}_{r}
$$

where S is the 2 × 2 matrix defined in $Eq.$ [\(3\).](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0003) The unit twist t_c associated with C_c is calculated using Eq. [\(1\).](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0001)

The procedure for a 5-joint serial mechanism [realization](https://www.sciencedirect.com/topics/engineering/realization) of compliance C presented below identifies the locations of the five joints and the corresponding joint compliances. Selections used in the procedure are illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0010) 10.

1. Select arbitrarily a line l_{12} and obtain the corresponding unit wrench w_{12} . Two joints J_1 and J_2 will lie on line l_{12} . The associated twist is calculated:

 (55) t₁₂ = Cw₁₂..

The unit twist associated with t_{12} , $\hat{\mathsf{t}}_{12}$,, is obtained and the center of $\hat{\mathsf{t}}_{12}$, T_{12} , is calculated using Eq. [\(2\).](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0002)

2. Select arbitrarily a line l_{34} (represented by unit wrench w_{34}). Two joints J_3 and J_4 will lie on line l_{34} .

3. Select the location of joint J_5 . To satisfy condition (ii) of **Proposition 3**, this point must be located on the same side of l_{34} as T_{12} . The unit twist centered at J_5 , t_5 , is obtained using Eq. [\(1\).](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0001)

4. Select a line l_{23} (represented by wrench w_{23}) that intersects l_{12} and l_{34} to obtain joint locations J_2 and J_3 . This line must be judiciously chosen such that T_{34} satisfies condition (ii) of [Proposition 3.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0003) The twist associated with w_{23} is calculated:

$$
(57)\mathbf{t}_{23} = \mathbf{C} \mathbf{w}_{23}.
$$

The unit twist associated with t_{23} , \hat{t}_{23} , is obtained, and the center of t_{23} , T_{23} , is calculated using Eq. [\(2\).](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0002) The distance ratio used in the equality condition corresponding to the selections of w_{12} , w_{34} , w_{23} and J_5 is calculated:

$$
(58)\,\gamma_{S} = \frac{(w_{12}^T t_c)(w_{34}^T \hat{\mathbf{t}}_{12})(w_{23}^T \mathbf{t}_5)}{(w_{12}^T \mathbf{t}_5)(w_{34}^T \mathbf{t}_5)(w_{23}^T \mathbf{t}_c)}.
$$

Note that in calculation of the ratio γ_s in Eq. [\(58\),](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0058) normalization of w_{12} , w_{34} and w_{23} to unit wrenches is not necessary since they appear in both the numerator and [denominator.](https://www.sciencedirect.com/topics/engineering/denominator)

5. Choose a line w_{14} such that the ratio of the distances from the line to T_{23} and J_5 is equal to γ_s . This can be accomplished by the following steps.

(a) On line l_p passing through T_{23} and J_5 , choose point P such that the ratio of the distances from T_{23} to P and from J_5 to P is $\gamma_s.$ This point (x_p, y_p) can be determined by the two equations:

$$
(59) \frac{\| \mathbf{t}_p - \hat{\mathbf{t}}_{23} \|}{\| \mathbf{t}_p - \mathbf{t}_5 \|} = |\gamma_s|, \mathbf{t}_p \cdot (\hat{\mathbf{t}}_{23} \times \mathbf{t}_5) = 0,
$$

where $\bm{{\rm t}}_p = \left[y_p, -x_p, 1\right]^T$ is the unit twist located at P , and $\hat{\bm{{\rm t}}}_{23}$ and $\bm{{\rm t}}_5$ are the unit twists at T_{23} and J_5 , respectively. Note that the set of equations in (59) yields two solutions on line l_p : one is inside line segment J_5T_{23} , and the other is outside. The selection of point *P*for the realization depends on the sign of γ_s : if $\gamma_s > 0$, the solution outside segment J_5T_{23} is selected; if $\gamma_s < 0$, the solution inside segment J_5T_{23} is selected. Thus P is uniquely determined.

(b) Any line passing through point P satisfies the ratio condition. Judiciously select a line l_{14} passing through P (denoted as w_{14}) that intersects w_{12} and w_{34} to obtain joint locations J_1 and J_4 . In the selection of this line, condition (ii) of **[Proposition 3](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0003)** for both T_{12} and T_{34} must be satisfied.

6. Calculate the joint compliances using Eq. [\(25\).](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0025)

With this final step, the five joint locations and compliances in the serial mechanism are identified.

Fig. 10. Compliance synthesis with a 5-joint serial mechanism using [geometric constraints.](https://www.sciencedirect.com/topics/engineering/geometric-constraint)

5.2. Synthesis with a 5-spring parallel mechanism

Suppose K is a given 3×3 [stiffness matrix](https://www.sciencedirect.com/topics/engineering/stiffness-matrix) having the form:

$$
\mathbf{K} = \begin{bmatrix} \mathbf{D} & \mathbf{v} \\ \mathbf{v}^T & k_{33} \end{bmatrix},
$$

the center of stiffness is calculated by:

$$
(60) \mathbf{r}_c = -\mathbf{SD}^{-1} \mathbf{v}
$$

where S is the 2 × 2 matrix defined in $\underline{Eq. (3)}$. The unit twist t_c associated with the stiffness center C_k is calculated using $Eq. (1)$.

Below, the synthesis procedure for stiffness realization with a 5-spring parallel mechanism is presented. Selections used in the process are shown in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0011) 11.

1. Select arbitrarily a point T_{12} and calculate the unit twist t_{12} centered at T_{12} using Eq. [\(1\).](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0001) Two spring axes w_1 and w_2 of the mechanism will intersect at this point. The wrench associated with t_{12} is calculated:

(61)
$$
w_{12} = Kt_{12}
$$
.

The unit wrench associated with w_{12} , \hat{w}_{12} , is obtained and the line of action of w_{12} is interpreted geometrically using $Eq. (4)$.

2. Select another point T_{34} arbitrarily. Calculate the unit twist t_{34} associated with T_{34} . The wrench corresponding to t_{34} is calculated:

(62) $w_{34} = Kt_{34}$.

With the unit wrench associated with w_{34} , the line of action of w_{34} is again interpreted geometrically using $\underline{Eq. (4)}$. Two spring axes w_3 and w_4 will intersect at T_{34} .

3. Select line l_5 , which will be the line of action of spring w_5 . To satisfy condition (ii) of [Proposition 4,](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0004) this line must be selected such that points T_{12} and T_{34} are on the same side of l_5 . The unit spring wrench w_5 is obtained using $Eq. (4)$.

4.

Judiciously select a point T_{23} , at which spring axes w_2 and w_3 will meet. This point, together with points T_{12} and T_{34} , determines wrenches w_2 and w_3 . In selecting this point, condition (ii) of [Proposition 4](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0004) must be satisfied. The wrench corresponding to the twist at T_{23} is:

(63)
$$
w_{23} = Kt_{23}
$$

Calculate the distance ratio used in the equality condition corresponding to the selections of T_{12} , T_{34} , T_{23} and w_5 :

(64)
$$
\gamma_p = \frac{(\mathbf{t}_{34}^T \mathbf{w}_{12})(\mathbf{t}_{c}^T \mathbf{w}_{23})(\mathbf{t}_{23}^T \mathbf{w}_{5})}{(\mathbf{t}_{c}^T \mathbf{w}_{12})(\mathbf{t}_{12}^T \mathbf{w}_{5})(\mathbf{t}_{34}^T \mathbf{w}_{5})}.
$$

5. Select a point T_{14} such that the ratio of the distances from the point to lines of w_{23} and w_5 is equal to γ_p , i.e.,

$$
(65) \frac{\mathbf{t}_{14}^T \mathbf{w}_{23}}{\mathbf{t}_{14}^T \mathbf{w}_5} = \gamma_p.
$$

This can be accomplished by the following steps.

(a) For the given w_{23} and w_5 , consider the equation defined by:

(66)
$$
\mathbf{t}^T \mathbf{w}_{23} = \gamma_p (\mathbf{t}^T \mathbf{w}_5),
$$

where t is a unit twist defined by $\mathbf{t}=[y,-x,1]^T$. Then $\underline{\sf Eq.$ [\(66\)d](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0066)efines a line l_t on which any point satisfies the equality condition.

(b) Judiciously choose one point T_{14} on line l_t such that condition (ii) of [Proposition 4](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0004) for both w_{12} and w_{34} are satisfied. The remaining four spring axes (w_i , $i = 1,2,3,4$) are determined by the four lines passing through points (T_{12}, T_{14}) , (T_{12}, T_{23}) , (T_{23}, T_{34}) , and (T_{14}, T_{34}) as illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0011) 11.

6. Calculate the stiffness for each spring using $Eq. (26)$.

With this final step, the five spring axes and their stiffnesses in the parallel mechanism are identified.

Fig. 11. Compliance synthesis with a 5-spring parallel mechanism based on geometry.

5.3. Discussion

In each step of the procedures presented in [Sections](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0023) 5.1 and [5.2,](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0024) a line or a point is selected. Since the elastic components (joints or springs) in a mechanism can be numbered arbitrarily, the order of these selections can be changed.

When the mechanism configuration is determined, each joint compliance in the serial mechanism (or each [spring constant](https://www.sciencedirect.com/topics/engineering/spring-constant) in the parallel mechanism) can be calculated using any permutation in Eq. [\(25\)o](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0025)r [\(26\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0026) provided that all twists and wrenches involved have finite (zero) pitch. Thus, any translational twist or [rotational](https://www.sciencedirect.com/topics/engineering/rotational) wrench (having infinite pitch) should not be used in Eq. (25) or (26) to calculate the joint compliances or [spring stiffnesses.](https://www.sciencedirect.com/topics/engineering/spring-stiffness)

In application, constraints on the mechanism configuration need to be considered. For example, if it is desired that the first joint J_1 and the last joint J_5 be located at given locations (e.g., when the location of the robot base and [end-effector](https://www.sciencedirect.com/topics/engineering/end-effector) pose are specified), the synthesis of a given compliance involves identifying the three remaining joint locations J_2 , J_3 and J_4 . The procedure presented in [Section](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0023) 5.1 can be modified to meet this requirement and is described below.

First, choose a line l_{12} to pass through the given point J_1 and choose J_5 at the location yielding the desired endeffector pose. Then select line l_{34} at a location that satisfies condition (ii) of [Proposition 3.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0003) Line l_{23} can be selected as described in Step 4 to determine J_2 and J_3 , and point P is calculated using the method described in Step 5a. By passing through points J_1 and P, line l_{14} is determined. The intersection of l_{34} and l_{14} is the location of J_4 . It can be seen that, since the locations of two joints are specified, line l_{14} is unique.

Note that in the compliance realization with a 5-joint mechanism, at most two joint locations can be specified arbitrarily. If three joint locations are constrained, the mechanism may not have sufficient mobility to achieve a configuration required by the realization, and thus would not be able to realize the given behavior. Once the first two joint locations are selected, the remaining joint locations must be selected such that geometric condition (ii) in [Proposition 3](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0003) is satisfied. The serial elastic [mechanism synthesis](https://www.sciencedirect.com/topics/engineering/mechanism-synthesis) example in the next section considers practical constraints.

In the synthesis process, the space of possible joint locations for each joint is identified. One can select any one from the available space based on its geometry. When five joint locations are selected, the corresponding joint compliances can be determined by $Eq. (25)$. Since the selection of each joint is not unique, if different configurations are selected in the process, a different set of joint compliances will be obtained using Eq. [\(25\).](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0025) It is guaranteed that all joint compliances calculated by $Eq. (25)$ $Eq. (25)$ are positive for any configuration selected using the procedure. Also, in selecting the location of each joint, mechanism [singularity](https://www.sciencedirect.com/topics/engineering/singularities) (for which three joints are located on a straight line) should be avoided. This can always be accomplished since there are infinitely many options in the space of joint locations.

Note that, once the joint locations are identified, the specified compliance is realized only at that configuration. Since a 5*J* serial mechanism has redundancy, the nullspace of the [Jacobian](https://www.sciencedirect.com/topics/engineering/jacobian) is nonzero and thus *kinematic* internal motion allows the mechanism configuration to change with the [endpoint](https://www.sciencedirect.com/topics/engineering/endpoint) pose relative to the base unchanged. The realization may not be maintained for a variation from the obtained configuration because a change of the configurations by an internal motion may violate the realization conditions.

It can be seen that to realize a given compliance using a serial mechanism with two specified joint locations, the mechanism must have *at least* 5 joints. This is because if the number of joints of a mechanism is less than 5, the mechanism does not have the necessary degree of freedom to satisfy the corresponding realization conditions.

Similar to the serial case, for parallel mechanism synthesis with two spring axes being specified, the mechanism must have *at least* five springs in order to realization an arbitrary stiffness. For a 5-spring parallel mechanism, the first two spring wrenches can always be selected arbitrarily. In the selection of the remaining springs, the previously selected spring wrenches and geometric condition (ii) in **[Proposition 4m](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0004)ust be satisfied**. Also, in selecting the spring axes, mechanism singularity (three springs intersecting at a single point or parallel to each

other) should be avoided. This can always be accomplished since there are infinitely many options in the space of acceptable spring wrenches.

Compared to previous results [30] for 4-component mechanisms, the advantages of the 5-component mechanisms considered in the paper are that the dimension of the space of realizable compliances at a configuration is increased by one and that more [geometric constraintso](https://www.sciencedirect.com/topics/engineering/geometric-constraint)n the mechanism can be accommodated. The disadvantage is that more components increase the complexity of the mechanism.

6. Synthesis procedure examples

[Numerical examples](https://www.sciencedirect.com/topics/engineering/numerical-example) for both serial and parallel mechanisms are presented to demonstrate the synthesis procedures. For serial [mechanism synthesis,](https://www.sciencedirect.com/topics/engineering/mechanism-synthesis) the procedure identifies a serial mechanism that realizes the given compliance C by selecting the locations of five joints and their corresponding joint compliances. For parallel mechanism synthesis, the procedure identifies a parallel mechanism that realizes the given stiffness K by selecting the axes of five springs and their corresponding [spring constants.](https://www.sciencedirect.com/topics/engineering/spring-constant)

In a [global coordinate frame,](https://www.sciencedirect.com/topics/engineering/global-coordinate-frame) the compliant behavior to be realized is specified by:

In the examples, the units of all wrenches and twists calculated in the synthesis process are consistent with that used in K and C. For a twist, the first two components have units of length in meters [m] and the third component has units of [radian.](https://www.sciencedirect.com/topics/engineering/radian) For a wrench, the first two components have units of force in Newton [N] and the third component has units of moment in $N \cdot m$. Also, all [Cartesian coordinates](https://www.sciencedirect.com/topics/engineering/cartesian-coordinate) are in meters.

Using [Eq. \(55\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0055) or [\(60\),](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0060) the stiffness/compliance center of this behavior is calculated to be located at $\left(\frac{17}{14}\right)$ $\frac{17}{14}, -\frac{8}{7}$ $\frac{8}{7}$. The unit twist at the center is

$$
\mathbf{t}_c = \left[-\frac{8}{7}, -\frac{17}{14}, 1 \right]^T.
$$

6.1. Synthesis with a 5-joint serial mechanism

In this example, it is desired that joints J_1 (the joint connected to the base of the manipulator) and J_5 (the joint connected to the manipulator end-effector) be at given locations. In the [coordinate frame](https://www.sciencedirect.com/topics/engineering/coordinate-frame) used to describe C, the two joints J_1 and J_5 are required to be located at

$$
\mathbf{r}_1 = [-2, -2]^T, \mathbf{r}_5 = [2, -2]^T.
$$

Below, the locations of the remaining three joints are identified using the modified synthesis procedure presented in [Section 5.3.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0025)

The unit twists associated with J_1 and J_5 are calculated using **[Eq.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0001)** (1) to be:

$$
\mathbf{t}_1 = [-2,2,1]^T, \mathbf{t}_5 = [-2,-2,1]^T.
$$

The modified serial mechanism synthesis procedure of [Section](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0025) 5.3 is used to determine the other three joint locations (J_2, J_3, J_4) , and all joint compliance values. The component geometry selections used in the process are illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0012) 12.

Fig. 12. Compliance synthesis with a 5-joint mechanism with J_1 and J_5 being specified. The desired compliance is realized by choosing the remaining three joint locations and all joint compliances in the serial mechanism.

The first line l_{12} is selected to pass through J_1 at $(-2, -2)$ with an arbitrary slope. Here, the slope is selected to be 1. The unit wrench representing l_{12} is:

$$
\mathbf{w}_{12} = \frac{\sqrt{2}}{2} [1,1,0]^T.
$$

The twist t_{12} corresponding to w_{12} is calculated to be:

$$
\hat{\mathbf{t}}_{12} = [-1.5758, -1.4848, 1]^T,
$$

Normalizing t_{12} yields the unit twist:

$$
\hat{\mathbf{t}}_{12} = [-1.5758, -1.4848, 1]^T,
$$

and the location of the center of t_{12} , T_{12} , is calculated to be (1.4848, -1.5758).

The second line l_{34} is where joints J_3 and J_4 will lie. Since the location of J_5 has been specified, l_{34} must be selected such that T_{12} and J_5 are on the same side of l_{34} ; otherwise T_{12} will be outside of the triangle formed by J_3 , J_4 and J_5 (which violates the inequality condition (ii) of [Proposition 3\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0003). In this example, l_{34} is chosen to pass through point (1,0) and parallel to *y*-axis. The unit wrench representing l_{34} is:

 $\mathbf{w}_{34} = [0,1,1]^T$..

The associated twist t_{34} is:

 $\mathbf{t}_{34} = \mathbf{C} \mathbf{w}_{34} = [0.14, 0.18, -0.06]^T.$

The twist center of t_{34} , T_{34} , is determined to be $(3, -2.3333)$.

Select a line l_{23} to meet lines l_{12} and l_{34} to determine the locations of J_2 and J_3 . This line should be chosen such that T_{34} is above line I_{25} (which passes J_2 and J_5 as shown in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0012) 12) to satisfy condition (ii) of [Proposition 3.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0003) Here, a line passing through point (0, −0.5) parallel to *x*-axis is selected. With this selection, joint locations of J_2 and J_3 are determined. The unit wrench associated with line l_{23} is:

 $\mathbf{w}_{23} = [1, 0, 0.5]^T$.

The associated twist t_{23} is:

 $\mathbf{t}_{23} = \mathbf{C} \mathbf{w}_{23} = [0.42, 0.29, -0.18]^T.$

The location of t_{23} , T_{23} , is determined to be $(1.6111, -2.3333)$.

The ratio γ_s in $\underline{\mathsf{Eq.}}$ [\(58\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0058) is calculated to be:

$$
\gamma_{s} = \frac{(\mathbf{w}_{12}^{T} \mathbf{t}_{c})(\mathbf{w}_{34}^{T} \hat{\mathbf{t}}_{12})(\mathbf{w}_{23}^{T} \mathbf{t}_{5})}{(\mathbf{w}_{12}^{T} \mathbf{t}_{5})(\mathbf{w}_{34}^{T} \mathbf{t}_{5})(\mathbf{w}_{23}^{T} \mathbf{t}_{c})} = 0.6667.
$$

Since $\gamma_s > 0$, point P must be outside line segment J_5T_{23} . Using the set of equations in [\(59\),](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0059) point P is calculated to be located at $(0.8333, -3.0001)$. The unit twist centered at this location is:

$$
\mathbf{t}_p = [-3.0001, -0.8333, 1]^T.
$$

Line l_{14} is uniquely determined by the two points J_1 and P. The unit wrench associated with l_{14} can be determined using Eq. [\(13\):](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0013)

$$
\mathbf{w}_{14} = [0.9430, -0.3328, 2.5516]^T.
$$

Finally, the intersection of l_{14} and l_{34} identifies the location of J_4 , which is calculated to be $(1, -3.0589)$. The three joint locations (J_2, J_3, J_4) illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0012) 12) are:

$$
\mathbf{r}_2 = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, \mathbf{r}_3 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}, \mathbf{r}_4 = \begin{bmatrix} 1 \\ -3.0589 \end{bmatrix}.
$$

Since T_{12} is inside triangle $J_3J_4J_5$, and T_{34} is outside triangle $J_1J_2J_5$ and is inside the area bounded by lines /₁₅ and /₂₅ (shown in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0012) 12), condition (ii) of [Proposition 3](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0003) is satisfied. Thus, all five joint compliances for this configuration are positive. The values of the joint compliances are obtained using Eq. [\(25\):](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0025)

 $c_1 = 0.0022$ m/N, $c_2 = 0.0089$ m/N, $c_3 = 0.1667$ m/N, $c_4 = 0.0221$ m/N, $c_5 = 0.0800$ m/N.

With this synthesis procedure, the serial mechanism configuration with specified locations of joints J_1 and J_5 (illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0012) 12) and the values of all 5 joint compliances are identified. The five joint twists at the configuration are:

$$
\mathbf{t}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix} -0.5 \\ 0.5 \\ 1 \end{bmatrix}, \mathbf{t}_3 = \begin{bmatrix} -0.5 \\ -1 \\ 1 \end{bmatrix}, \mathbf{t}_4 = \begin{bmatrix} -3.0589 \\ -1 \\ 1 \end{bmatrix}, \mathbf{t}_5 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}.
$$

The result is verified by adding the joint compliance components using $Eq. (14)$:

$$
\mathbf{C}_{S} = c_{1} \mathbf{t}_{1} \mathbf{t}_{1}^{T} + c_{2} \mathbf{t}_{2} \mathbf{t}_{2}^{T} + c_{3} \mathbf{t}_{3} \mathbf{t}_{3}^{T} + c_{4} \mathbf{t}_{4} \mathbf{t}_{4}^{T} + c_{5} \mathbf{t}_{5} \mathbf{t}_{5}^{T} = \begin{bmatrix} 0.58 \text{m/N} & 0.46 \text{m/N} & -0.32 \text{N}^{-1} \\ 0.46 \text{m/N} & 0.52 \text{m/N} & -0.34 \text{N}^{-1} \\ -0.32 \text{N}^{-1} & -0.34 \text{N}^{-1} & 0.28 (\text{N} \cdot \text{m})^{-1} \end{bmatrix}.
$$

It can be seen that the [compliance matrix](https://www.sciencedirect.com/topics/engineering/compliance-matrix) C_S obtained for the synthesized serial mechanism is the desired compliance matrix C in $Eq. (67)$.

Note that with this procedure, the serial mechanism configuration (with J_1 and J_5 specified) is determined by choosing the locations of the other three joints. In construction, the joint connecting sequence does not affect the compliant behavior realized by the serial mechanism, i.e., the same compliance is achieved with joint sequence of $J_1J_4J_2J_3J_5$ or $J_1J_2J_4J_3J_5$.

6.2. Synthesis with a 5-spring parallel mechanism

Following the synthesis procedure presented in [Section](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#sec0024) 5.2, the given stiffness is achieved by a 5-spring parallel mechanism. The component geometry selections used in the process are illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0013) 13.

Fig. 13. Stiffness synthesis with a 5-spring mechanism. The desired stiffness is realized by choosing the axes of five springs and their corresponding spring rates in the parallel mechanism.

First, point T_{12} can be chosen arbitrarily. Two spring axes (w_1 and w_2) will meet at this point. In this example, T_{12} is selected to be $(1, -2)$. The unit twist at T_{12} is:

$$
\mathbf{t}_{12} = [-2, -1, 1]^T.
$$

The associated wrench w_{12} is:

$$
\mathbf{w}_{12} = \mathbf{K} \mathbf{t}_{12} = [-6, 6, 4]^T.
$$

Normalizing w_{12} yields the corresponding unit wrench:

$$
\widehat{\mathbf{w}}_{12} = \frac{\sqrt{2}}{2} \left[-1, 1, \frac{2}{3} \right]^T.
$$

The wrench axis of w_{12} is interpreted geometrically by $\underline{Eq. (4)}$ $\underline{Eq. (4)}$ $\underline{Eq. (4)}$ and is illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0013) 13.

The second point T_{34} (where two additional spring axes w_3 and w_4 will meet) can also be chosen arbitrarily. In this example, T_{34} is chosen to be $(2, -2)$. Using Eq. [\(3\),](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0003) the unit twist centered at T_{34} is:

$$
\mathbf{t}_{34} = [-2, -2, 1]^T,
$$

and the associated wrench is:

$$
\mathbf{w}_{34} = \mathbf{K} \mathbf{t}_{34} = [-2, -6, -6]^T.
$$

The wrench axis of w_{34} is shown in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0013) 13.

Next, line l_5 is selected. To satisfy condition (ii) of [Proposition 4,](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0004) T_{12} and T_{34} must be on the same side of l_5 . Also, since T_{12} and T_{34} are located below the stiffness center C_k , w_5 should be located above C_k . Here, l_5 is chosen to pass through point $(0, -0.5)$ and parallel to the *x*-axis. The unit spring wrench w_5 associated with this line is:

 $\mathbf{w}_5 = [1, 0, 0.5]^T$.

Next, choose point T_{23} to determine the spring wrenches w_2 and w_3 . Here, to reduce the range spanned by the springs, point $(2, -4)$ is selected so that w_3 is vertical (parallel to the *y*-axis). With this selection, the [realization](https://www.sciencedirect.com/topics/engineering/realization) condition of [Proposition 4](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0004) (ii) is satisfied. The unit twist at T_{23} is calculated to be:

$$
\mathbf{t}_{23} = [-4, -2.1]^T,
$$

and the corresponding wrench is calculated to be:

$$
\mathbf{w}_{23} = \mathbf{K} \mathbf{t}_{23} = [-14.2, -10]^T.
$$

The distance ratio in $\underline{Eq. (64)}$ $\underline{Eq. (64)}$ $\underline{Eq. (64)}$ corresponding to the selections of T_{12} , T_{34} , T_{23} and l_5 is calculated to be:

$$
\gamma_p = \frac{(\mathbf{t}_{34}^T \mathbf{w}_{12})(\mathbf{t}_c^T \mathbf{w}_{23})(\mathbf{t}_{23}^T \mathbf{w}_5)}{(\mathbf{t}_c^T \mathbf{w}_{12})(\mathbf{t}_{12}^T \mathbf{w}_5)(\mathbf{t}_{34}^T \mathbf{w}_5)} = -6.2222.
$$

Using Eq. [\(66\),](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0066) the equation for line *l^t* is obtained:

$$
(68) y = -0.2571x - 0.8857.
$$

The final point T_{14} must be on line l_t . Here, T_{14} is selected so that w_1 is also vertical. The location of T_{14} can be obtained by solving $Eq. (68)$ $Eq. (68)$ with $x = 1$. The location of the point is determined to be at $(1, -1.1428)$. The unit twist at this point is:

$$
\mathbf{t}_{14} = [-1.1428, -1.1]^T.
$$

The axes of the remaining 4 springs are identified by lines $T_{12}T_{14}$, $T_{12}T_{23}$, $T_{23}T_{34}$, and $T_{14}T_{34}$ as illustrated in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0013) 13. The 5 spring wrenches in the mechanism are:

$$
\mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 0.4472 \\ -0.8944 \\ 0 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{w}_4 = \begin{bmatrix} 0.7592 \\ -0.6508 \\ 0.2186 \end{bmatrix}, \mathbf{w}_5 = \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix}.
$$

It can be seen that, for (w_1, w_2) and (w_3, w_4) , the realization condition in [Proposition 4\(](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#enun0004)ii) is satisfied. Thus all [spring stiffnesses](https://www.sciencedirect.com/topics/engineering/spring-stiffness) must be positive. Since twists t_{12} , t_{23} , t_{14} and t_{34} are already known, to determine the values of the spring stiffnesses using Eq. [\(26\),](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0026) only t_{35} and t_{45} are needed. It is easy to determine that T_{35} is located at $(2, -0.5)$,, and T_{45} is located at $(0.25, -0.5)$. The two unit twists needed are:

$$
\mathbf{t}_{35} = [-0.5, -2.1]^T, \mathbf{t}_{45} = [-0.5, -0.25, 1]^T.
$$

Using these twists, the spring stiffnesses calculated using $Eq. (26)$ $Eq. (26)$ are:

 $k_1 = 4.6677N/m$, $k_2 = 1.6672N/m$, $k_3 = 3.1427N/m$, $k_4 = 6.7460N/m$, $k_5 = 1.7778N/m$.

With this synthesis procedure, the five spring axes (shown in [Fig.](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#fig0013) 13) and the associated spring constants are identified. The realization of the desired stiffness with the 5-spring mechanism is verified using $Eq. (50)$:

$$
\mathbf{K}_{P} = k_{1}\mathbf{w}_{1}\mathbf{w}_{1}^{T} + k_{2}\mathbf{w}_{2}\mathbf{w}_{2}^{T} + k_{3}\mathbf{w}_{3}\mathbf{w}_{3}^{T} + k_{4}\mathbf{w}_{4}\mathbf{w}_{4}^{T} + k_{5}\mathbf{w}_{5}\mathbf{w}_{5}^{T} = \begin{bmatrix} 6N/m & -4N/m & 2N \\ -4N/m & 12N/m & 10N \\ 2N & 10N & 18N \cdot m \end{bmatrix}.
$$

It can be seen that the [stiffness matrix](https://www.sciencedirect.com/topics/engineering/stiffness-matrix) K_p obtained for the synthesized parallel mechanism is the desired stiffness matrix K in Eq. [\(67\).](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0067)

Note that the procedure only identifies the axis of each spring. In parallel mechanism construction, a line spring can be placed at any location along its axis. Also note that, since w_1 and w_3 are selected to be parallel in this

example, t_{13} calculated in Eq. [\(22\)](https://www.sciencedirect.com/science/article/pii/S0094114X18319438#eq0022) has infinite pitch (centered at infinity). Thus, in calculating spring constants k_2 , k_4 and k_5 using $\underline{Eq. (26)}$, permutations not containing t_{13} should be used.

7. Summary

In this paper, a geometric approach to passive [realization](https://www.sciencedirect.com/topics/engineering/realization) of any given planar compliance with a simple 5-joint serial or 5-spring parallel mechanism is presented. Realization conditions for a 5-component redundant mechanism to achieve an arbitrarily specified planar compliance are presented. Since these conditions impose requirements on the mechanism configuration, they can be geometrically interpreted and illustrated on the plane. Geometry based synthesis procedures to realize a compliance are developed for both types of mechanisms. For serial [mechanism synthesis,](https://www.sciencedirect.com/topics/engineering/mechanism-synthesis) the procedure identifies the five joint locations and the corresponding joint compliances for the compliance realization. A 5-joint serial mechanism can be constructed based on the component geometry and joint elastic property. For parallel mechanism synthesis, the procedure identifies the five spring axes (lines of action) and the corresponding spring rates for the stiffness realization. A 5-spring parallel mechanism can be constructed based on the geometric and elastic properties of each component. Unlike most of the previous realization approaches, the procedures allow one to choose each component, based on its geometry, from the space of admissible options. Because the space of admissible options is restricted by inequalities, the realization of a compliant behavior with constraints on the mechanism configuration can be accomplished. Any given compliance can be achieved with a mechanism having at most two specified components (i.e., two joint locations of a serial mechanism, or two spring axis lines of a parallel mechanism). The selection of the remaining components is restricted in some way. This ability makes 5 component redundant [compliant mechanisms](https://www.sciencedirect.com/topics/engineering/compliant-mechanism) more practical in [robotic applications,](https://www.sciencedirect.com/topics/engineering/robotics-application) especially for tasks when the mechanism geometry and/or workspace are constrained. Since the approach developed in the paper is completely geometry based, graphic tools can be used to better design an elastic mechanism.

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