9-1-2014

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Published version. *Journal of Statistical Theory and Applications*, Vol. 13, No. 3 (September 2014): 222-234. DOI.

Journal of Statistical Theory and Applications, Vol. 13, No. 3 (September 2014), 222-234

Characterizations of a Class of Distributions by Dual Generalized Order Statistics and Truncated Moments

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Received 11 June 2014

Accepted 21 August 2014

The problem of characterizing a distribution is an important problem which has recently attracted the attention of many researchers. Thus, various characterizations have been established in many different directions. The present work deals with the characterizations of a general class of distributions. These characterizations are based on: (i) a simple relationship between two truncated moments, (ii) truncated moment of certain functions of the n^{th} order statistic, (iii) single truncated moment of certain functions of the random variable and (iv) moments of dual generalized order statistics. We like to mention that the characterization (i) which is expressed in terms of the ratio of truncated moments is stable in the sense of weak convergence. We also study the recurrence relations between moments of dual generalized order statistics of this class of distributions.

Keywords: Moments; Dual generalized order statistics; Recurrence relations; Characterizations of distributions.

2000 Mathematics Subject Classification: 62E10, 62G30

1. Introduction

The problem of characterizing a distribution is an important problem which has recently attracted the attention of many researchers. Thus, various characterizations have been established in many different directions. The present work deals with the characterizations of a general class of distributions. These characterizations are based on: (i) a simple relationship between two truncated moments, (ii) truncated moment of certain functions of the n^{th} order statistic, (iii) single truncated moment of certain functions of the random variable and (iv) moments of dual generalized order statistics. We also study the recurrence relations between moments of dual generalized order statistics of this class of distributions.

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An investigator will be vitally interested to know if their model fits the requirements of any member of the class of distributions mentioned above. To this end, one will depend on the characterizations of these distributions which provide conditions under which the underlying distribution is indeed one of these distributions. Although in many applications an increase in the number of parameters provides a more suitable model, in characterization problems a lower number of parameters (without seriously affecting the suitability of the model) is mathematically more appealing (see Glänzel and Hamedani [9]).

The cumulative distribution function (*cdf*) and the probability density function (*pdf*) corresponding to general class of distributions studied in this work are given, respectively, by

$$F(x) = F(x;a,b,c) = [a\xi(x) + b]^c, \tag{1.1}$$

and

$$f(x) = f(x;a,b,c) = ac\xi'(x)[a\xi(x) + b]^{c-1} \tag{1.2}$$

where $a \neq 0$, b and $c \neq 0$ are parameters, and $\xi(x)$ is a monotonic and differentiable function on $\left(F^{-1}(0), F^{-1}(1)\right)$ so that $F(x)$ is indeed a *cdf*. We observe that Malinowska and Szynal [21] proposed certain characterizations of the class of distributions defined in (1.1) using the conditional expectation of the kth lower (upper) record values.

We note that for this class of $cdf's$ given by (1.1) and (1.2), we have

$$F(x;a,b,c) = \frac{[a\xi(x) + b]}{ac\xi'(x)} f(x;a,b,c). \tag{1.3}$$

For proper choices of a, b, c and $\xi(x)$, various well known distributions such as the power function, the Dagum (Burr Type III), the Pareto, the Inverse Weibull, the Reflected Exponential and the Rectangular, can be obtained.

2. Characterization Results

As we mentioned in the *Introduction*, the class of distributions considered in this paper includes some important distributions which are suitable models in many different applications. So, an investigator will be vitally interested to know if their model fits the requirements of the distribution from this class. To this end, one will depend on the characterizations of this class of distributions which provide conditions under which the underlying distribution indeed belongs to this class.

2.1. *Characterization based on two truncated moments*

In this subsection we present characterizations of (1.1) in terms of a simple relationship between two truncated moments. We like to mention here the works of Galambos and Kotz [4], Kotz and Shanbhag [20], Glänzel [5 – 7], Glänzel et al. [8], Glänzel and Hamedani [9] and Hamedani [10 – 14] are in this direction. Our characterization results presented here will employ an interesting result due to Glänzel [6] (Theorem 2.1.1 below).

Theorem 2.1.1. Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a given probability space and let $H = [a,b]$ be an interval for some $a < b$ $(a = -\infty, b = \infty$ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with the cumulative distribution function F and let g and h be two real functions defined on H such that

$$\mathbf{E}\left[g(X)\,|X \geq x\right] = \mathbf{E}\left[h(X)\,|X \geq x\right]\,\eta(x), \qquad x \in H\ ,$$

is defined with some real function η. Assume that $g\ ,\ h \in C^1(H)\ ,\ \eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $h\eta = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions $g\ ,\ h$ and η , particularly

$$F(x) = \int_a^x C\left|\frac{\eta'(u)}{\eta(u)\,h(u) - g(u)}\right|\exp(-s(u))\,du,$$

where the function s is a solution of the differential equation $s' = \frac{\eta' h}{\eta h - g}$ and C is a constant, chosen to make $\int_H dF = 1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence, in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions g_n, h_n and η_n $(n \in \mathbb{N})$ satisfy the conditions of Theorem 2.1.1 and let $g_n \to g\ ,\ h_n \to h$ for some continuously differentiable real functions g and h. Let, finally, X be a random variable with distribution F . Under the condition that $g_n(X)$ and $h_n(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if η_n converges to η , where

$$\eta(x) = \frac{E\left[g(X)\,|X \geq x\right]}{E\left[h(X)\,|X \geq x\right]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions $g\ ,\ h$ and η , respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if $\alpha \to \infty$, as was pointed out in [9] .

A further consequence of the stability property of Theorem 2.1.1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions g , h and, specially, η should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose η as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.

Remarks 2.1.2. (a) In *Theorem 2.1.1*, the interval H need not be closed. (b) The goal is to have the function η as simple as possible. For a more detailed discussion on the choice of η , we refer the reader to Glänzel and Hamedani [8] and Hamedani [10 − 14].

Proposition 2.1.3. . Let $X:\Omega\to\left(F^{-1}(0),F^{-1}(1)\right)$ be a continuous random variable and let $h(x)=[a\xi(x)+b]^{2-c}$ and $g(x)=[a\xi(x)+b]^{4-c}$ for $x\in\left(F^{-1}(0),F^{-1}(1)\right)$. The pdf of X is (1.2) if and only if the function η defined in *Theorem 2.1.1* has the form

$$\eta(x)=\frac{1}{2}\left\{1+[a\xi(x)+b]^2\right\},\qquad x\in\left(F^{-1}(0),F^{-1}(1)\right).$$

Proof. Let X have pdf (1.2), for $x\in\left(F^{-1}(0),F^{-1}(1)\right)$, then

$$(1-F(x))\,\mathbf{E}[h(X)|X\geq x]=\frac{c}{2}\left\{1-[a\xi(x)+b]^2\right\},$$

$$(1-F(x))\,\mathbf{E}[g(X)|X\geq x]=\frac{c}{4}\left\{1-[a\xi(x)+b]^4\right\}$$

and finally

$$\eta(x)h(x)-g(x)=\frac{1}{2}[a\xi(x)+b]^{2-c}\left\{1-[a\xi(x)+b]^2\right\}>0.$$

Conversely, if η is given as above, for $x\in\left(F^{-1}(0),F^{-1}(1)\right)$, then

$$s'(x)=\frac{\eta'(x)h(x)}{\eta(x)h(x)-g(x)}=\frac{2a\xi'(x)[a\xi(x)+b]}{1-[a\xi(x)+b]^2}$$

and hence

$$s(x)=-\ln\left\{1-[a\xi(x)+b]^2\right\}.$$

Now, in view of *Theorem 2.1.1*, X has cdf (1.1) and pdf (1.2) .

Corollary 2.1.4. Let $X:\Omega\rightarrow\left(F^{-1}(0),F^{-1}(1)\right)$ be a continuous random variable and let $h(x)$ be as in *Proposition 2.1.3* . The pdf of X is (1.2) if and only if there exist functions g and η defined in *Theorem 2.1.1* satisfying the differential equation

$$\frac{\eta'(x)\left[a\xi(x)+b\right]^{2-c}}{\eta(x)\left[a\xi(x)+b\right]^{2-c}-g(x)}=\frac{2a\xi'(x)\left[a\xi(x)+b\right]}{1-\left[a\xi(x)+b\right]^{2}},\quad x\in\left(F^{-1}(0),F^{-1}(1)\right).$$

Remarks 2.1.5. (c) The general solution of the differential equation in *Corollary 2.1.4* is

$$\eta(x)=\left\{1-\left[a\xi(x)+b\right]^{2}\right\}^{-1}\left[-\int 2a\xi'(x)\left[a\xi(x)+b\right]^{c-1}g(x)\,dx+D\right],$$

for $x\in\left(F^{-1}(0),F^{-1}(1)\right)$, where D is a constant. One set of appropriate functions is given in *Proposition 2.1.3* with $D=\frac{1}{2}$.

(d) Clearly there are other triplets of functions (h,g,η) satisfying the conditions of *Theorem 2.1.1.* We presented one such triplet in *Proposition 2.1.3*. Here are two more triplets of functions: $h(x)=1$, $g(x)=\left[a\xi(x)+b\right]^{c}$, $\eta(x)=\frac{1}{2}\left\{1+\left[a\xi(x)+b\right]^{c}\right\}$ and $h(x)=a\xi(x)+b$, $g(x)=\left[a\xi(x)+b\right]^{c+2}$, $\eta(x)=\frac{1}{2}\left\{1+\left[a\xi(x)+b\right]^{c+1}\right\}$. The function $\eta(x)$ in *Proposition 2.1.3*, however, is simpler and hence more desirable.

2.2. *Characterization based on truncated moment of certain functions of the n^{th} order statistic*

Let $X_{1:n}\leq X_{2:n}\leq\ldots\leq X_{n:n}$ be n order statistics from a continuous cdf F. We state here a characterization of (1.1) based on certain functions of the n^{th} order statistic. Our characterization of (1.1) will be a consequence of the following proposition, which is similar to the one appeared in our previous work. We give a brief proof of it here for the sake of completeness.

Proposition 2.2.1. Let $X:\Omega\rightarrow\left(F^{-1}(0),F^{-1}(1)\right)$ be a continuous random variable with cdf F . Let $\psi(x)$ and $q(x)$ be two differentiable functions on $(0,\infty)$ such that $\lim_{x\rightarrow F^{-1}(0)}\psi(x)\left[F(x)\right]^{n}=0$ and $\int_{F^{-1}(0)}^{F^{-1}(1)}\frac{q'(t)}{\left[\psi(t)-q(t)\right]}dt=\infty$. Then

$$E\left[\psi(X_{n:n})\,|X_{n:n}<t\right]=q(t),\quad t\in\left(F^{-1}(0),F^{-1}(1)\right),\tag{2.1}$$

implies

$$F(x) = \exp\left\{-\int_x^{F^{-1}(1)} \frac{q'(t)}{n[\psi(t)-q(t)]} dt\right\}, \quad x \geq F^{-1}(0). \tag{2.2}$$

Proof. If (2.1) holds, then using integration by parts on the left hand side of (2.1) and the assumption $\lim_{x \to F^{-1}(0)} \psi(x)[F(x)]^n = 0$, we have

$$\int_{F^{-1}(0)}^{t} \psi'(x)(F(x))^n dx = [\psi(t) - q(t)](F(t))^n.$$

Differentiating both sides of the above equation with respect to t, we arrive at

$$\frac{f(t)}{F(t)} = \frac{q'(t)}{n[\psi(t)-q(t)]}, \quad t > F^{-1}(0). \tag{2.3}$$

Now, integrating (2.3) from x to $F^{-1}(1)$, we have, in view of $\int_{F^{-1}(0)}^{F^{-1}(1)} \frac{q'(t)}{[\psi(t)-q(t)]} dt = \infty$, a cdf F given by (2.2).

Remark 2.2.2. (e) Taking, e.g. (for simplicity), $\psi(x) = [a\xi(x) + b]^{nc}$ with $\lim_{x \to F^{-1}(0)} \psi(x) = 0$, $\lim_{x \to F^{-1}(1)} \psi(x) = 1$ and $q(x) = \frac{1}{2}\psi(x)$ in *Proposition 2.2.1*, equation (2.3) will be

$$\frac{f(x)}{F(x)} = \frac{ac\xi'(x)[a\xi(x)+b]^{c-1}}{[a\xi(x)+b]^c}, \quad x > F^{-1}(0)$$

from which, we have $F(x) = [a\xi(x) + b]^c$.

2.3. *Characterization based on single truncated moment of certain functions of the random variable*

We like to point out that *Proposition 2.2.1* holds true (with of course appropriate modification) if we replace $X_{n:n}$ with the base random variable X. In this subsection we employ a single function ψ of X and characterize the distribution of X in terms of the truncated moment of $\psi(X)$. The following propositions have already appeared in our previous work, so we will just state them here and then use them to characterize cdf (1.1).

Proposition 2.3.1. Let $X : \Omega \to \left(F^{-1}(0), F^{-1}(1)\right)$ be a continuous random variable with cdf F. Let $\psi(x)$ be a differentiable function on $\left(F^{-1}(0), F^{-1}(1)\right)$ with $\lim_{x \to F^{-1}(0)} \psi(x) = 1$. Then for $\delta \neq 1$,

$$E\left[\psi(X)\,|X>x\right]=\delta\,\psi(x),\quad x\in\left(F^{-1}(0),F^{-1}(1)\right), \tag{2.4}$$

if and only if

$$\psi(x)=(1-F(x))^{\frac{1}{\delta}-1},\quad x\in\left(F^{-1}(0),F^{-1}(1)\right). \tag{2.5}$$

Proposition 2.3.2. Let $X:\Omega\to\left(F^{-1}(0),F^{-1}(1)\right)$ be a continuous random variable with cdf F. Let $\psi_1(x)$ be a differentiable function on $\left(F^{-1}(0),F^{-1}(1)\right)$ with $\lim_{x\to F^{-1}(1)}\psi_1(x)=1$. Then for $\delta_1\neq 1$,

$$E\left[\psi_1(X)\,|X<x\right]=\delta_1\,\psi_1(x),\quad x\in\left(F^{-1}(0),F^{-1}(1)\right), \tag{2.6}$$

if and only if

$$\psi_1(x)=(F(x))^{\frac{1}{\delta_1}-1},\quad x\in\left(F^{-1}(0),F^{-1}(1)\right). \tag{2.7}$$

Remarks 2.3.3. (g) Taking $\psi(x)=\{1-[a\xi(x)+b]^c\}^{\frac{1-\delta}{\delta}}$, we obtain from (8) the cdf (1.1). (h) Taking $\psi_1(x)=[a\xi(x)+b]^{\frac{c(1-\delta_1)}{\delta_1}}$, we obtain from (2.7) the cdf (1.1).

3. Dual generalized order statistic

Kamps [16] introduced the concept of generalized order statistics as an unified approach to a variety of models of random variables arranged in ascending order of magnitude. However, as highlighted by Burkschat et al. [3] , random variables that are decreasingly ordered cannot be integrated into this framework. For this reason, Burkschat et al. [3] introduced so-called dual generalized order statistics as a systematic approach to models of decreasingly ordered random variables. Specifically, let $X_1,X_2,...,X_n$ be a random sample of size n from an absolutely continuous cdf $F(x)$ and pdf $f(x)$. Let $n\in\mathbb{N}$, $k\geq 1$ and $m\in\mathbb{R}$ be parameters such that $\gamma_r=k+(n-r)(m+1)\geq 1$ for $1\leq r\leq n$. The random variables $\widetilde{X}_{1,n,m,k}$, $\widetilde{X}_{2,n,m,k},...,\widetilde{X}_{n,n,m,k}$ are said to be dual generalized order statistics $(dgos)$ from $F(x)$ if their joint pdf is given by

$$\begin{aligned}\widetilde{f}_{1,2,...,n,n,m,k}(x_1,x_2,...,x_n) &= \left(\Pi_{j=1}^{n}\gamma_j\right)\left(\Pi_{j=1}^{n-1}[F(x_j)]^{\gamma_j-\gamma_{j+1}-1}f(x_j)\right)\\ &\quad\times[F(x_n)]^{k-1}f(x_n),\end{aligned}$$

for $F^{-1}(1)\geq x_1\geq x_2\geq...\geq x_n\geq F^{-1}(0)$ (see, [3] or [1]).

The marginal pdf of the r^{th} $dgos$, $\widetilde{X}_{r,n,m,k}$, is

$$\widetilde{f}_{r,n,m,k}(x)=\frac{C_{r-1}}{(r-1)!}\left[F(x)\right]^{\gamma_r-1}\left(g_m\left(F(x)\right)\right)^{r-1}f(x), \tag{3.1}$$

where $C_{r-1}=\Pi_{j=1}^{r}\gamma_j$ and

$$\begin{cases} g_m(F(x))=\frac{1}{m+1}\left[1-F(x)^{m+1}\right] & for \quad m\neq -1 \\ g_m(F(x))=-\ln\left[F(x)\right] & for \quad m=-1, F(x)\in[0,1). \end{cases}$$

It is worthwhile pointing out that for $m=0$ and $k=1$ the $dgos$ $\widetilde{X}_{r,n,0,1}$ reduce to the order statistics $X_{n-r+1:n}$, while for $m=-1$ and $k=1$, $\widetilde{X}_{r,n,-1,1}$ becomes the r^{th} lower record value. Consequently, results for order statistics and lower record values can be deduced as special cases from results obtained in this *section* for $dgos$.

3.1. *Recurrence Relation for Moments*

In this subsection, we derive recurrence relations for moments of dual generalized order statistics from distribution function defined in (1.1). First, we observe that, for $m\neq -1$, from (3.1) and (1.2), we can write

$$\begin{aligned} \tilde{f}_{r,n,m,k}(x) &= \frac{acC_{r-1}}{(r-1)!(m+1)^{r-1}}\xi'(x)\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^j\left[a\xi(x)+b\right]^{c(\gamma_r+j(m+1))-1} = \\ &= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{j=0}^{r-1}\frac{\binom{r-1}{j}(-1)^j}{(\gamma_r+j(m+1))}f(x;a,b,c_j) \end{aligned} \tag{3.2}$$

where $f(x;a,b,c_j)$, with $c_j=c\left(\gamma_r+j(m+1)\right)$ for $j=0,1,...,r-1$, are probability density functions defined in (1.2). Consequently, using (3.2), the $\ell-th$ moment of $dgos$ is

$$E\left[\tilde{X}^{\ell}_{r,n,m,k}\right]=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{j=0}^{r-1}\frac{\binom{r-1}{j}(-1)^j}{(\gamma_r+j(m+1))}E\left[X^{\ell}|a,b,c_j\right] \tag{3.3}$$

where $E(.|a,b,c_j)$ is the expectation of pdf defined in (1.2) with parameters a, b and c_j.
In order to determine the recurrence relations for moments of $dgos$ from distribution function

defined in (1.1), we use the following general result

$$E\left[\tilde{X}^{\ell}_{r,n,m,k}\right]-E\left[\tilde{X}^{\ell}_{r-1,n,m,k}\right]=-\frac{\ell C_{r-2}}{(r-1)!}\int_{\Re}x^{\ell-1}\left[F(x)\right]^{\gamma_r}\left[g_m\left(F(x)\right)\right]^{r-1}dx \tag{3.4}$$

(see, Bieniek and Szynal, [2]). Using (1.3), after simple algebra, we can write

$$\begin{aligned}I&=\int_{\Re}x^{\ell-1}\left[F(x)\right]^{\gamma_r}\left[g_m\left(F(x)\right)\right]^{r-1}dx\\&=\frac{(r-1)!}{acC_{r-1}}\int_{\Re}\frac{x^{\ell-1}\left[a\xi(x)+b\right]}{\xi'(x)}\frac{C_{r-1}}{(r-1)!}\left[F(x)\right]^{\gamma_r-1}\left[g_m\left(F(x)\right)\right]^{r-1}f(x)dx\\&=\frac{(r-1)!}{acC_{r-1}}E\left\{\frac{\tilde{X}^{\ell-1}_{r,n,m,k}\left[a\xi\left(\tilde{X}_{r,n,m,k}\right)+b\right]}{\xi'\left(\tilde{X}_{r,n,m,k}\right)}\right\}\end{aligned} \tag{3.5}$$

and, therefore the recurrence relation is

$$E\left[\tilde{X}^{\ell}_{r,n,m,k}\right]-E\left[\tilde{X}^{\ell}_{r-1,n,m,k}\right]=-\frac{\ell}{ac\gamma_r}E\left\{\frac{\tilde{X}^{\ell-1}_{r,n,m,k}\left[a\xi\left(\tilde{X}_{r,n,m,k}\right)+b\right]}{\xi'\left(\tilde{X}_{r,n,m,k}\right)}\right\} \tag{3.6}$$

In particular, by using the binomial expansion, for $m\neq-1$, we can write (3.5) as

$$\begin{aligned}I&=\frac{1}{(m+1)^{r-1}}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^j\int_{\Re}x^{\ell-1}\left[a\xi(x)+b\right]^{c[\gamma_r+j(m+1)]}dx\\&=\frac{1}{(m+1)^{r-1}}\sum_{j=0}^{r-1}\frac{\binom{r-1}{j}(-1)^j}{a\left\{c_j+1\right\}}E\left\{\frac{X^{\ell-1}}{\xi'(X)}|a,b,c_j+1\right\}.\end{aligned} \tag{3.7}$$

Moreover, in the case of lower record values, that is for $m=-1$ and $k=1$, given that $\gamma_r=1$, we have

$$I=\frac{1}{a(c+1)}E\left\{\frac{X^{\ell-1}}{\xi'(X)}\left[-\ln F(X;a,b,c)\right]|a,b,c+1\right\}. \tag{3.8}$$

Special cases. It seems useful to report the recurrence relations between the moments of *lgos* for some distribution functions belonging to the class of distributions defined by (1.1).

- **Power distribution**. Setting $a=x_0^{-1}$, $\xi(x)=x$, $c=p$ and $b=0$ in (1.1), we obtain the Power distribution function $F(x)=x_0^{-p}x^p$, for $0\leq x\leq x_0$. It is simple to verify that, for $m\neq-1$ we have

$$I=\frac{1}{(m+1)^{r-1}}\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{(-1)^j x_0^{\ell}}{\{p[\gamma_r+j(m+1)]+\ell\}}.$$

  Consequently, after simple algebra, we get

$$E\left[\tilde{X}^{\ell}_{r,n,m,k}\right]-E\left[\tilde{X}^{\ell}_{r-1,n,m,k}\right]=\frac{-\ell x_0^{\ell}C_{r-2}}{p(r-1)!(m+1)^r}B\left(\frac{p\gamma_r+\ell}{p(m+1)},r\right)$$

  where $B(\varepsilon,\kappa)=\int_0^1 w^{\varepsilon-1}(1-w)^{\kappa-1}dw$ is the Beta function. Hence, since $E\left[\tilde{X}^{\ell}_{1,n,m,k}\right]=\frac{p\gamma_1 x_0^{\ell}}{\ell+p\gamma_1}$, we can determine the moments $E\left[\tilde{X}^{\ell}_{r,n,m,k}\right]$, for $r=2,3,...,n$.

For $m = -1$ and $k = 1$, after simple algebra, we get

$$I = \frac{p^{r-1} x_0^{\ell}}{(\ell + p)^r} \Gamma(r)$$

where $\Gamma(\varepsilon) = \int_0^\infty w^{\varepsilon-1} e^{-w} dw$ is the Gamma function. Observed that, $E\left[\tilde{X}^{\ell}_{1,n,-1,1}\right] = \frac{p x_0^p}{p+\ell}$, we can determine the moments $E\left[\tilde{X}^{\ell}_{r,n,-1,1}\right]$, for $r = 2, 3, ..., n$.

- **Dagum distribution**. Setting $a = \lambda$, $\xi(x) = x^{-\delta}$, $c = -\beta$ and $b = 1$ in (1.1), we obtain the Dagum distribution (Burr III) $F(x) = (1 + \lambda x^{-\delta})^{-\beta}$, for $x > 0$. For this distribution, for $\ell < 0$, we have

$$I = \frac{\lambda^{\frac{\ell}{\delta}}}{\delta(m+1)^{r-1}} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j B\left(\beta[\gamma_r + j(m+1)] + \frac{\ell}{\delta}, -\frac{\ell}{\delta}\right).$$

- **Pareto distribution**. Setting $a = -x_0^{\beta}$, $\xi(x) = x^{-\beta}$, $c = 1$ and $b = 1$ in (1.1), we get the Pareto distribution $F(x) = 1 - x_0^{\beta} x^{-\beta}$ for $x_0 \le x$. For $\ell < 0$, we have

$$I = \frac{x_0^{\ell}}{\beta(m+1)^{r-1}} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j B\left(-\frac{\ell}{\beta}, [\gamma_r + j(m+1)] + 1\right).$$

- **Inverse Weibull distribution**. Setting $a = 1$, $\xi(x) = exp(-x^{-p})$, $c = \theta^p$ and $b = 0$, we obtain the Inverse Weibull distribution $F(x) = exp(-\theta^p x^{-p})$. For $\ell < 0$, we have

$$I = \frac{\theta^{\ell} \Gamma\left(\frac{-\ell}{p}\right)}{p(m+1)^{r-1}} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j [\gamma_r + j(m+1)]^{\frac{\ell}{p}}.$$

3.2. *Characterization based on moments of dgos*

In this subsection, we present a characterization of (1.1) based on the application of Hwang and Lin [15] result related to the sequence of functions complete.

Proposition 3.2.1. Let X be a random variable having an absolutely continuous cdf $F(x)$ with $0 < F(x) < 1$ for all x. Then

$$E\left[\frac{\widetilde{X}^{l-1}_{r,n,m,k}\left[a\xi\left(\widetilde{X}_{r,n,m,k}\right) + b\right]}{\xi'\left(\widetilde{X}_{r,n,m,k}\right)}\right] = \frac{ac\gamma_r}{l}\left\{E\left[\widetilde{X}^{l}_{r-1,n,m,k}\right] - E\left[\widetilde{X}^{l}_{r,n,m,k}\right]\right\}, \tag{3.9}$$

if and only if $F(x)$ is given by (1.1) .

Proof. Observed that $\frac{C_{r-1}}{C_{r-2}} = \gamma_r$, by (3.4) and (3.5), we obtain (3.9). On the other hand, given that $\gamma_{r-1} - 1 = \gamma_r + m$ and $\gamma_r C_{r-2} = C_{r-1}$, we can write

$$E\left[\tilde{X}^{\ell}_{r-1,n,m,k}\right] - E\left[\tilde{X}^{\ell}_{r,n,m,k}\right] = \frac{C_{r-2}(r-1)\gamma_r}{(r-2)!(r-1)\gamma_r}\int_{\Re} x^{\ell}[F(x)]^{\gamma_r+m}[g_m(F(x))]^{r-2} f(x)dx +$$
$$- \frac{C_{r-1}}{(r-1)!}\int_{\Re} x^{\ell}[F(x)]^{\gamma_r-1}[g_m(F(x))]^{r-1} f(x)dx =$$
$$= \frac{C_{r-1}}{(r-1)!\gamma_r}\int_{\Re} x^{\ell}[F(x)]^{\gamma_r}[g_m(F(x))]^{r-2} f(x)$$
$$\left\{(r-1)[F(x)]^m - \frac{\gamma_r g_m(F(x))}{F(x)}\right\}dx. \quad (3.10)$$

Let

$$h(x) = -[F(x)]^{\gamma_r}[g_m(F(x))]^{r-1},$$

for $m \neq -1$, it is simple to verify that

$$h'(x) = -\left\{\gamma_r[F(x)]^{\gamma_r-1} f(x)[g_m(F(x))]^{r-1} - (r-1)[F(x)]^{\gamma_r+m}[g_m(F(x))]^{r-2} f(x)\right\}$$
$$= [F(x)]^{\gamma_r}[g_m(F(x))]^{r-2} f(x)\left\{(r-1)[F(x)]^m - \frac{\gamma_r g_m(F(x))}{F(x)}\right\}$$

Consequently, we have

$$E\left[\tilde{X}^{\ell}_{r-1,n,m,k}\right] - E\left[\tilde{X}^{\ell}_{r,n,m,k}\right] = \frac{C_{r-1}}{(r-1)!\gamma_r}\int_{\Re} x^{\ell} h'(x)dx,$$

and integrating by parts, we obtain

$$E\left[\tilde{X}^{\ell}_{r-1,n,m,k}\right] - E\left[\tilde{X}^{\ell}_{r,n,m,k}\right] = \frac{\ell C_{r-1}}{(r-1)!\gamma_r}\int_{\Re} x^{\ell-1}[F(x)]^{\gamma_r}[g_m(F(x))]^{r-1} dx. \quad (3.11)$$

Substituting (3.11) in (3.9), we get

$$\int_{\Re} x^{\ell-1}\frac{[a\xi(x)+b]}{\xi'(x)}\frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r-1}[g_m(F(x))]^{r-1} f(x)dx =$$
$$\frac{acC_{r-1}}{(r-1)!}\int_{\Re} x^{\ell-1}[F(x)]^{\gamma_r}[g_m(F(x))]^{r-1} dx$$

which reduces to

$$\frac{C_{r-1}}{(r-1)!}\int_{\Re} x^{\ell-1}[F(x)]^{\gamma_r-1}[g_m(F(x))]^{r-1}\left\{\frac{[a\xi(x)+b]}{\xi'(x)} f(x) - acF(x)\right\}dx = 0. \quad (3.12)$$

Now, applying a generalization of the Muntz-Szasz theorem (Hwang and Lin, [15]) to equation (3.12), we get

$$\frac{[a\xi(x)+b]}{\xi'(x)} f(x) = acF(x)$$

that is

$$F(x) = \frac{[a\xi(x)+b]}{ac\xi'(x)} f(x).$$

This complete the proof of the proposition.

Remark 3.2.2. By suitable choice of a, b, c and $\xi(x)$, *Proposition 3.2.1* characterizes different important distribution functions such as Power, Pareto, Dagum, Inverse Weibul ect..

Remark 3.2.3. The *Proposition 3.2.1* is a generalization of the characterizations based on moments of *lgos* of the exponentiated Pareto distribution, exponentiated Gamma distribution and generalized exponential distribution obtained by Khan and Kumar [17], [18] and [19], respectively.

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