# The Derivation of Euler's Equations of Motion in Cylindrical Vector Components To Aid in Analyzing Single Axis Rotation

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# THE DERIVATION OF EULER'S EQUATIONS OF MOTION IN CYLINDRICAL VECTOR COMPONENTS TO AID IN ANALYZING SINGLE AXIS ROTATION

by

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# ABSTRACT

# THE DERIVATION OF EULER'S EQUATIONS OF MOTION IN CYLINDRICAL VECTOR COMPONENTS TO AID IN ANALYZING SINGLE AXIS ROTATION

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The derivation of Euler's equations of motion in using cylindrical vector components is beneficial in more intuitively describing the parameters relating to the balance of rotating machinery. Using the well established equation for Newton's equations in moment form and changing the position and angular velocity vectors to cylindrical vector components results in a set of equations defined in radius-theta space rather than X-Y space. This easily allows for the graphical representation of the intuitive design parameters effect on the resulting balance force that can be used to examine the robustness of a design. The sensitivity of these parameters and their influence on the dynamic balance of the machine can then be quantified and minimized by adjusting the parameters in the design. This gives a theoretical design advantage to machinery that requires high levels of precision such as a Computed Tomography (CT) scanner.

# DEDICATION

To my parents, brother, and extended family for their consistent support and encouragement and with forever love to my fiancè Cristina.

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## Introduction

In dynamics, there are many different types of problems which each have their own intricacies, uniqueness, and differences. Many times, each type of problem has its own elegant way to define, set up, solve and analyze the results of that problem. These differences has spawned over the years many different approaches to dynamics with their own notation, rules, and usefulness. Like any tool in a carpenter's toolbox, each of these approaches have a certain application for which they were designed and for which they excel at. As a carpenter would not use a handsaw when a router would be appropriate, neither should a dynamist use a certain notation when another is more appropriate. In either case that tool could get to the desired result, but it might not be as elegant nor may it result in a useful or understandable solution.

A few specific examples of these can start out in even the most obvious of senses such as the difference between classical (e.g., Newtonian) dynamics and analytical (e.g., Lagrangian) dynamics. These two approaches have their own benefits depending upon what is important to know in the solution and analysis of the problem. If an expedient path to the equations of motion are the desired output, then the Lagrangian method may be the best approach to the problem [13]. However, if there is a desire to know the constraint forces between bodies in the system (possibly in the analysis of the strength needed for a joint in a robotic system) then the Newtonian approach may be the more appropriate choice of procedure.

All of these notations and tools are suited for a certain application and can all provide the same answer when applied correctly. Most of these require a substantial knowledge of complicated notation, formulas and application rules that most engineers or even dynamists may not have the knowledge that they exist or how and when it is appropriate to apply them. Many only use the standard Cartesian coordinate system and a standard formulation of Euler's equations of motion to set up and solve a particular problem. Many times it is not convenient to use a Cartesian coordinate system, such as in the case of a body or bodies rotating about a single axis, and it is much more convenient and intuitive to apply a cylindrical coordinate system to the problem. The issue with this approach is that Euler's equations of motion are defined in Cartesian coordinates and any system defined in a cylindrical coordinate system needs to be converted before it can be analyzed using Euler's equations. The conversion often times adds an unnecessary disconnect between the important parameters in an analysis and the results that are desired.

The goal of this thesis is to create another tool for an engineer or dynamist to use to solve a very common yet specific problem: to analyze the dynamic balance of a multi-body single axis rotational system. This type of problem is central to many different applications and the analysis and study of dynamic balance of a multi-body single axis rotational system can offer a great deal of benefit. Current methods and notation used in analyzing single axis rotation problems seem to provide a very general method and notation which is able to be applied to a wide variety of situations and scenarios. The issue with applying those methods to a single axis rotation problem is that it can overly complicate the problem and cloud the simplicity and intuitiveness of the answer and analysis. A new tool, designed specifically for solving the dynamic balance of a multi-body single axis rotational system, created to simplify the notation and also give a route to an elegant solution and better understanding is going to be derived and explained in the subsequent chapters.

## 1.1 Literature Review

One main area where there is a wide variety of notation and tools is in the definition of a rotation in classic Cartesian space. Many specific formulations have been developed to create an easy way to notate and calculate a rotation in space.

One of the most famous and earliest of these is the use of Euler angles [4]. Euler angles represent any rotation in space by decomposing it into a composition of three elemental rotations (precession, nutation, and intrinsic rotation or spin) starting from a known standard orientation. The classic definition is a precession rotation ($\alpha$) from and XYZ coordinate system about the Z axis to a temporary orientation denoted as $(X_1, Y_1, Z_1)$, then a nutation rotation ($\beta$) about the new X axis (commonly called the N axis) to a secondary position $(X_2, Y_2, Z_2)$, and finally an intrinsic rotation or spin ($\gamma$) about the Z axis again to a final $(X_3, Y_3, Z_3)$ orientation. This sequence can provide a means to rotate a Cartesian coordinate system to any other possible orientation without translation.

Along the same lines of the Euler angles is the rotation matrix approach [4]. This uses the same type of reasoning as the Euler angle approach by using a succession of three rotations to get change from one orientation to another by use of matrix multiplication. The matrices are generated by using simple rotations about a single axis and three predetermined arrays of trigonometric functions. These three matrices can be combined into a single matrix which is used to multiply the unit vectors to create the new orientation of the coordinate system.

The Euler angle approach was further generalized to show that any rotation in space can be expressed by a single rotation about some axis. This axis is called the axis angle and greatly simplifies the conversion from one orientation of a coordinate system to another. The main issue with this is that the combination of two successive Euler axis angles is not straightforward and can be shown to not satisfy the law of vector addition [7]. This angle-axis rotation equation is also called the Euler-Lexell-Rodrigues formula in certain texts and presented as one of a few equations which use the rotation angle and rotation axis as inputs to calculate the axis-angle rotation matrix [8].

Many of these approaches to rotate a coordinate system deal with complex matrices that are filled with a variety of sine and cosine terms which sometimes can lead to some rounding errors, singularities or discontinuities. Another approach was developed to alleviate some of those issues and are called quaternions. A quaternion provides the same function as a rotation matrix but is a more compact representation and does not require the use of trigonometric functions in the matrices. With quaternions, the round off error is generally less due to not using trigonometric functions and also avoids discontinuous jumps and singularities [8]. Because of this, they have become a popular way to calculate rotations in higher complex systems. These highly complex systems make use of a computer algorithm which perform a high number of calculations which is where these issues can occur.

To treat special cases where a rotation is about only a single axis, a formulation based on vectors was developed by Olinde Rodrigues and is called the Rodrigues' rotation formula. It is a specialized version of the angle-axis rotation equation and uses a projection and cross product of a vector and a unit vector about which a rotation occurs to quickly calculate the new vector's position [8]. This calculation is a very convenient and simple way to calculate a vector's new position if the rotational axis' position is known with respect to the vector being rotated. It also does not require the use of a rotation matrix and makes the math a little simpler.

The notations that are listed in this introduction are great and robust ways of representing and calculating any number of complicated rotations. However, they are too bulky and often confusing to simply apply to a simple problem and handle single axis rotations in an elegant manner. The goal of this thesis is to re-evaluate Euler's equations of motion with a new notion that will create a direct link between single axis rotation and the analysis of Euler's equations of motion.

A major catalyst for this type of effort is from a real example of a problem using single axis rotation in which it was difficult to access how the parameters directly affect the balance of the system. In a thesis written by Lindsay Rogers, the balance of a computed tomography (CT) scanner was analyzed using Euler's equations of motion [15]. A major conclusion from this thesis is that it would be greatly beneficial if Euler's equations of motion were defined in pure cylindrical coordinates. It was discovered that a cylindrical coordinate system in its purest sense was not ideal for use in the derivation of Euler's equations of motion. However, by revamping the notation in the derivation with cylindrical vector components, the main design parameters radius to a part, $R$, and angle relative to a datum, $\theta$, can be present throughout the analysis, from set up to solution and beyond.

The concept of analysis a systems's dynamic balance using cylindrical coordinates is not a new or novel idea. A literature search was performed to try to find some research that was related to the problem that Rogers was trying to solve. The goals of this search was to find applications that analyzed dynamic balance using a similar approach that was focused on the resultant balance of the system. Finding the ways that dynamic balance has been approached in the past would give a good indication on where to focus the efforts in developing a new way at looking at dynamic balance using Euler's equations using cylindrical vector components.

There is an abundance of research relating to aerospace applications concerning the dynamics of rotating artificial satellites. This research focuses on how the inertia of these satellites affect their motion and how they can be controlled [10] [11]. It has a great value for the study of angular momentum in a three dimensional and free to translate and rotate environment that an artificial satellite would be in, but they do not translate well into rotating machinery on earth. This research does not deal with bearing forces that are required for rotating machinery on earth and also can generally assume that inertias and designs are "perfect" because they are empirically calculated. These factors make it difficult to apply the work done in the aerospace field to problems of dynamic balance analysis for earthy rotating machinery.

Another area of related study is in the analysis and design of automatic balancers for rotating machinery. Automatic dynamic balancers, ADB, are generally designed as a circular bearing race constraining two masses which can rotate about the center of mass while submerged in a damping fluid. ADB's work very well in rotating systems that can have a variable loading response and need flexibility in the design with respect to dynamic balance. Due to the nature of these systems requiring the analysis of every possible loading case at once, most of the research regarding the design and use of these systems has been based in the Lagrangian method [6] [2] or using vibration analysis [14] [5] rather than the Newton-Euler method. Because of the freedom of choice of the generalized coordinates, the natural use of the radius and angles to describe the motion of the balancing masses is employed in these models and is very beneficial to the analysis.

High speed spindle design was also seen to be focused on a Lagrangian approach to the system's governing equations [12] [1]. These systems are intuitively set up to be used with coordinates employing radius and angle components and the Lagrangian method is extremely attractive when using these coordinates. The goal of this thesis is to follow the same principle of using the natural radius and angle coordinates in a rotating machinery dynamic balance application while also being able to easily apply the Newton-Euler method.

## 1.2 Organization of Thesis

The creation of the link between the design parameters and the outcome of the dynamic balance analysis is accomplished by using cylindrical vector components to relate one coordinate system with another with the assumption that they would only different by one rotation about one axis. It is performed by using a method that is similar to the Rodrigues' rotation formula by using vector projections but in a much simpler and intuitive form. The next chapter (Chapter 2) will directly integrate the new notation in a rederivation of Euler equations of motion to provide a direct and simple link from the problem's definition through it's solution and subsequent analysis.

After the derivation in Chapter 3, a few key differences and benefits will be enumerated such as the differences in the calculation of the inertia matrix as well as an alternative form of the parallel axis theorem will be discussed. In Chapter 4, a couple of examples of the benefit of a consistent problem definition and analysis will be presented with a look into how this new formulation could be useful in the study of dynamic balance. A simple example of the full use of the Cylindrical Vector Component (CVC) approach will then be presented with a discussion of its benefits in Chapter 5. This is followed by a more detailed comparison of this new notation with the notation listed above and where this specific notation could be especially useful in Chapter 6. Finally, a look at some perspective avenue of future work on this topic is discussed in the concluding Chapter 7.

# CHAPTER 2

## Derivation of Euler's Equations Using Cylindrical Vector Components

The biggest problem that was discovered when looking at the different notations used in most rotational problems is that they generally perform the rotation transformations, and then apply Euler's equations which are set up for standard Cartesian coordinates. With the rotation aspect more directly integrated into Euler's equation the link between the problem setup and the analysis or outcome of the problem are more directly linked. To create this link, the new notation development will need to start with a rederivation of Euler's equation.

### 2.1 Angular Momentum Defined Using Vector Components

The derivation of Euler's equations using cylindrical vector components will follow the same structure as the standard X-Y derivation as described in [3]. The starting point is the base equation for angular momentum of several particles which is defined as

$$\vec{H}_A = \sum_{j=1}^{N} m_j[\vec{r}_{j/A} \times (\vec{\omega} \times \vec{r}_{j/A})]. \tag{2.1}$$

The angular momentum, $\vec{H}_A$, of a collection of particles at point A is the summation of the mass ($m_j$) of particle $j$ times the cross product of particle $j$'s position vector from point A to the particle ($\vec{r}_{j/A}$) and the particle's velocity which is written as the cross product between its position and the angular velocity of the body ($\vec{\omega}_j \times \vec{r}_{j/A}$) over the total particles $N$. It is critical to remember that this equation is only valid under a number of special restrictions, namely, that the body is rigid, and that point A fixed relative to the body.[3]

In Fig. 2.1, a rigid body in fixed the global coordinate system of indiscriminate shape is broken down into a series of differential masses. A reference coordinate system

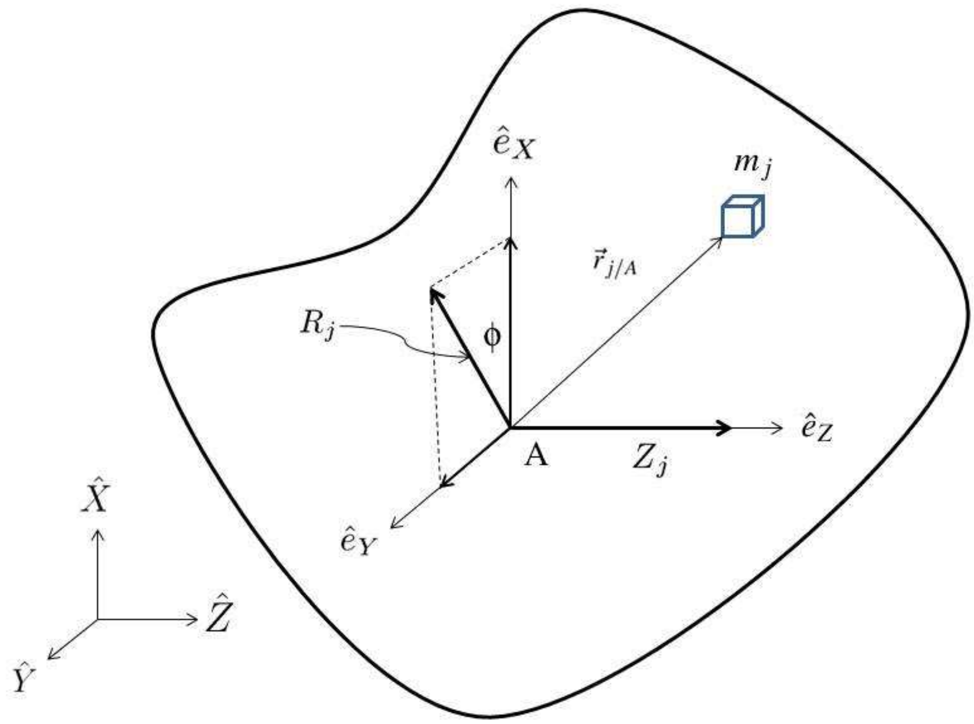


Figure 2.1: Differential element of mass $m_j$ relative to a body-fixed $XYZ$ reference frame

is attached to the body and each differential mass has a unique position vector defining its position in the body. These vectors have classically been expressed in standard Cartesian components ($XYZ$) and subsequently the results of the derivation would be formatted in these standard Cartesian components. However, these vectors are free to be expressed in any acceptable manner. This opens a possibility of deriving Euler's equations using a more convenient notation for single axis rotating systems. For the purposes of this thesis, the position vector ($\vec{r}_{j/A}$) is presented as a combination of the components of the position vector that are projected onto the XY plane and the relative angle from the X axis and the component of the position vector along the Z axis. Specifically,

$$\vec{r}_{j/A} = R_j \cos\phi_j \hat{e}_X + R_j \sin\phi_j \hat{e}_Y + Z_j \hat{e}_Z \tag{2.2}$$

$$\vec{\omega} = \omega_X \hat{e}_X + \omega_Y \hat{e}_Y + \omega_Z \hat{e}_Z \tag{2.3}$$

It is important to note, for purposes of clarity, that the angle shown, $\phi$, is measured from the $\hat{e}_X$ vector to the $\hat{e}_Y$ vector and is in the $XY$ plane. $R_j$ in Fig. 2.1 is the projection of the position vector to the differential mass $m_j$ onto the XY plane. It is then broken up into components along the $\hat{e}_X$ and $\hat{e}_Y$ axis using the angle $\phi$. $Z_j$ is the position of the particle along the $\hat{e}_Z$ axis.

This notation is a hybrid approach between a cylindrical coordinate system and a Cartesian coordinate system. The main issue with trying to define the angular momentum of a rigid body with a purely cylindrical coordinate system is that the coordinate directions are not fixed in space relative to each other. This causes a problem with trying to define an inertia in cylindrical coordinates. It is unclear how the inertias of each differential mass can be combined with each other since the coordinate axes have a different orientations for each individual differential mass.

The difficulty in many dynamic problems is relating two Cartesian coordinate systems that are rotated. By introducing the cylindrical vector components into the derivation, an easy way to relate coordinate systems to each other around an axis is created. As long as all of the coordinate systems are referenced back to an main fixed coordinate system using the angle $\phi$, the rigidness of the Cartesian system is mitigated while keeping the constraint of having a body fixed coordinate system for the inertia is satisfied. In essence, all of the bodies in the system have their own coordinate systems which rotate with them yet when the kinematics are derived for the bodies, they will be put in terms of a single chosen coordinate system. The coordinate systems are easily communicated by the fixed angle that exists between each coordinate system.

Substituting in the $\vec{\omega}$ and $\vec{r}_{j/A}$ from Eq. 2.2 and Eq. 2.3 into the inner cross product for $\vec{H}_A$ (Eq. 2.1) results in

$$\vec{\omega} \times \vec{r}_{j/A} = [\omega_X \hat{e_X} + \omega_Y \hat{e_Y} + \omega_Z \hat{e_Z}] \times [R_j \cos\phi_j \hat{e_X} + R_j \sin\phi_j \hat{e_Y} + Z_j \hat{e_Z}] \tag{2.4}$$

Performing the cross product of $\vec{\omega}$ and $\vec{r}_{j/A}$ is straightforward. This is followed by a rearranging to group each component of the vector by unit vector.

$$\begin{aligned} \vec{\omega} \times \vec{r}_{j/A} \;&= \\ & [\omega_Y Z_j - \omega_Z R_j \sin\phi_j]\hat{e_X} + [\omega_Z R_j \cos\phi_j - \omega_X Z_j]\hat{e_Y} \\ &+ \; [\omega_X R_j \sin\phi_j - \omega_Y R_j \cos\phi_j]\hat{e_Z} \end{aligned} \tag{2.5}$$

The next step is to cross $\vec{r}_{j/A}$ into the equation above as shown below.

$$\begin{aligned} \vec{r}_{j/A} \times [\vec{\omega} \times \vec{r}_{j/A}] \;&= \\ & [R_j \cos\phi_j \hat{e_X} + R_j \sin\phi_j \hat{e_Y} + Z_j \hat{e_Z}] \times \\ & [\omega_Y Z_j - \omega_Z R_j \sin\phi_j \hat{e_X} + \omega_Z R_j \cos\phi_j - \omega_X Z_j \hat{e_Y} \\ &+ \; \{\omega_X R_j \sin\phi_j - \omega_Y R_j \cos\phi_j\}\hat{e_Z}] \end{aligned} \tag{2.6}$$

The cross product $\vec{r}_{j/A}$ and $\vec{\omega} \times \vec{r}_{j/A}$, after rearranging into different components and then factoring out the angular velocities, results in the following.

$$\begin{aligned} \vec{r}_{j/A} \times [\vec{\omega} \times \vec{r}_{j/A}] \;&= \\ & [\omega_X (R_j^2 \sin^2\phi_j + Z_j^2) - \omega_Y (R_j^2 \cos\phi_j \sin\phi_j) - \omega_Z (R_j Z_j \cos\phi_j)]\hat{e_X} \\ &+ \; [\omega_Y (Z_j^2 + R_j^2 \cos^2\phi_j) - \omega_X (R_j^2 \cos\phi_j \sin\phi_j) - \omega_Z (R_j Z_j \sin\phi_j)]\hat{e_Y} \\ &+ \; [\omega_Z (R_j^2) - \omega_X (R_j Z_j \cos\phi_j) - \omega_Y (R_j Z_j \sin\phi_j)]\hat{e_Z} \end{aligned} \tag{2.7}$$

Using the assumption that the masses $m_j$ are infinitesimally small, the summation of these changes to an integral over the differential mass $dm$. Also, since the angular velocity is not dependent on the location of the different "masses" and is a property of the entire body, the angular velocity components can be taken out of the integrals. This leaves a set of nine separate (six unique) triple integrals (shown below) of body geometries over the differential mass. These integrals, known as moments and products of inertia, result in the same value as in the standard Cartesian coordinates only now are defined using angular vector component notation. This is important because the inertias that are listed in tables and in CAD programs are still valid when using this approach. It would have been a huge hinderance to reformulating Euler's equations if all of the inertias had to be recalculated. In contrast, these actually give an alternate, yet equivalent, approach in calculating moments of inertia that may become useful for difficult curvilinear shapes. Specifically they are

$$I_{XX} = \int\int\int (R^2 \sin^2\phi + Z^2)dm \tag{2.8}$$

$$I_{YY} = \int\int\int (Z^2 + R^2 \cos^2\phi)dm \tag{2.9}$$

$$I_{ZZ} = \int\int\int (R^2)dm \tag{2.10}$$

$$I_{XY} = I_{YX} = \int\int\int (R^2 \cos\phi \sin\phi)dm \tag{2.11}$$

$$I_{XZ} = I_{ZX} = \int\int\int (RZ \cos\phi)dm \tag{2.12}$$

$$I_{ZY} = I_{YZ} = \int\int\int (RZsin\phi)dm \tag{2.13}$$

Substituting those inertias into Eq. 2.1 greatly simplifies the equation for angular momentum ($\vec{H}_A$) and also brings it into a form that is more recognizable.

$$\begin{aligned}\vec{H}_A &= \\ & (I_{XX}\omega_X - I_{XY}\omega_Y - I_{XZ}\omega_Z)\hat{e}_X \\ &+ (I_{YY}\omega_Y - I_{YX}\omega_X - I_{YZ}\omega_Z)\hat{e}_Y \\ &+ (I_{ZZ}\omega_Z - I_{ZX}\omega_X - I_{ZY}\omega_Y)\hat{e}_Z \end{aligned} \tag{2.14}$$

## 2.2 The Change in Angular Momentum Using Cylindrical Vector Components

Under specific circumstances [3], the sum of the moments about a point is equal to the change in angular momentum ($\sum M = \dot{\vec{H}}_A$) about that same point. The certain condition that must be met for this to be true is that point A must be either fixed, the center of mass of the system, or always accelerating towards the center of mass. Most dynamic problems require these conditions in order to use this central element in kinetics which is the companion to the sum of the forces is equal to the change in linear momentum $\sum F = \dot{\vec{P}}_G$ where $\vec{P}_G = m\vec{v}_G$. Similar to the derivation of Euler's equations in standard Cartesian coordinates, the angular momentum has been solved for previously and all that remains is to take the derivative of this to get the change in angular momentum.

Taking the derivative is accomplished by splitting the full derivative of the angular momentum into the sum of the partial derivative with respect to time and the angular velocity crossed with the angular momentum using the partial derivative technique. This is a common and proven technique used when taking derivatives of vectors [9] and is allowable because the coordinate system in Fig. 2.1 is attached to the body. In other words,

$$\frac{d}{dt}\vec{H}_A = \frac{\partial}{\partial t}\vec{H}_A + \vec{\Omega} \times \vec{H}_A. \tag{2.15}$$

Splitting up the derivative allows for solving of the two parts separately. Starting with the first half, the partial derivative with respect to time of angular momentum equation, $\frac{\partial}{\partial t}\vec{H}_A$, and substituting in the equation for angular momentum from Eq. 2.14 results in the following.

$$\begin{aligned}\frac{\partial}{\partial t}\vec{H}_A &= \\ & [\frac{d}{dt}(I_{XX}\omega_X - I_{XY}\omega_Y - I_{ZZ}\omega_Z)]\hat{e_X} \\ &+ [\frac{d}{dt}(I_{YY}\omega_Y - I_{YX}\omega_X - I_{YZ}\omega_Z)]\hat{e_Y} \\ &+ [\frac{d}{dt}(I_{ZZ}\omega_Z - I_{ZX}\omega_X - I_{ZY}\omega_Y)]\hat{e_Z} \qquad (2.16)\end{aligned}$$

At this point, a quick aside to reiterate the point of this derivation is appropriate. Many problems in dynamics deal with a rotation about a single axis. To better represent that type of motion in Euler's equations, the main factors in a rotation about a single axis, $R$ (the radius of the rotating body from the rotational axis) and $\theta$ (the angle of the rotating body from the coordinate axis) are directly inserted into the equations. This substitution will better highlight the sensitivity of these variables and allow for an easier simplification and solving of the system. The main aspect of this type of problem is that the angular velocity vector will only be about a single axis and thus only have one component in X, Y or Z. That means that Eq. 2.16 would be greatly simplified as only one of the angular velocity terms would stay in the equation and would reduce down to three terms. However, to ensure that the equation is general, those terms will remain in the derivation and carried out to the end.

It is assumed that there is no deformation of the body ($[I] \neq [I](t)$) and the coordinate system is attached to the body. With the inertia being independent of time, it can be factored out of the derivative and reduces it to a very straightforward derivative, $\frac{d\vec{\omega}}{dt} = \vec{\alpha}$. Also, it is worth stating that the angle between the position vector and the X unit vector ($\phi$) must not change with time ($\phi \neq \phi(t)$) due to the rigid body assumption. As stated before, this is just another way to represent the position vector to each differential mass. Since the body is not deforming, the vector, nor any of its components (i.e., $\phi$), may not change with respect to time. This results in a clean and manageable equation.

$$
\begin{aligned}
\frac{\partial}{\partial t}\vec{H}_A &= \\
& [(I_{XX}\alpha_X - I_{XY}\alpha_Y - I_{XZ}\alpha_Z)]\hat{e_X} \\
&+ [(I_{YY}\alpha_Y - I_{YX}\alpha_X - I_{YZ}\alpha_Z)]\hat{e_Y} \\
&+ [(I_{ZZ}\alpha_Z - I_{ZX}\alpha_X - I_{ZY}\alpha_Y)]\hat{e_Z}
\end{aligned} \tag{2.17}
$$

For the second half of Eq. 2.15, the cross product, an assumption is required. By forcing the $XYZ$ coordinate system to be fixed to the body requires the angular velocity of the coordinate system $\vec{\Omega}$ be the same as the coordinate system of the body $\vec{\omega}$. By doing this, the $\vec{\Omega}$ can be replaced with $\vec{\omega}$ and will allow for grouping of terms later in the derivation. This also means that there is no relative motion between the body and the coordinate system, which reduces unnecessary complexity. Thus,

$$
\begin{aligned}
\vec{\omega} \times \vec{H}_A &= \\
& (\omega_X\hat{e_X} + \omega_Y\hat{e_Y} + \omega_Z\hat{e_Z}) \times \\
& [(I_{XX}\omega_X - I_{XY}\omega_Y - I_{ZZ}\omega_Z)\hat{e_X} \\
&+ (I_{YY}\omega_Y - I_{YX}\omega_X - I_{YZ}\omega_Z)\hat{e_Y} \\
&+ (I_{ZZ}\omega_Z - I_{ZX}\omega_X - I_{ZY}\omega_Y)\hat{e_Z}]
\end{aligned} \tag{2.18}
$$

Performing this cross product and rearranging to group like terms together yields.

$$
\begin{aligned}
\vec{\omega} \times \vec{H}_A &= \\
& [(I_{ZZ} - I_{YY})\omega_Z\omega_Y + I_{ZY}(\omega_Z^2 - \omega_Y^2) + I_{YX}\omega_X\omega_Z - I_{ZX}\omega_X\omega_Y]\hat{e_X} \\
&+ [(I_{XX} - I_{ZZ})\omega_X\omega_Z + I_{XZ}(\omega_X^2 - \omega_Z^2) + I_{ZY}\omega_Y\omega_X - I_{ZY}\omega_Y\omega_Z]\hat{e_Y} \\
&+ [(I_{YY} - I_{XX})\omega_X\omega_Y + I_{XY}(\omega_Y^2 - \omega_X^2) + I_{XZ}\omega_Y\omega_Z - I_{YZ}\omega_X\omega_Z]\hat{e_Z}
\end{aligned} \tag{2.19}
$$

Substituting Eq. 2.17 and Eq. 2.19 into Eq. 2.15 and again rearranging to combine like terms yields.

$$\begin{aligned}
\dot{\vec{H}}_A &= \\
& [I_{XX}\alpha_X + (I_{ZZ} - I_{YY})\omega_Z\omega_Y - I_{XY}(\alpha_Y - \omega_X\omega_Z) \\
& -I_{XZ}(\alpha_Z + \omega_X\omega_Y) + I_{ZY}(\omega_Z^2 - \omega_Y^2)]\hat{e}_X \\
&+ [I_{YY}\alpha_Y + (I_{RX} - I_{ZZ})\omega_X\omega_Z - I_{YZ}(\alpha_Z - \omega_Y\omega_X) \\
& -I_{YX}(\alpha_X + \omega_Y\omega_Z) + I_{XZ}(\omega_X^2 - \omega_Z^2)]\hat{e}_Y \\
&+ [I_{ZZ}\alpha_Z + (I_{YY} - I_{XX})\omega_X\omega_Y - I_{ZX}(\alpha_X - \omega_Y\omega_Z) \\
& -I_{ZY}(\alpha_Y + \omega_X\omega_Z) + I_{XY}(\omega_Y^2 - \omega_X^2)]\hat{e}_Z
\end{aligned} \tag{2.20}$$

Many dynamic problems utilize rigid bodies that are symmetric around two or more planes. This greatly simplifies the problem because $I_{XY} = I_{XZ} = I_{YX} = I_{YZ} = I_{ZX} = I_{ZY} = 0$. This is what is known as having principal axes [3]. Applying the principal axes as an assumption simplifies the equation to the familiar format of Euler's equations.

$$\begin{aligned}
\dot{\vec{H}}_A &= \\
& [I_{XX}\alpha_X + (I_{ZZ} - I_{YY})\omega_Z\omega_Y]\hat{e}_X \\
&+ [I_{YY}\alpha_Y + (I_{XX} - I_{ZZ})\omega_X\omega_Z]\hat{e}_Y \\
&+ [I_{ZZ}\alpha_Z + (I_{YY} - I_{XX})\omega_X\omega_Y]\hat{e}_Z.
\end{aligned} \tag{2.21}$$

## 2.3 Derivation Summary

This derivation results in a set of equations that are already set up for an intuitive application to a single axis rotation problem. The new convention and notation in assigning coordinate axes provides a very simple and accurate way to describe a body's location in space very naturally using a radius and angle which also makes it very easy and convenient to add multiple bodies to the system. The inertias for the problem are all described using a radius and angle and keeps everything in the problem concise and consistent.

The last step that needs to be completed before a problem can be solved using these equations is to calculate the inertias of general shapes. It is taken for granted when solving Euler's equation in Cartesian coordinates that all of the triple integral inertias have been already solved (and memorized) in many cases. This looks like a large stumbling block for this new notation since one would have to now recalculate all of the inertias and create another table to keep track of them. However, as long as the coordinate system is orthogonal and centered at the same point, the inertias will come out the same in this notation as in a standard Cartesian coordinate system. Because inertia is merely the measure of an object's resistance to any change in rotation about an axis, the value of the inertia will be the same as long as the axes are coincident. The route to get to the solution will be different but they will end up in the same place. An example of the calculation of the inertia in this new notation is discussed in Chapter 3.

Another issue that arises is that many times it is favorable to use the parallel axis theorem to get inertias of bodies using non-standard coordinate centers. However, like Euler's equations, this can be derived using this notation and works in generally the same manner as in the standard Cartesian coordinates. This new derivation will allow for the application of the parallel axis theorem with the benefit of using the cylindrical vector component notation. All of these tools together create a direct and elegant link from the problem setup and input parameters to the final result and analysis by preserving the important design parameters with respect to dynamic balance.

# CHAPTER 3

## Inertia of Rigid Bodies Using Cylindrical Vector Components

During the derivation of Euler's equations of motion using cylindrical vector components, the new notation for the position vectors made their way into the inertia equations and changed their form. This seems problematic since the inertia needs to be calculated for any rigid body that is going to be analyzed using this method. While this is also true for Euler's equations of motion using Cartesian coordinates, the inertia values for most shapes are already calculated and listed in many engineering resources. If this new notation is going to create a whole separate list of inertias and is going to require a recalculation of inertia then its value is somewhat lessened. Fortunately, these new inertia equations will result in the exact same value for the inertia as its Cartesian counterparts. This means that instead of being a large negative for using this new notation, it actually adds a great benefit to this notation and to dynamics in itself. These new formulations are an entire set of equivalent equations to calculate inertia for any body. This is especially important and beneficial because these equations involve triple integrals which can be extremely difficult or impossible to solve depending upon the input variables. It is possible that this new formulation could provide for an exact solution to an inertia when the standard Cartesian formulation would have had to have been approximated due to being unsolvable.

In the following sections, sample calculations of the inertia matrix for a cylinder will be presented along with a derivation of the parallel axis theorem using cylindrical vector notation. Because the premise of this thesis is that keeping the problem in the same type of notation is extremely helpful to the understanding of the problem, the parallel axis theorem needs to be derived to utilize cylindrical vector components. Additionally, a few examples of the parallel axis theorem in cylindrical vector components which will highlight some benefits and particularly interesting aspects of this approach.

## 3.1 Calculation of Inertia Matrix for a Cylinder Using Cylindrical Vector Components

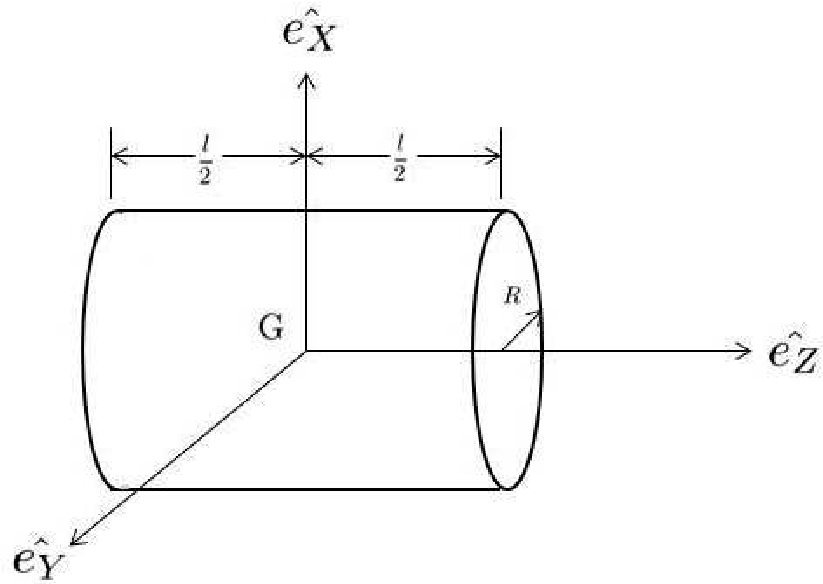


Figure 3.1: Inertia Calculation Example

Inertias are a measure of the distribution of the mass of a body. To calculate the moments of inertia of a body, a triple integral of the mass over the body's geometric dimensions is calculated. The hardest part in many of these calculations is to find the easiest way to describe the geometric dimensions to make the triple integral easy and possible to solve. In a body such as a cylinder, it may be unclear using a Cartesian system how to break up the geometry to easily calculate the inertia. However, using cylindrical vector components it is apparent to use $R, \phi, Z$. These vectors easily describe the shape and have direct substitutions into the inertia calculations in Chapter 2.

$$\begin{aligned} 0 \leq & \quad R & \leq r \\ 0 \leq & \quad \phi & \leq 2\pi \\ -\frac{l}{2} \leq & \quad Z & \leq \frac{l}{2} \end{aligned}$$

The differential mass term for a cylinder is described as [3]

$$dm = \rho R dR d\phi dZ$$

To solve the inertias in terms of mass the density of a cylinder can be defined as the mass of the cylinder divided by the standard formulation for the volume of a cylinder $\pi r^2 l$.

$$\rho = \frac{m}{\pi r^2 l} \tag{3.1}$$

These terms are substituted into the inertia integrals defined by equations 2.8 - 2.13. For brevity, the solution to these triple integrals are detailed in Appendix A, but the solutions are listed below.

$$\begin{aligned} I_{XX} = I_{YY} &= \frac{1}{12} m[3r^2 + l^2] \\ I_{ZZ} &= \frac{1}{2} m r^2 \end{aligned} \tag{3.2}$$

These are the same results that have classically been associated with a cylinder which was calculated using the standard Cartesian formulation of the inertias. The main point is that the new formulations could be very useful in calculating the inertia for shapes that are more conveniently described using cylindrical vector components. For example, a computer program using some iterative method for calculating deformations or stresses might find these formulations useful in reducing calculation time or getting around complicated or unsolvable integrals.

Similarly, calculating the products of inertia using the new formulation results in the following. Again, the details of this calculation is presented in Appendix A.

$$I_{XZ} = I_{ZX} = I_{XY} = I_{YX} = I_{ZY} = I_{YZ} = 0 \tag{3.3}$$

These results are consistent with the inertia that is classically associated with a cylinder. This makes sense because the inertia should be consistent if the coordinate system is attached at the same point in the body and the axis are in the same direction. This is merely a new path to arrive at the same result.

## 3.2 Derivation of the Parallel Axis Theorem Using Cylindrical Vector Components

A major constraint in using the change in angular momentum to solve dynamic problems is the strict rules as to where the coordinate system of a body may be located. This may serve as a problem in the case where the inertia of a body is known at one coordinate axes location yet it does not allow for the formulation of the equations for rotational motion [3]. Because of this problem, a way to translate the aforementioned coordinate axes to the latter coordinate axes location was developed. This is known as the parallel axis theorem and is an extremely useful tool in dynamic analysis. This chapter will rederive the parallel axis theorem, just as Euler's equations were derived in Chapter 2, using the cylindrical vector components throughout the derivation.

Fig. 3.2. shows the typical starting figure for the parallel axis theorem derivation with two coordinate systems attached to a rigid body. Coordinate system G is attached at the body's center of mass and coordinate system B is shown at a random yet known vector $r_{B/G}$ from coordinate system G. This vector would be defined simply as

$$\vec{r}_{B/G} = x_o e_{\hat{X}_G} + y_o e_{\hat{Y}_G} + z_o e_{\hat{Z}_G} \tag{3.4}$$

in the standard derivation. However, like before, the vector from B to G is split into the components of the projected vector $r_o$ along with the angle $\phi_o$ from the X axis on the XY plane and the vector $z_o$ along the Z axis shown in Fig. 3.3. This new formulation of the vector $r_{B/G}$ is

$$\vec{r}_{B/G} = r_o \cos\phi_o e_{\hat{X}_G} + r_o \sin\phi_o e_{\hat{Y}_G} + z_o e_{\hat{Z}_G} \tag{3.5}$$

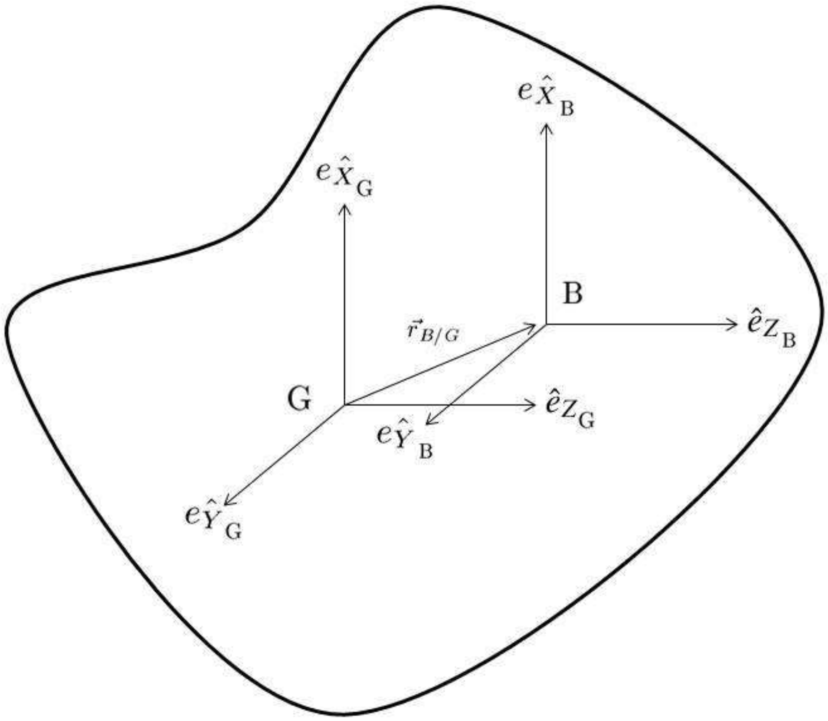


Figure 3.2: Parallel Axis Theorem Diagram

In this example, it is assumed that angular momentum of the body is expressed in coordinate system G and is known. It is desired that the angular momentum of the body be expressed in coordinate system B. Normally, the vector from coordinate system G to coordinate system B is expressed using standard Cartesian components in the $XYZ$ directions. However, in this example the cylindrical vector components are used, as in the Chapter 2 derivation. This will change the final result for the parallel axis theorem to be in cylindrical vector components, further allowing the dynamist to keep the problem entirely in cylindrical vector component notation. The angular momentum expressed expressed in coordinate system G can be transferred to coordinate system B by adding on the vector between the coordinate systems crossed with the cross product of the vector between the systems and the angular momentum

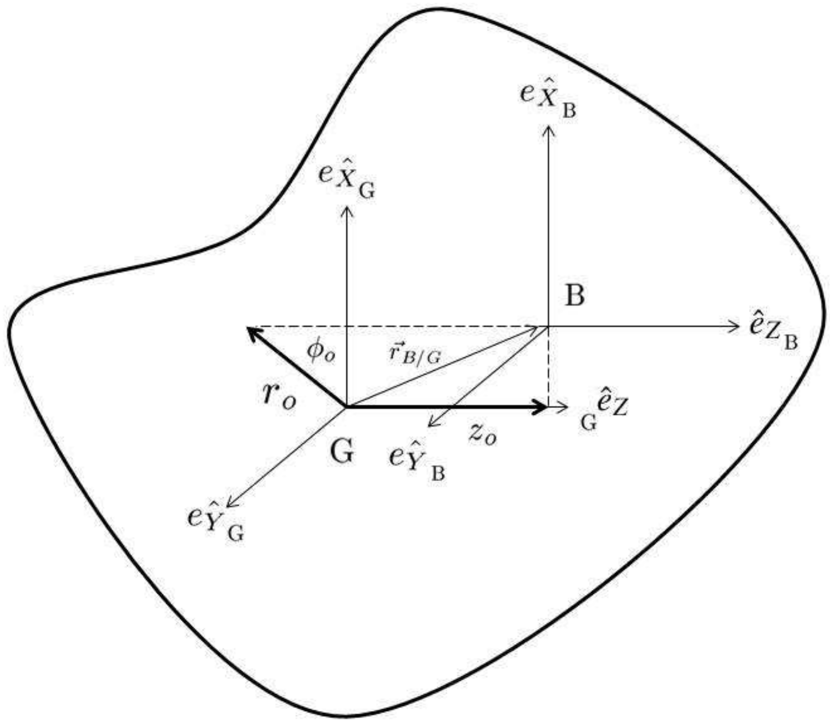


Figure 3.3: Parallel Axis Theorem Diagram with Cylindrical Vector Components

of the body. Defining the angular momentum for coordinate system B with $\vec{H}_B$ and the angular momentum of coordinate system G as $\vec{H}_G$, this equation can be written as

$$\vec{H}_B = \vec{H}_G + m(\vec{r}_{B/G} \times [\vec{\omega} \times \vec{r}_{B/G}]). \tag{3.6}$$

This equation, when solved, results in the new formulation of the Parallel Axis Theorem using cylindrical vector components. The details of this derivation is presented in Appendix B and the results are as follows.

$$
\begin{aligned}
I_{XX_B} &= I_{XX_G} + m(r_o^2 \sin^2 \phi_o + z_o^2) \\
I_{YY_B} &= I_{YY_G} + m(r_o^2 \cos^2 \phi_o + z_o^2) \\
I_{ZZ_B} &= I_{ZZ_G} + mr_o^2
\end{aligned}
$$

$$
\begin{aligned}
I_{XY_B} &= I_{YX_B} = I_{XY_G} + mr_o^2 \cos \phi_o \sin \phi_o \\
I_{XZ_B} &= I_{ZX_B} = I_{XZ_G} + mr_o z_o \cos \phi_o \\
I_{ZY_B} &= I_{YZ_B} = I_{XY_G} + mr_o z_o \sin \phi_o
\end{aligned} \tag{3.7}
$$

These equations will allow for the calculation of inertia for any shape with known inertias at a point other than the center of mass provided that the coordinate systems have parallel unit vectors. This formulation of the equations will make it much more convenient in a single rotationally biased problem to find these parallel inertias because the values for $r_o, \phi_o$ and $z_o$ will be obvious.

## 3.3 Examples of Parallel Axis Theorem Using Cylindrical Vector Components

### 3.3.1 Parallel Axis Theorem Transformation Along Two Axes

In this section, examples of the parallel axis theorem derived using cylindrical vector components will be presented. These examples will exemplify a few interesting points in this process and how this method is extremely simple for this type of problem.

A cylinder as shown in Fig. 3.4 has known inertia consistent with a regular cylinder with the coordinate system centered on the center of mass G. It is desired that the inertia be expressed at a point on the edge of the radius of the cylinder directly along the $\hat{e}_{Y_G}$ axis as shown with coordinate system B. This example is used to both show the validity of this new derivation but also how it easily handles a transformation that is generally associated with a cylindrical type problem or single

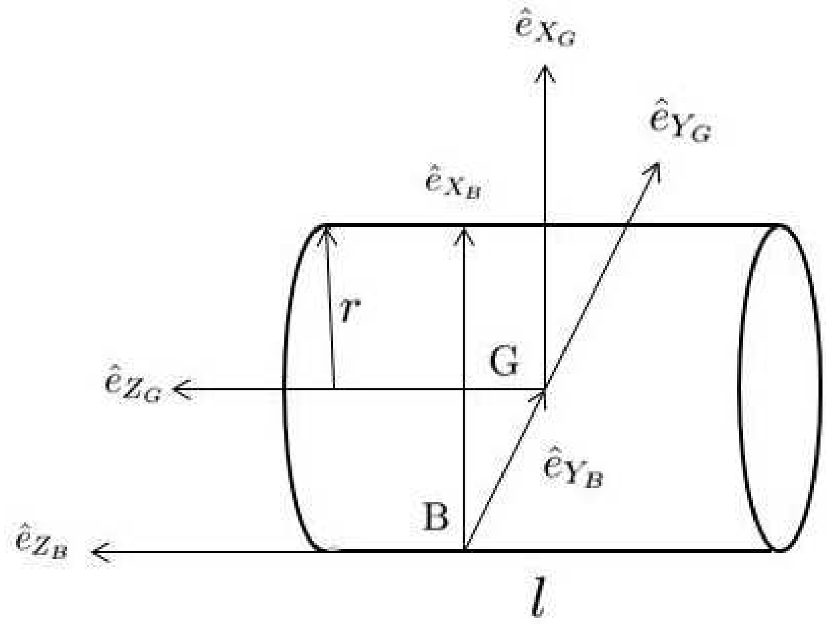


Figure 3.4: Parallel Axis Theorem Example 1

axis rotational problem. The variables associated with the transformation are listed below.

$$\begin{aligned} r_o &= r \\ \phi_o &= -\frac{\pi}{2} \\ z_o &= 0 \end{aligned} \tag{3.8}$$

It is important to note that even though the coordinate system was moved along the $\hat{e}_Y$ vector, the value for $\phi$ is NOT 0 as it would be initially thought. The point that the secondary coordinate system is at $(X'Y'Z')$ corresponds to $\phi = -\frac{\pi}{2}$ for the projected vector on the XY plane. Fig. 3.5 shows the position vector $\hat{r}_{B/G}$ and its components $r_o$ and $\phi_o$ to illustrate this point. This is a major difference between the Cartesian coordinate system way of thinking and how the cylindrical vector notion needs to be used. All vectors and movements must be treated as a radius and angle, even if the move is about one axis.

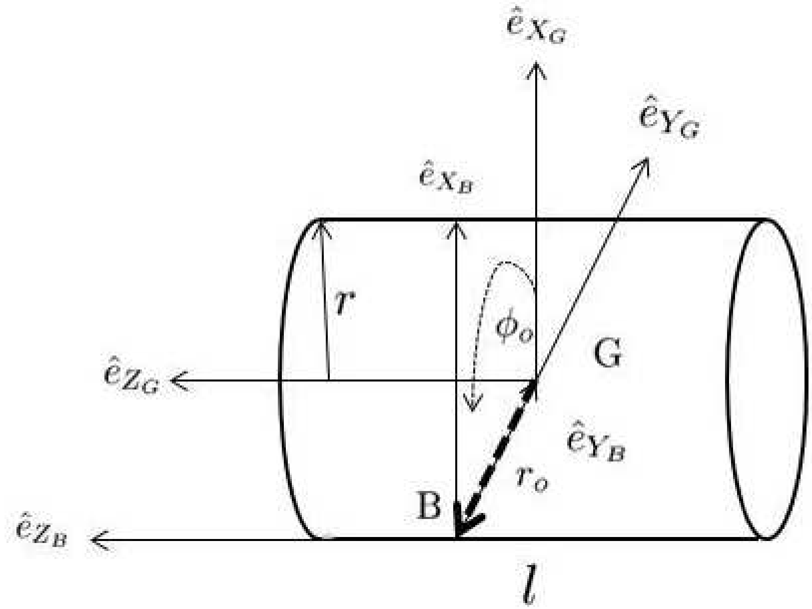


Figure 3.5: Parallel Axis Theorem Example 1 with Position Vector Shown

$$\begin{aligned} I_{XX_B} &= I_{XX_G} + m(r^2 \sin^2(-\frac{\pi}{2}) + (0)^2) \\ I_{YY_B} &= I_{YY_G} + m(r^2 \cos^2(-\frac{\pi}{2}) + (0)^2) \\ I_{ZZ_B} &= I_{ZZ_G} + mr^2. \end{aligned} \tag{3.9}$$

The inertia for a cylinder with the coordinate system at the center of mass is listed below.

$$\begin{aligned} I_{XX_G} &= \frac{1}{12}m[3r^2 + l^2] \\ I_{YY_G} &= \frac{1}{12}m[3r^2 + l^2] \\ I_{ZZ_G} &= \frac{1}{2}mr^2. \end{aligned} \tag{3.10}$$

Combining 3.9 with 3.10 and simplifying the results leads to the following.

$$\begin{aligned} I_{XX_B} &= \frac{1}{12}m[15r^2 + l^2] \\ I_{YY_B} &= \frac{1}{12}m[3r^2 + l^2] \\ I_{ZZ_B} &= \frac{3}{2}mr^2. \end{aligned} \tag{3.11}$$

Again, the very beneficial aspect of this is that this is identical to the result that would be obtained by using the parallel axis theorem in Cartesian coordinates. The main point of this new notation and application for this parallel axis theorem is that it keeps the problem in a consistent notation for cylindrical vector components.

### 3.3.2 Parallel Axis Theorem Transformation Along Three Axes

Making the example a little more complicated will show that even with the added complexity this transformation is still easily performed. Moving the coordinate system out to the end of the cylinder in both the X and Z directions and changing the angle ($\phi$) by $-\frac{\pi}{6}$ as shown in Fig. 3.6 will exemplify this. Specifically, for this example the parameters are:

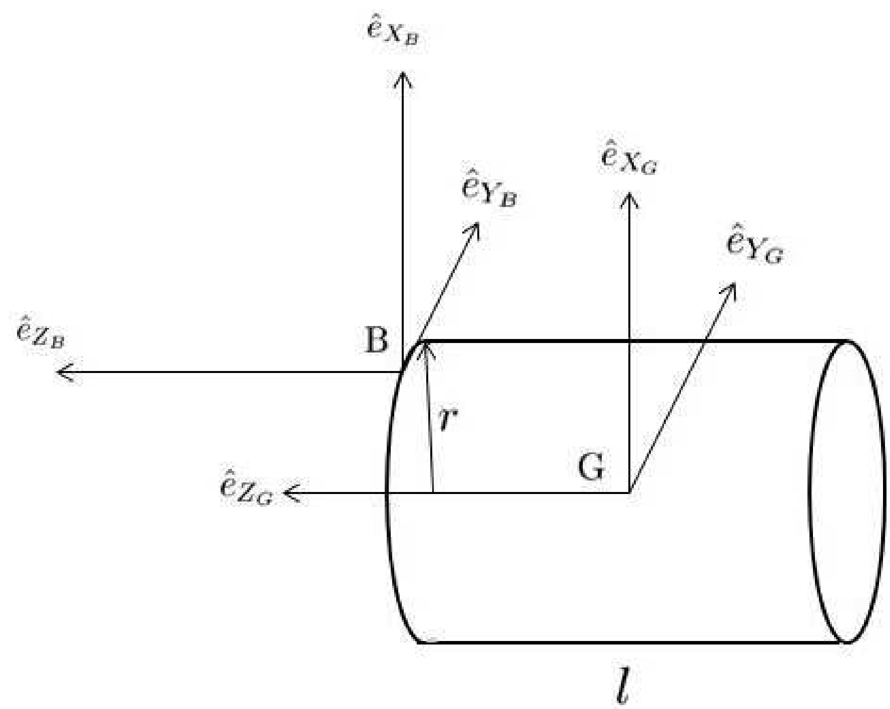


Figure 3.6: Parallel Axis Theorem Example 2

$$
\begin{aligned}
r_o &= r \\
\phi_o &= -\frac{\pi}{6} \\
z_o &= \frac{l}{2}
\end{aligned}
\tag{3.12}
$$

Substituting the above variables into the cylindrical coordinate form of the parallel axis theorem.

$$
\begin{aligned}
I_{XX_B} &= I_{XX_G} + m(r^2 \sin^2(-\frac{\pi}{6}) + \frac{l}{2}^2) \\
I_{YY_B} &= I_{YY_G} + m(r^2 \cos^2(-\frac{\pi}{6}) + \frac{l}{2}^2) \\
I_{ZZ_B} &= I_{ZZ_G} + mr^2
\end{aligned}
\tag{3.13}
$$

Substituting in the moments of inertia for a cylinder.

$$
\begin{aligned}
I_{XX_B} &= \frac{1}{12}m[3r^2 + l^2] + m(r^2 \sin^2(-\frac{\pi}{6}) + \frac{l}{2}^2) \\
I_{YY_B} &= \frac{1}{12}m[3r^2 + l^2] + m(r^2 \cos^2(-\frac{\pi}{6}) + \frac{l}{2}^2) \\
I_{ZZ_B} &= \frac{1}{2}mr^2 + mr^2
\end{aligned}
\tag{3.14}
$$

Rearranging the terms and simplifying

$$
\begin{aligned}
I_{XX_B} &= \frac{1}{2}mr^2 + \frac{1}{3}ml^2 \\
I_{YY_B} &= mr^2 + \frac{1}{3}ml^2 \\
I_{ZZ_B} &= \frac{3}{2}mr^2.
\end{aligned}
\tag{3.15}
$$

In the previous examples, the coordinate system was placed at a point where principal axes were obtained. However, the placement of the coordinate system for this example requires the calculation of the products of inertia. These equations are detailed below

$$
\begin{aligned}
I_{XY_B} &= I_{YX_B} = 0 + mr^2 \cos(-\frac{\pi}{6}) \sin(-\frac{\pi}{6}) \\
I_{XZ_B} &= I_{ZX_B} = 0 + mr\frac{l}{2} \cos(-\frac{\pi}{6}) \\
I_{ZY_B} &= I_{YZ_B} = 0 + mr\frac{l}{2} \sin(-\frac{\pi}{6}).
\end{aligned}
\tag{3.16}
$$

Simplifying these equations results in the following

$$
\begin{aligned}
I_{XY_B} &= I_{YX_B} = -\frac{\sqrt{3}}{4}mr^2 \\
I_{XZ_B} &= I_{ZX_B} = \frac{\sqrt{3}}{4}mrl \\
I_{ZY_B} &= I_{YZ_B} = -\frac{1}{4}mrl
\end{aligned}
\tag{3.17}
$$

As it was for the simpler example, this result is exactly the same as it would be from the Cartesian coordinates version of the parallel axis theorem. This example shows how a system that is naturally set up using cylindrical vector notation can be described at any point on the cylinder easily using this parallel axis theorem formulation using cylindrical vector components. Since most single axis rotational problems are described easily and most completely using a cylindrical type coordinate system, this parallel axis formulation is a great tool that can be used to describe a body at any location in the system.

### 3.4 Discussion on Cylindrical Vector Component Inertia

In the previous sections, the new formulation for inertia and the parallel axis theorem were presented. These new formulations give a clear advantage to dynamists studying single axis rotational motion in that the formulation of the inertia and parallel axis theorem have direct connections to the main variables $R$ and $\phi$. These formulations also provide an additional means of calculating the inertias of bodies, for any type of problem, that would otherwise be difficult to calculate using the Cartesian coordinate formulation. Yet, because they will result in the exact same value as the Cartesian coordinate formulation, it is not required to recalculate the inertia for any body for which the inertia is already known. The parallel axis theorem examples highlight the benefits to this new formulation. These problems show that when dealing with a problem that is naturally described using cylindrical vector components, this formulation has direct substitutes for the important and most often known variables. Even with a seemingly complicated coordinate axis transformation in Example 2 requires only a direct substitution and relatively easy solving of an equation. These formulations do not give the ability to solve an otherwise unsolvable equation but gives a more elegant route to a solution which would generally lead to a more comprehensive understanding of the equation and the roles in which each variable has in the equation. This elegance can help in the understanding of variable sensitivity for these types of problems since there are relatively few steps between the substitution and useable results of these equations.

# CHAPTER 4

# Example Particle Problems

This chapter takes a step back to illustrate the benefit of the cylindrical vector notation in pure particle problems. It is much more natural to think of some problems in cylindrical terms and when doing so, it is easier to take the problem to depths and understand that are difficult to get to via the purely Cartesian coordinates model. These problems are not meant to show that it is impossible to reach the level of depth or understand using only Cartesian coordinates, but that it is sufficiently difficult to warrant the need for a better more direct way. This more direct and elegant way is to define the problem and to perform the calculations in cylindrical vector components throughout the problem. This is basic premise of this thesis, not that it is impossible to perform some problems without Euler's equations in cylindrical vector components, but that it is easier and that will open the doors to more complex analysis and a deeper understanding of single axis rotation.

## 4.1 Two Particle System in Plane Using Cylindrical Vector Components

The simplest problem that will exemplify the benefits of cylindrical vector components is a two particle problem in a rotating system. Taking two point masses and placing them at a radius away from an axis of rotation will demonstrate many principles of dynamic balance, generally a major concern in a single axis rotational problem. Using this system, it will show how far this can be taken to enhance the understanding of single axis rotation. In these problems, there is no need to use the full Euler's equations because particles, by definition, do not have any inertia around axes through them. However, these are presented to show that looking at a single axis rotational problem using cylindrical vector components is beneficial. This idea can then be expanded to more complex systems that require the use of rigid bodies and the full Euler's equation or motion.

In these problems, a commonly analyzed aspect of rotating machinery and single axis rotation systems, dynamic balance, is discussed. Dynamic balance in critically important for rotating machinery design because it affects the system's structural integrity, its performance, allowable operating conditions, and even its feasibility in many cases. These problems will take a look at the notion that a generally set up problem can be used to find ways to design smarter and better systems to create a more robust dynamic balance in a system. These problem will try to see if the input parameters of a problem be used in such a way to create rotating machinery that resists the trend to dynamic imbalance.

In Figure 4.1, the bodies, (body A and body B) have variable masses ($m_1$ and $m_2$) and are at variable radii ($R_1$ and $R_2$) from the rotational axes. These masses are also at a variable angle from the other. The reason for keeping so many aspects of the problem a variable (i.e., arbitrary, known constant parameter) is to keep the problem as general as possible. During a real world design problem, many of these variables may be predetermined, fixed or completely free to change so keeping the analysis as general as possible is very important. A major concern for rotating systems is the dynamic balance of the entire system. It is obvious that all of these variables have a direct influence on dynamic balance. However, many times the effect on the dynamic balance may not be clear when setting these parameters, or what steps should be taken to reduce these effects on the balance of the system. Using these generalities, and pushing the problem as far as is practical, the relationship between the system parameters and their sensitivity with respect to the resultant force on the bases can be quantified and analyzed.

A coordinate system ($XYZ$) is attached at the center of the rotating axis with $\hat{e}_X$ pointing to body A. Similarly, a coordinate system ($X_\phi Y_\phi Z_\phi$) is attached at the center of the rotating axes with $\hat{e}_{X_\phi}$ pointing to body B which is positioned at a constant angle $\phi$ from coordinate system $XYZ$. These bodies rotate about an axis of rotation $\hat{e}_Z$ at a constant angular velocity $\omega$ (i.e., $\alpha = 0$) and are contained in the same plane, i.e., plane $XY$ is coincident with plane $X_\phi Y_\phi$. Gravity in this problem acts perpendicular to the starting reference of angle $\theta$, the angle of coordinate system $XYZ$.

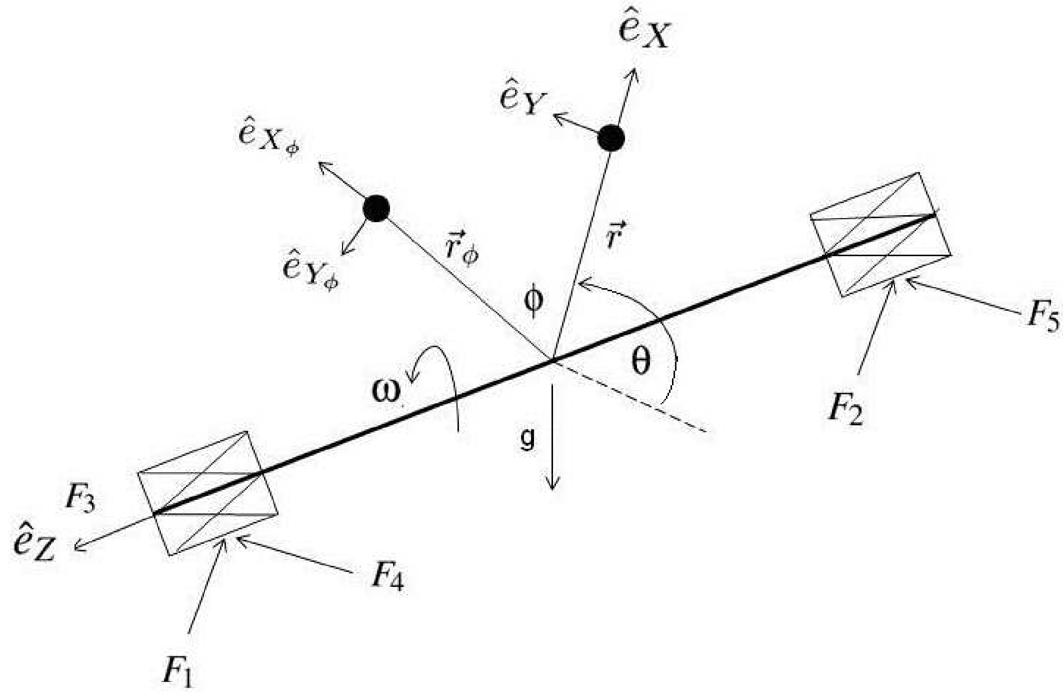


Figure 4.1: Simple Two Particle System

Calculating the kinematics for this problem is simple because of the natural use of cylindrical vector components. The coordinate system for body A ($XYZ$) will be used as the main coordinate system and all of the parameters will be defined in that coordinate system. Once the coordinate transformation is created, this becomes a very easy task. Specifically

$$
\begin{aligned}
\vec{r} &= R_1\hat{e}_X \\
\vec{r}_\phi &= R_2\hat{e}_{X_\phi} = R_2(\cos\phi\hat{e}_X + \sin\phi\hat{e}_Y) \\
\vec{v} &= R_1\omega\hat{e}_Y \\
\vec{v}_\phi &= R_2\omega(-\sin\phi\hat{e}_X + \cos\phi\hat{e}_Y) \\
\vec{a} &= -R_1\omega^2\hat{e}_X \\
\vec{a}_\phi &= -R_2\omega^2(\cos\phi\hat{e}_X + \sin\phi\hat{e}_Y)
\end{aligned}
$$

The axis that the forces are summed along do not necessarily have to be the same about which moments are summed but in this case they will end up being the same. Summing the forces along the main body fixed axes of $\hat{e}_X, \hat{e}_Y$, and $\hat{e}_Z$ yields the following results.

$$
\begin{aligned}
\sum F_X &= F_1 + F_2 - m_1 g\cos\theta - m_2 g\cos(\theta+\phi) && (4.1)\\
\sum F_Y &= F_4 + F_5 + m_1 g\sin\theta + m_2 g\sin(\theta+\phi) && (4.2)\\
\sum F_Z &= F_3 && (4.3)
\end{aligned}
$$

The moments are now summed about the same $XYZ$ coordinate system as above. Because the forces are aligned with these axes, the choice of this coordinate system is logical.

$$
\begin{aligned}
\sum M_X &= F_5(\frac{L}{2}) - F_4(\frac{L}{2}) = 0 \\
&\Rightarrow F_5 = F_4 && (4.4)\\
\sum M_Y &= F_1(\frac{L}{2}) - F_2(\frac{L}{2}) = 0 \\
&\Rightarrow F_1 = F_2 && (4.5)\\
\sum M_Z &= -m_2 g\sin(\theta+\phi) - m_1 g\sin\theta
\end{aligned}
$$

Notice that because the masses are located in the center of the bar the sum of the moment equations result in only requiring that $F_1 = F_2$ and $F_4 = F_5$. Also, the moment about Z does not cancel out unless $\phi = \pi$ and $m_1 = m_2$ in which case the sign of the sine term would switch and cancel out. Since the moment does not cancel out for all other $\phi$ the sum of the moments about the Z axis will result in a torque about the shaft. This torque is what one would instinctively think of when a nonsymmetric body is spinning about an axis (which this system would emulate with $\phi \neq \pi$). Next, combining 4.1-4.5 and solving for the individual forces yields the following:

$$\begin{aligned}
F_1 &= \frac{m_1}{2}(g\cos\theta - R_1\omega^2) + \frac{m_2}{2}(g\cos(\theta+\phi) - R_2\omega^2\cos\phi) && (4.6)\\
F_2 &= \frac{m_1}{2}(g\cos\theta - R_1\omega^2) + \frac{m_2}{2}(g\cos(\theta+\phi) - R_2\omega^2\cos\phi) && (4.7)\\
F_3 &= 0 && (4.8)\\
F_4 &= -\frac{m_1}{2}g\sin\theta - \frac{m_2}{2}(g\sin(\theta+\phi) + R_2\omega^2\sin\phi) && (4.9)\\
F_5 &= -\frac{m_1}{2}g\sin\theta - \frac{m_2}{2}(g\sin(\theta+\phi) + R_2\omega^2\sin\phi) && (4.10)
\end{aligned}$$

### 4.1.1 Examining the Resultant Balance Force

Equations 4.6 to 4.10 are a general solution for the system in Figure 4.1 and can be used to extensively examine the balance of the system. Any variable can be singled out and studied to see its effects on a single final balance force. It is easier, however, to simplify this further and combine these forces into one resultant force that can be minimized to find the balance point. Then, examining how much the resultant force changes with a change of a variable will show that variable's sensitivity in the system with respect to dynamic balance. This is done by taking these forces and resolving them into one force thus combining the forces ($F_1, F_2, F_4$ and $F_5$).

$F_1$ and $F_2$ both lie in the same plane and in the same direction and can be added together directly. The same procedure of adding directly is true with $F_4$ and

$F_5$ since these forces both lie in the $\theta$ plane and have the same direction.

$$F_R = F_1 + F_2$$
$$F_\theta = F_4 + F_5 \tag{4.11}$$

Having two independent forces, they can be combined into one resultant force by treating them as components of a vector. Taking the square root of the sum of the squares of these vectors will result in the new resultant vector.

$$F_{res} = \sqrt{{F_R}^2 + {F_\theta}^2} \tag{4.12}$$

Removing the problem from the gravitational field (i.e., setting $g = 0$) puts the focus on the dynamic balance and removes the static balance forces from the problem. Gravity is the main driver in the static balance of a problem and while gravity does affect the dynamic balance of the system, the dynamic imbalance can be seen and studied without the gravity directly acting on the system. Performing this substitution and calculating the above equation results in the balance force below.

$$\begin{aligned} F_{res} = \quad & [\left(\omega^2 R_1 m_1 + \omega^2 R_2 m_2 \cos(\phi)\right)^2 \\ & + \omega^4 {R_2}^2 {m_2}^2 \sin(\phi)^2]^{\frac{1}{2}} \end{aligned} \tag{4.13}$$

At this point, a set a system parameters will be applied to the equation above and it will be graphed in three dimensional space. Without getting into advanced means of depicting multi-dimensional graphs, traditional graphing is limited to only three variables. Because of this, body A will be fixed and assigned parameters while the radius of body B, $R_2$, from the axis of rotation and angle from body A to body B, $\phi$, will remain variables. To simplify the results, the masses will be set to be identical and, like above, the masses will be taken out of a gravity field (i.e., $g = 0$). The figure will show how the resulting balance $F_{res}$ varies with respect to $R_2$ and $\phi$. Specifically, for this problem the parameters will be set as the following. These values have been chosen based upon the general design of a CT scanner.

$$m_1 = 40kg \quad , \quad m_2 = 40kg$$

$$R_1 = 1m \quad , \quad \omega = 12rad/s$$

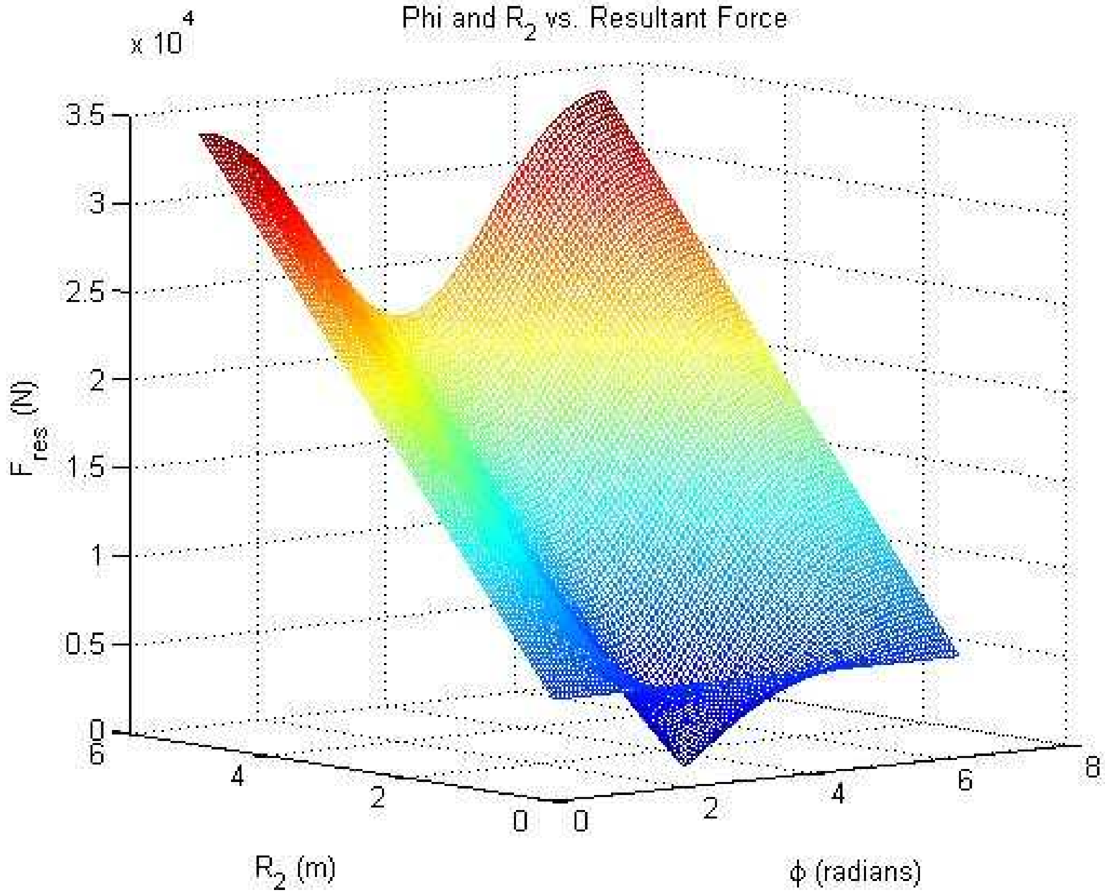


Figure 4.2: Simple Two Particle System Balance Force

This graph clearly shows two main aspects of this problem. The first is where the system is balanced, (where $F_{res} = 0$) which will give the values for $R_2$ and $\phi$ that balances the system. The second and possibly more interesting is how the change in $R_2$ and $\phi$ affects the balance force $F_{res}$. This is what has been referenced as the sensitivity of the design parameters with respect to the output, the balance force.

### 4.1.2 Manipulation of Design Parameters to Create a Zero Derivative at the Balance Point

Using the theoretical model, it would be very easy to design a system that was perfectly balanced. However, the model of any problem has assumptions and the results are never exactly matched to reality. Achieving the exact point on Figure 4.2 where $F_{res} = 0$ is not practically feasible in many cases. Also, the more troubling part of this graph is that the slope of the surface near the balance point is very severe, entailing that any slight imperfection in radius or angle will cause a large affect on the balance force. This also is assuming that the mass of the body is perfectly in line with the mathematical model. It is easy to see how this can cause many problems in the practical design of a rotating machine with respect to dynamic balance.

It is intuitive that this problem would have only one point where the resultant force would be zero. The system itself is limited by degrees of freedom, since there are only two masses. However, it is unclear whether the slope of the surface near the balance point can be manipulated to be closer to zero. Having a smaller slope near the balance point would give greater leeway in design decisions and would result in a much more robust design.

To examine the derivative at the balance point, the balance point must first be solved for. To keep the problem as general as possible, the balance point will be solved for every case of $R_2$ and $\phi$. Using MuPad, a symbolic mathematical solver in Matlab, to solve Eq. 4.13 for $R_2$ to make $F_{res} = 0$ yields the following.

$$R_2 = \begin{cases} -\frac{R_1\, m_1\, \cos(\phi) + R_1\, m_1 \sqrt{-\sin(\phi)^2}}{m_2 \cos(\phi)^2 + m_2 \sin(\phi)^2} & \text{if} \quad \phi \in \{\pi\, k |\, k \in \mathbb{Z}\} \\ \textit{empty set} & \text{if} \quad \phi \ni \{\pi\, k |\, k \in \mathbb{Z}\} \end{cases} \tag{4.14}$$

This seemingly complicated equation states a very obvious result when looked at in detail. The first solution states that if $\phi$ is any multiple of $\pi$, there is a solution for which a value for $R_2$ can cause the $F_{res} = 0$. Otherwise, there is no solution ($\phi$

is the null set). Interestingly, $\phi = 2\pi$ is a solution because the "$\cos\pi$" term in the top equation will change the sign of $R_2$ to negative which put it at the same point as a positive radius and an odd multiple for $k$ in $\phi = k\pi$. Stated plainly, if $\phi \neq k\pi$, there is no way to make $F_{res} = 0$. The top option for $R_2$ also seems very complicated, however, when substituting $\phi = \pi$ into the equation (as is the requirement), it reduces down to a very simple and intuitive equation.

$$R_2 = \frac{R_1\, m_1}{m_2} \tag{4.15}$$

It is intuitive that for a two particle system to be dynamically balanced, the configuration must be for the two masses to be perfectly opposed to each other and at a distance away from the axis of rotation by a factor of the ratio of the masses. This is consistent with Fig. 4.2 as the balance point falls on the intersection of $\phi = \pi$ and where $R_1 = R_2$, the factor of the ratio of the masses being 1 ($\frac{40}{40}$).

The derivative of Eq. 4.13 in both directions at the balance point are the equations that are the most telling in the sensitivity analysis. If these equations can be manipulated to be 0, or close to 0, the design will be much more robust. These derivatives are as follows.

$$\frac{\partial}{\partial R_2} F_{res}|_{\phi=\pi} = -\frac{\omega^2\, m_2\, \left(\omega^2\, R_1\, m_1 - \omega^2\, R_2\, m_2\right)}{\sqrt{\left(\omega^2\, R_1\, m_1 - \omega^2\, R_2\, m_2\right)^2}} \tag{4.16}$$

$$\frac{\partial}{\partial \phi} F_{res}|_{R_2=\frac{R_1 m_1}{m_2}} = \frac{P}{Q} \tag{4.17}$$

where

$$P = -2\,\omega^2\, R_1\, m_1\, \sin(\phi)\, \left(\omega^2\, R_1\, m_1 + \omega^2\, R_1\, m_1\, \cos(\phi)\right) - 2\,\omega^4\, {R_1}^2\, {m_1}^2\, \cos(\phi)\, \sin(\phi) \tag{4.18}$$

$$Q = \sqrt{\left(\omega^2\, R_1\, m_1 + \omega^2\, R_1\, m_1\, \cos(\phi)\right)^2 + \omega^4\, {R_1}^2\, {m_1}^2 \sin(\phi)^2} \tag{4.19}$$

On the surface, it is unclear in these equations if there is any possibility to manipulate them to make them zero. As a starting point, these equations will be graphed using the same parameters as in Section 3.1.2.

$$m_1 = 40kg \quad , \quad m_2 = 40kg$$

$$R_1 = 1m \quad , \quad \omega = 12rad/s$$

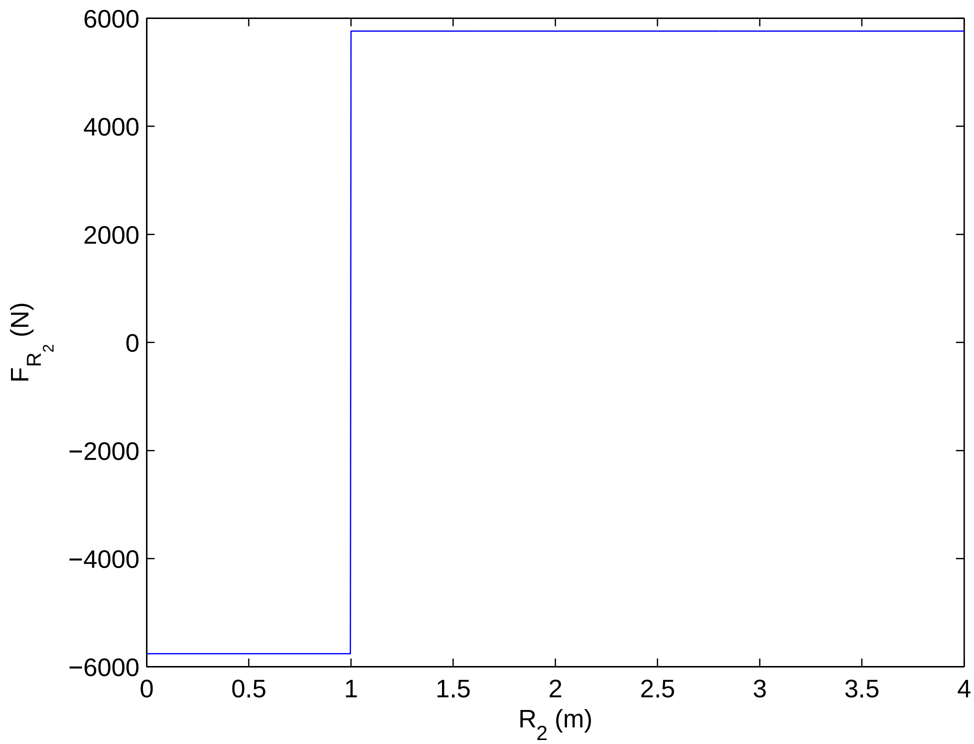


Figure 4.3: Derivative of Balance Force in $R_2$

The unfortunate result in this graph is that the balance point in both of these graphs is a discontinuity and that the derivatives are not easily changed around the discontinuity. Fig. 4.3 shows that the derivative is constant approaching the balance point from both sides only opposite in sign on either side. It can be seen that the value that the derivative stays constant at is the product of the angular velocity squared times the mass of the second particle. This implies that the only way to get a zero derivative in this equation is to either have zero mass for particle 2 or to have zero angular velocity. It does seem to suggest, however, that the speed has a much greater effect on this derivative than the mass does because it is squared. Although, due to

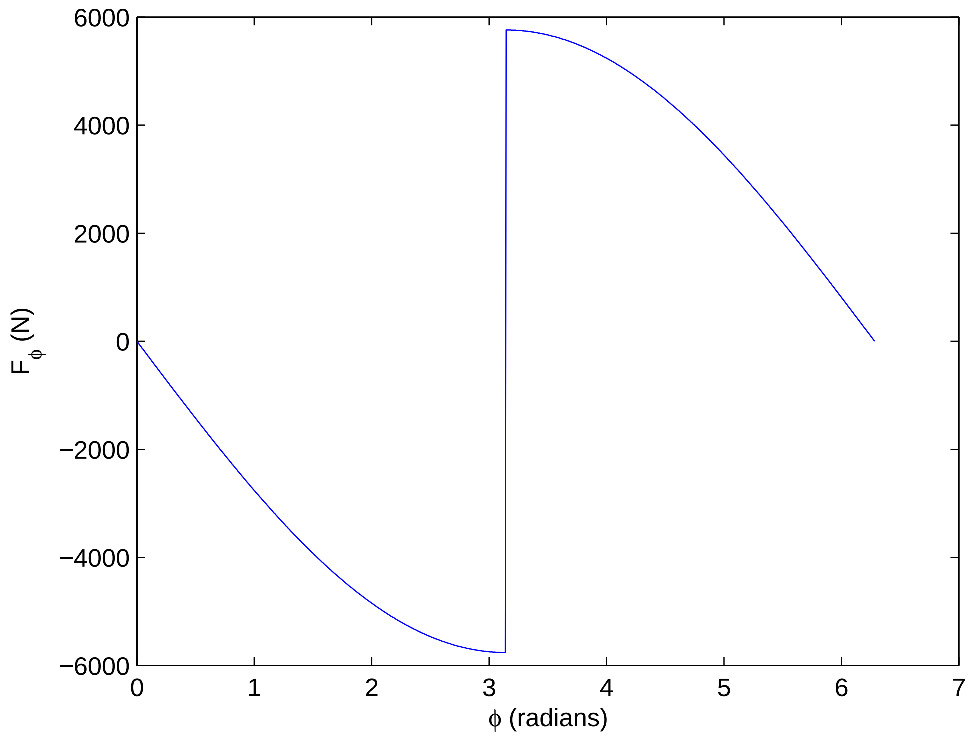


Figure 4.4: Derivative of Balance Force in $\phi$

the difference in units between the angular velocity and the mass, it is difficult to say this is true. Reiterating, in Fig. 4.4, the derivative gets larger as it approaches the balance point, implying that it is the most sensitive near the balance point. This is the opposite of what is desired and this looks to be difficult to change as the whole shape of the graph must change.

Another similar method that could be used to analyze this system would be to use bifurcation. Bifurcation would discover how the independent variables $R_2$ and $\phi$ affect the balance point solution in Eq. 4.13. These solutions are then analyzed to see if they are stable or not and how any slight variations in the input parameters may affect the total solution. This bifurcation method is used in other dynamic balance applications such as the ADB's and can help find optimal designs for balance [6]. This analysis was deemed to be out of scope for this thesis and is work that can be examined in the future.

## 4.2 Three Particle System in Plane Using Cylindrical Vector Components

Adding another mass to the system is an easy step to add a bit more complexity and reality to the problem. Most engineering problems deal with many bodies and cannot be simplified to a problem with only few variables to be studied in detail. Adding one mass exemplifies how even minor additions to a problem can significantly increase the problem's difficulty. However, this will give an extra degree of freedom to the system that can be an avenue to a more flexible resultant balance force. A flexible balance force could allow for a reduction in the slope around the balance point or possibly multiple balance points in the system.

This problem, shown in Fig. 4.5 is basically the same as in Section 4.1 except with a mass C that is placed at a variable position $(R_3, \gamma)$ from the reference mass A but within the same plane as A and B. This will add another coordinate system at mass C that can be used to calculate the kinematics. All of the forces and parameters for body B and body A will remain the same as well as the $\omega$ and the $\alpha$ of the system. The kinetics for this system are again easily calculated because of the natural use of cylindrical coordinates. Again using the cylindrical coordinate system at body A ( $\hat{e}_X, \hat{e}_Y, \hat{e}_Z$ ) as the reference coordinate system and solving for all of the kinematic parameters is a simple task once the coordinate transformation is calculated. These

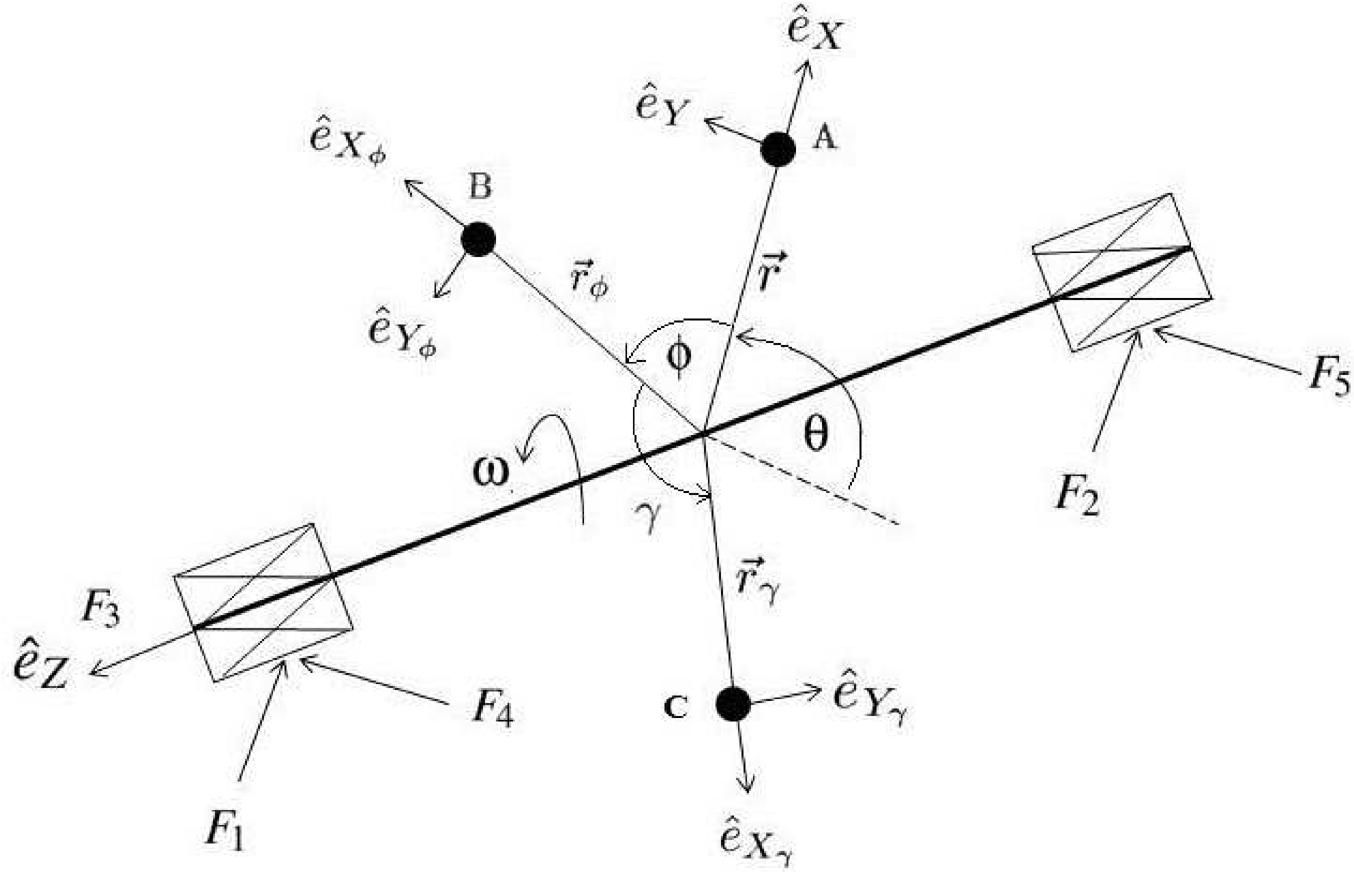


Figure 4.5: Simple Three Particle System

are the same equations as above just adding in the parameters for the added particle.

$$\begin{aligned}
\vec{r} &= R_1\hat{e}_X \\
\vec{r}_\phi &= R_2\hat{e}_{X_\phi} = R_2(\cos\phi\hat{e}_X + \sin\phi\hat{e}_Y) \\
\vec{r}_\gamma &= R_3\hat{e}_{X_\gamma} = R_3(\cos\gamma\hat{e}_X + \sin\gamma\hat{e}_Y) \\
\vec{v} &= R_1\omega\hat{e}_Y \\
\vec{v}_\phi &= R_2\omega(-\sin\phi\hat{e}_X + \cos\phi\hat{e}_Y) \\
\vec{v}_\gamma &= R_3\omega(-\sin\gamma\hat{e}_X + \cos\gamma\hat{e}_Y) \\
\vec{a} &= -R_1\omega^2\hat{e}_X \\
\vec{a}_\phi &= -R_2\omega^2(\cos\phi\hat{e}_X + \sin\phi\hat{e}_Y) \\
\vec{a}_\gamma &= -R_3\omega^2(\cos\gamma\hat{e}_X + \sin\gamma\hat{e}_Y)
\end{aligned}$$

Like in Section 4.1 the forces will be summed along the main body fixed axes of X Y and Z. These are also very similar to the equations listed above with the added term for the weight of the added third particle.

$$\sum F_X = F_1 + F_2 - m_1 g \cos\theta - m_2 g \cos(\theta + \phi) - m_3 g \cos(\theta + \gamma) \quad (4.20)$$

$$\sum F_Y = F_4 + F_5 + m_1 g \sin\theta + m_2 g \sin(\theta + \phi) + m_3 g \sin(\theta + \gamma) \quad (4.21)$$

$$\sum F_Z = F_3 \quad (4.22)$$

When summing the moments, the only change is again the added weight of the third particle in the sum of moments about the Z axis. From intuition, the way to have the terms in the moment about the Z axis to equal zero is if they are evenly spaced (i.e., $\phi = \frac{\pi}{3}$ and $\gamma = \frac{2\pi}{3}$) and for the masses to be equal. This will put each mass at 60 degrees from each other and this symmetry will eliminate the imbalance when the system is rotating.

$$\begin{aligned} \sum M_X &= F_5(\frac{L}{2}) - F_4(\frac{L}{2}) = 0 \\ &\Rightarrow F_5 = F_4 \quad (4.23) \\ \sum M_Y &= F_1(\frac{L}{2}) - F_2(\frac{L}{2}) = 0 \\ &\Rightarrow F_1 = F_2 \quad (4.24) \\ \sum M_Z &= -m_3 g \sin(\gamma + \phi) - m_2 g \sin(\theta + \phi) - m_1 g \sin\theta \end{aligned}$$

Combining 4.20-4.24 and solving for the individual forces yields the following set of equations for the individual forces. These equations have the same terms as the two body problem except with an additional term for the third body.

$$\begin{aligned}
F_1 &= \frac{m_1}{2}(g\cos\theta - R_1\omega^2) + \frac{m_2}{2}(g\cos(\theta+\phi) - R_2\omega^2\cos\phi) \\
&+ \frac{m_3}{2}(g\cos(\theta+\gamma) - R_3\omega^2\cos\gamma) && (4.25)\\
F_2 &= \frac{m_1}{2}(g\cos\theta - R_1\omega^2) + \frac{m_2}{2}(g\cos(\theta+\phi) - R_2\omega^2\cos\phi) \\
&+ \frac{m_3}{2}(g\cos(\theta+\gamma) - R_3\omega^2\cos\gamma) && (4.26)\\
F_3 &= 0 && (4.27)\\
F_4 &= -\frac{m_1}{2}g\sin\theta - \frac{m_2}{2}(g\sin(\theta+\phi) + R_2\omega^2\sin\phi) - \frac{m_3}{2}(g\sin(\theta+\gamma) \\
&+ R_3\omega^2\sin\gamma) && (4.28)\\
F_5 &= -\frac{m_1}{2}g\sin\theta - \frac{m_2}{2}(g\sin(\theta+\phi) + R_2\omega^2\sin\phi) - \frac{m_3}{2}(g\sin(\theta+\gamma) \\
&+ R_3\omega^2\sin\gamma) && (4.29)
\end{aligned}$$

### 4.2.1 Examination the Resultant Balance Force

Taking the same approach as in the two body case, the forces can be combined into a single resultant balance force $F_{res}$ that is in the same form. Specifically,

$$F_{res} = \sqrt{{F_R}^2 + {F_\theta}^2}. \qquad (4.30)$$

Combining $F_1$ and $F_2$ to be $F_R$ and $F_4$ and $F_5$ to be $F_\theta$ results in the following.

$$\begin{aligned}
F_{res} &= \\
&\quad [\left(\omega^2\,R_3\,m_3\,\sin(\gamma) + \omega^2\,R_2\,m_2\,\sin(\phi)\right)^2 \\
&+ \left(\omega^2\,R_1\,m_1 + \omega^2\,R_3\,m_3\,\cos(\gamma) + \omega^2\,R_2\,m_2\,\cos(\phi)\right)^2]^{\frac{1}{2}} && (4.31)
\end{aligned}$$

In order to depict this equation in a similar fashion to the three-dimensional graph provided in Section 4.1.1, a set of parameters must be chosen to limit the variables to two independent variables ($R_3$ and $\gamma$) and one dependent variable ($F_{res}$). The radius and angle for mass A and mass B, along with all three particles' mass and angular velocity, need to be established in this system in order to fully define the system. With this in mind, it must be understood that this is a representation of one configuration among an infinite number. However, this does not undermine the results as a good deal of understanding can still be attained with the analysis of this single case. This also shows the challenge of many if not all design processes, it is generally not possible to know how a design will work or perform until after the design parameters are established and chosen. In most cases the mass of the particles are given and known, but their placement may have more freedom. But a choice must be made for their placement and analyzed before the performance can be evaluated. The point of this exercise is to put this decision as far as possible into the analysis and to keep as much as possible a variable to a notably better degree than the standard method of Cartesian coordinate system based analysis. Listed below are the chosen parameters for this example.

$$m_1 = m_2 = m_3 = 40kg \quad , \quad \omega = 12rad/s$$
$$R_1 = R_2 = 1m \quad , \quad \phi = \frac{2\pi}{3}rad$$

Figure 4.6 shows a similar result as in the two body case with similar parameters. There is a single point at which the system has a zero resultant balance force and occurs when the masses are arranged in a perfectly symmetrical pattern, each 120 degrees apart from one another, and at an equal distance from the axis of rotation. Again this result is not a surprise because of the simplicity of the problem. It is expected that there only be one configuration for mass C to cause the system to be balanced.

#### 4.2.2 Manipulation of Design Parameters to Create a Zero Derivative at the Balance Point

Using the same method as in Sec. 4.1.2, the values for $R_3$ and $\gamma$ at the zero point can be found to be

$$R_3 = R_2 = 40kg \quad , \quad \gamma = \frac{4\pi}{3}rad. \tag{4.32}$$

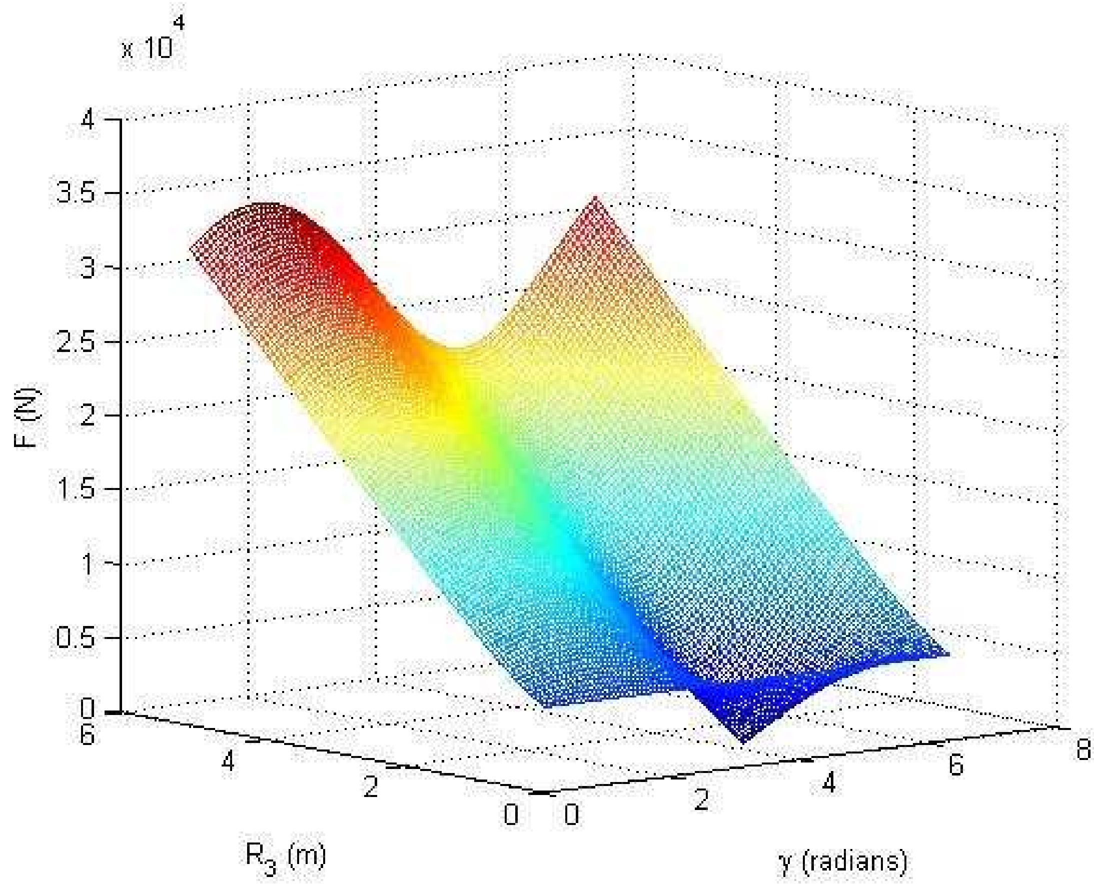


Figure 4.6: Simple Three Particle System Balance Force

Evaluating the resultant balance force at this point and taking the derivative will give an indication if the sensitivity of the balance point can be manipulated. First, substituting in $\gamma = \frac{4\pi}{3} rad$ and taking the partial derivative of the resultant balance force $F_{res}$ with respect to $R_3$ will give the equation for the slope of the line intersecting the balance point in the $R_3$ direction.

$$\frac{\partial}{\partial R_3} F_{res}|_{\gamma=\frac{4\pi}{3}} = \frac{P}{Q} \tag{4.33}$$

where

$$\begin{aligned} P &= -\omega^2\, m_3\, (\omega^2\, R_1\, m_1 - \frac{\omega^2\, R_3\, m_3}{2} + \omega^2\, R_2\, m_2\, \cos(\phi)) \\ &\quad -\sqrt{3}\,\omega^2\, m_3\, (\frac{\sqrt{3}\,\omega^2\, R_3\, m_3}{2} - \omega^2\, R_2\, m_2\, \sin(\phi)) \end{aligned} \tag{4.34}$$

$$
\begin{aligned}
Q &= 2\,[\frac{\sqrt{3}\,\omega^2\,R_3\,m_3}{2} - \omega^2\,R_2\,m_2\,\sin(\phi)^2 \\
&\quad +(\omega^2\,R_1\,m_1 - \frac{\omega^2\,R_3\,m_3}{2} + \omega^2\,R_2\,m_2\,\cos(\phi)^2]^{\frac{1}{2}}
\end{aligned} \tag{4.35}
$$

Similarly, substituting $R_3 = R_2$ and taking the partial derivative of the resultant balance force $F_{res}$ with respect to $\gamma$ gives the slope of the line intersecting the balance point in the $\gamma$ direction.

$$
\frac{\partial}{\partial \gamma} F_{res}|_{R_3=R_2} = \frac{M}{N} \tag{4.36}
$$

where

$$
\begin{aligned}
M = &\quad 2\,\omega^2\,R_2\,m_3\,\cos(\gamma)\,\left(\omega^2\,R_2\,m_3\,\sin(\gamma) + \omega^2\,R_2\,m_2\,\sin(\phi)\right) \\
- &\quad 2\,\omega^2\,R_2\,m_3\,\sin(\gamma)\,\left(\omega^2\,R_1\,m_1 + \omega^2\,R_2\,m_3\,\cos(\gamma) + \omega^2\,R_2\,m_2\,\cos(\phi)\right)
\end{aligned} \tag{4.37}
$$

and

$$
\begin{aligned}
N = &\quad 2\,[\left(\omega^2\,R_2\,m_3\,\sin(\gamma) + \omega^2\,R_2\,m_2\,\sin(\phi)\right)^2 \\
+ &\quad \left(\omega^2\,R_1\,m_1 + \omega^2\,R_2\,m_3\,\cos(\gamma) + \omega^2\,R_2\,m_2\,\cos(\phi)\right)^2]^{\frac{1}{2}}
\end{aligned} \tag{4.38}
$$

Like in Section 4.1.2, the equations are very complicated and it is hard to see if these can be manipulated in any advantageous way. An easy way to get more insight is to graph them and try to deduce any information from these graphs. They are shown in Figure 4.7 and Figure 4.8.

The results from Figure 4.7 are the same as in the two body case. The balance point for this system has a very unstable solution with regards to sensitivity and it does not appear that the derivative can be changed at that point. The discrete nature of these particle problems is the main contributing factor in the balance point being so unstable and singular. With all of the mass being distributed in exact particles about the axis of rotation, it leaves only a single arrangement to balance the system and

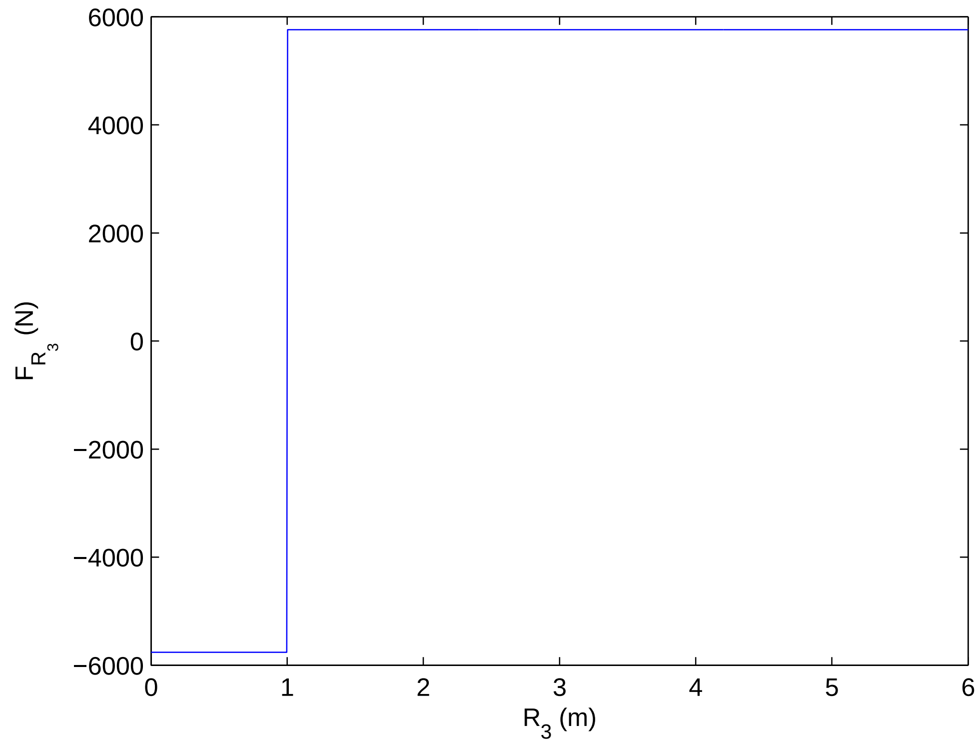


Figure 4.7: Derivative of Balance Force in $R_3$

any variation, however small, causes the system to be out of balance in a relatively large way.

It should be noted that there are an infinite number of different cases which will yield a different solution for $R_3$ and $\gamma$. The other cases have been found to only have a single solution and with a similar if not exact derivative. The only case which results in a different type of solution is where $\phi = \pi$. In this scenario, if $R_3$ or $\gamma$ are non-zero, then the system is out of balance and the balance force varies linearly with $R_3$.

### 4.2.3 General Form of Individual Forces for Multiple Particle Problems

If it is necessary or beneficial to expand this problem to a higher number of discrete particles, a method of expanding the equation to an infinite number of particles can be obtained. From the individual force equations from Section 4.1, Eq's

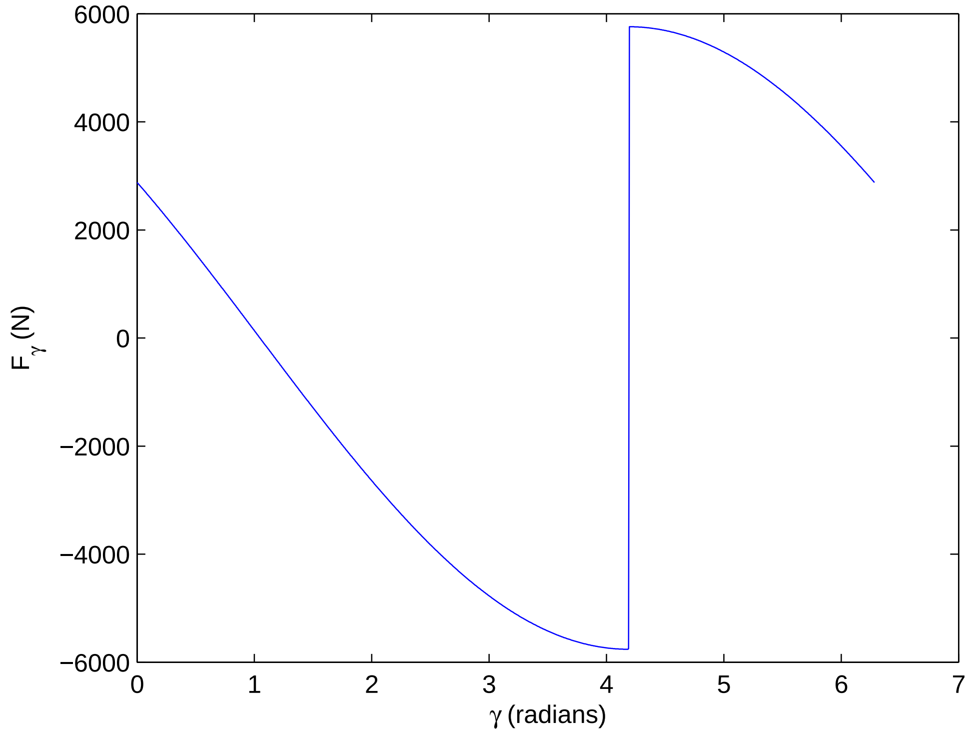


Figure 4.8: Derivative of Balance Force in $\gamma$

4.25-4.29, it can be deduced that any subsequent masses will merely be added onto the end of the equation. Each of the particles adds a specifically formatted term in the force equation and can be easily formulated into a sum. By rewriting the individual force equations as a sum, this problem can theoretically be looked at with any number of particles (n) for a more complex and deeper look into the problem. Below are the forces written as a general sum.

$$F_1 = F_2 = \sum_{i=1}^{n} \frac{m_i}{2}[g\cos(\theta + \Theta_i) - R_i\omega^2\cos(\Theta_i)] \tag{4.39}$$

$$F_3 = 0 \tag{4.40}$$

$$F_4 = F_5 = \sum_{i=1}^{n} -\frac{m_i}{2}[g\sin(\theta + \Theta_i) + R_i\omega^2\sin(\Theta_i)] \tag{4.41}$$

The variable $\Theta_i$ is the corresponding angle from the reference coordinate system to that particles coordinate system. To better present this variable for clarity, it is written below in a vector format for the three particle problem.

$$\Theta = [0, \phi, \gamma] \tag{4.42}$$

This form of equation could be useful as the governing equation in a computer simulation. Possibly adding an increasing number of discrete particles can result in some unforseen results in terms of balance. Essentially, when enough particles are added to the system, it should begin to behave like any rigid body the particles are arranged to resemble. This opens a door to the study of the grey area between a system with a large number of discrete particles and a system composed of a single rigid body. When approaching an infinite number of discrete particles, at what point or number of particles can the system be regarded as a single rigid body? It can be a door to some new and interesting research.

## 4.3 Summary of Results from Particle Problems

These particle problems in Chapter 4 show important aspects relating to the goal of this research. First of all it shows an example of a purely cylindrical vector component problem being solved. While this is not a proof in the purest sense, it is still a good validation of the use of this type of formulation. Second, it details how the formulation of this type of problem will be more convenient to use in problem with single axis rotation. These problems show that even using the particle version of the Euler's equations, using the more intuitive and appropriate cylindrical vector component notation can lead to a more elegant solution. Lastly, while it does not yet achieve the goal of finding a way to decrease the slope of the resultant balance force around the single balance point, it does give an avenue to use on similar problems. These are just two problems used as an example of this method. There may be different applications or another strategy to use on the method described here to achieve the desired result.

# CHAPTER 5

## Rigid Body Problem Using Cylindrical Vector Components

In this chapter an example problem will be solved using Euler's equations of motion derived in cylindrical vector components. The problem presented will include the inertias of the rotating bodies and not merely treat them as point particles with only mass. By including the inertias, the problem will show how the shape and size of the rotating body affects the dynamic balance or torque requirements. All of the major points used in the previous chapters will be utilized in this problem making it the generalization of the previous work. The problem itself is also a great representation of many different types of real-world applications in engineering and industry. Like the particle problems in Chapter 4, this can be thought of as a CT scanner with more fidelity in the model.

### 5.1 Rigid Body Problem Example

Two rigid bodies, body A and body B, in the form of identical cylinders are placed at distance ($R_1$ and $R_2$) from a rotating axis and are situated at an angle $\phi$ apart from each other similar to the two particle problem in Chapter 4. Each cylinder has a mass $m$, length of $l$, and a radius of $r$ and are positioned so that the $I_{ZZ}$ axis is always parallel to the rotating axis. Each has a coordinate system attached to them that rotates with the cylinders. The bodies are rotating about the axis at an angular velocity of $\omega$ and angular acceleration $\alpha$ and the angle between the bodies does not change, i.e., they are both rotating at the same angular velocity.

It is important to note that in this example, the $R_1$ and $R_2$ vectors only go to the outside edge of the cylinder and not to the center of mass as depicted in Fig. 5.2. This decision was made to keep the more generality in the problem. If the shape of

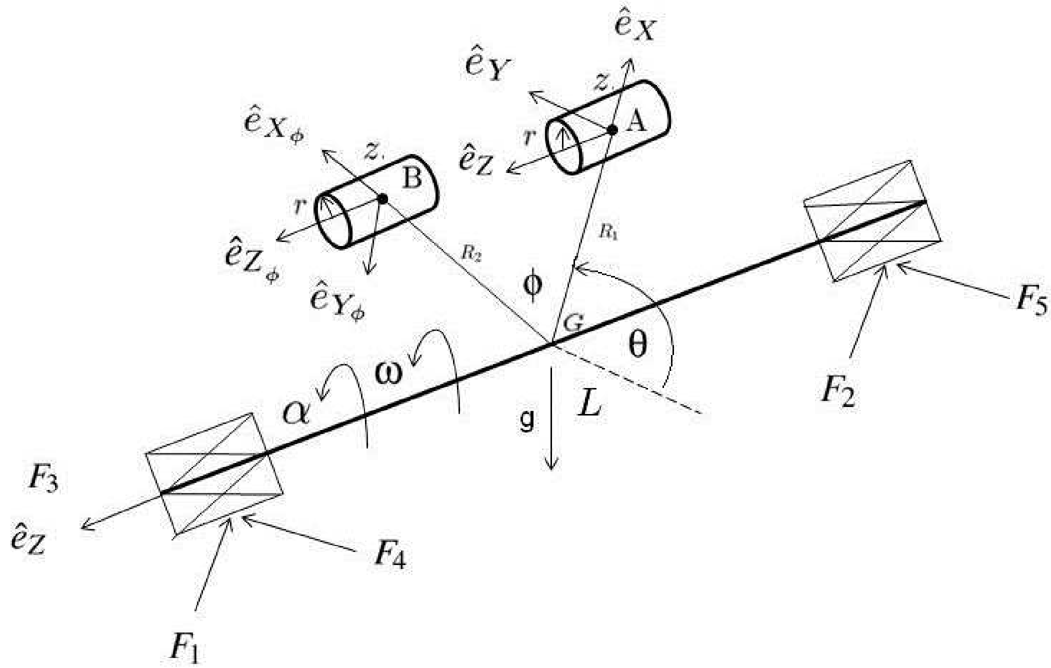


Figure 5.1: Rigid Body Single Axis Motion Example

the cylinder changes, it is easy to quantify the effect by preserving the $r$ value in the problem.

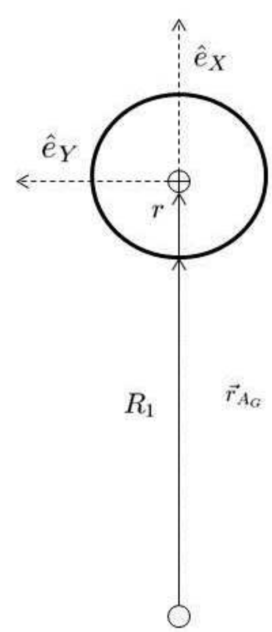


Figure 5.2: Profile View of Cylinder

Calculating the kinematics of the problem is very similar to how it was done in Chapter 3. All of the values are expressed in terms of a main coordinate system that is attached to body A.

$$\begin{aligned}
\vec{r}_{A_G} &= (R_1 + r)\hat{e}_X \\
\vec{r}_{B_G} &= (R_2 + r)\hat{e}_{X_\phi} = (R_2 + r)(\cos\phi\hat{e}_X + \sin\phi\hat{e}_Y) \\
\vec{v}_{A_G} &= (R_1 + r)\omega\hat{e}_Y \\
\vec{v}_{B_G} &= (R_2 + r)\omega(-\sin\phi\hat{e}_X + \cos\phi\hat{e}_Y) \\
\vec{a}_{A_G} &= -(R_1 + r)\omega^2\hat{e}_X \\
\vec{a}_{B_G} &= -(R_2 + r)\omega^2(\cos\phi\hat{e}_X + \sin\phi\hat{e}_Y) \\
\vec{\omega} &= \omega\hat{e}_Z \\
\vec{\alpha} &= \dot{\omega}\hat{e}_Z = 0
\end{aligned}$$

Next, summing the forces along the $XYZ$ coordinate system that is attached to body A.

$$\begin{aligned}
\sum F_X &= F_1 + F_2 - m_1 g\cos\theta - m_2 g\cos(\theta + \phi) \\
\sum F_Y &= F_4 + F_5 - m_1 g\sin\theta - m_2 g\sin(\theta + \phi) \\
\sum F_Z &= F_3
\end{aligned}$$

The goal is to create a scenario where the sum of the moments can be taken for both bodies at one time. To accomplish this, the bodies have to be expressed at a mutual point that satisfies the conditions for the change in angular momentum. The point that makes the most sense is the center of the axis of rotation where the bodies are connected, point G.

To express the inertia of body A at G, the parallel axis theorem, Eq. 3.7, derived in Chapter 3 will be used. Below, in the same format they are listed in

Chapter 3, are the parameters that will transfer the inertia of body A to point G. Again, it is important to note that $\phi$ is $\pi$ in this example and not 0 and that the shift along the X axis is the summation of $r$ and $R_1$ based on how the problem is set up.

$$\begin{aligned} r_{A_o} &= -(r + R_1) \\ \phi_{A_o} &= \pi \\ z_{A_o} &= 0 \end{aligned}$$

Taking these values and substituting them into the parallel axis theorem described in Chapter 3 results in a new inertia matrix defined as $[I]'_A$ with components shown below.

$$\begin{aligned} I_{X'X'} &= I_{XX} + m[(r + R_1)^2 \sin^2(\pi)] = \frac{1}{12}m[3r^2 + l^2] \\ I_{Y'Y'} &= I_{YY} + m[(r + R_1)^2 \cos^2(\pi)] = \frac{1}{12}m[3r^2 + l^2] + m(r + R_1)^2 \\ I_{Z'Z'} &= I_{ZZ} + m(r + R_1)^2 = \frac{1}{2}mr^2 + m(r + R_1)^2 \\ I_{X'Y'} &= I_{XY} + m[(r + R_1)^2 \sin(\pi)\cos(\pi)] = 0 \\ I_{X'Z'} &= I_{XZ} + m(-r - R_1)(0)\cos\pi = 0 \\ I_{Y'Z'} &= I_{YZ} + m(-r - R_1)(0)\sin\pi = 0 \end{aligned}$$

Taking these results and expressing them in an inertia matrix shows the effect of moving the coordinate system to G. There are extra terms on the $I_{YY}$ and $I_{ZZ}$ values which are due to the radius between the center of mass of the cylinder and the axis of rotation.

$$[I]'_A = \begin{bmatrix} \frac{1}{12}m[3r^2 + l^2] & 0 & 0 \\ 0 & \frac{1}{12}m[3r^2 + l^2] + m(r + R_1)^2 & 0 \\ 0 & 0 & \frac{1}{2}mr^2 + m(r + R_1)^2 \end{bmatrix}$$

Now, the exact same calculation for body B to bring it to the center of rotation needs to be performed. The values listed below are consistent with the form from Chapter 3 and are very similar to that of Body A.

$$\begin{aligned} r_{B_o} &= -(r + R_2) \\ \phi_{B_o} &= \pi \\ z_{B_o} &= 0 \end{aligned}$$

The calculation of the parallel axis theorem for body B is carried out in the same fashion as body A. Specifically:

$$\begin{aligned} I_{X'X'_B} &= I_{XX_B} + m[(r + R_2)^2 \sin^2(\pi)] = \frac{1}{12} m[3r^2 + l^2] \\ I_{Y'Y'_B} &= I_{YY_B} + m[(r + R_2)^2 \cos^2(\pi)] = \frac{1}{12} m[3r^2 + l^2] + m(r + R_2)^2 \\ I_{Z'Z'_B} &= I_{ZZ_B} + m(r + R_2)^2 = \frac{1}{2} mr^2 + m(r + R_2)^2 \\ I_{X'Y'_B} &= I_{XY_B} + m[(r + R_2)^2 \sin(\pi) \cos(\pi)] = 0 \\ I_{X'Z'_B} &= I_{XZ_B} + m(-r - R_2)(0) \cos \pi = 0 \\ I_{Y'Z'_B} &= I_{YZ_B} + m(-r - R_2)(0) \sin \pi = 0 \end{aligned}$$

At this point, both cylinders are expressed at the center of rotation, G. However, body B is still rotated by the angle $\phi$. This prevents the inertias being summed together so that the sum of the moments can be expressed for both bodies in one calculation. To fix this issue, body B needs to be rotated by the negative angle of $\phi$ to align its axes with the axes of body A. This is accomplished by the use of a similarity transformation about the Z axis of an angle of $-\phi$ [3].

$$[I]''_B = [R][I]'_B[R]^T$$

Substituting in the standard rotation transformation of $-\phi$ about the Z axis in for [R] results in the following.

$$[I]''_B = \begin{bmatrix} \cos(-\phi) & \sin(-\phi) & 0 \\ -\sin(-\phi) & \cos(-\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} [I]' \begin{bmatrix} \cos(-\phi) & -\sin(-\phi) & 0 \\ \sin(-\phi) & \cos(-\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying the three matrices and performing some simplification results in the following inertia matrix for body B.

$$[I]''_B = \begin{bmatrix} I_{X''X''_B} & I_{X''Y''_B} & I_{X''Z''_B} \\ I_{Y''X''_B} & I_{Y''Y''_B} & I_{Y''Z''_B} \\ I_{Z''X''_B} & I_{Z''Y''_B} & I_{Z''Z''_B} \end{bmatrix} \tag{5.1}$$

$$\begin{aligned}
I_{X''X''_B} &= \frac{1}{12}m[3r^2 + l^2] + m(r + R_2)^2 \sin^2(-\phi) \\
I_{Y''Y''_B} &= \frac{1}{12}m[3r^2 + l^2] + m(r + R_2)^2 \cos^2(-\phi) \\
I_{Z''Z''_B} &= \frac{1}{2}mr^2 + m(r + R_2)^2 \\
I_{X''Y''_B} &= = I_{Y''X''_B} = \frac{1}{2}m(r + R_2)^2 \sin(-2\phi) \\
I_{X''Z''_B} &= = I_{Z''X''_B} = 0 \\
I_{Y''Z''_B} &= = I_{Z''Y''_B} = 0
\end{aligned}$$

This matrix defines the inertia for the cylinder that was rotated an angle $\phi$ and at a distance $R_2 + r$ away from the center of rotation at the point G. The rotation transformation that was performed aligns the axes for body A and body B and allows for the summation of inertias of both bodies. This step greatly simplifies the problem because only one summation of moments and application of Euler's equations needs to be performed. Also, with this result, all of the parameters for both cylinders and the problem are still visible during any analysis. Performing any sensitivity analysis should be straightforward since each parameters has clear (and direct) influence on the inertia or moment equations. The individual elements in the inertia matrix for body A are added to the corresponding element in the inertia matrix for body B. This addition is allowable since the bodies are not moving with respect to each other and can be considered one object.

$$[I]_{total} = [I]'_A + [I]''_B$$

$$\begin{aligned}
I_{XX_{total}} &= \frac{1}{6}m[3r^2 + l^2] + m(r + R_2)^2 \sin^2(\phi) \\
I_{YY_{total}} &= \frac{1}{6}m[3r^2 + l^2] + m(r + R_2)^2 \cos^2(\phi) \\
I_{ZZ_{total}} &= mr^2 + m(r + R_1)^2 + m(r + R_2)^2 \\
I_{XY_{total}} &= I_{YX_{total}} = -\frac{1}{2}m(r + R_2)^2 \sin(2\phi)
\end{aligned} \tag{5.2}$$

Performing the analysis for dynamic stability now only requires one application of Euler's equations. Summing the moments about the X Y and Z axes of body A results in the following. To see the many quantities that were omitted because they were zero, refer to Chapter 2.

$$\begin{aligned}
\sum M_X &= F_5(\frac{L}{2}) - F_4(\frac{L}{2}) = -I_{XZ}\alpha_Z + I_{ZY}\omega_Z^2 \\
\sum M_Y &= F_1(\frac{L}{2}) - F_2(\frac{L}{2}) = -I_{YZ}\alpha_Z + I_{XZ}\omega_Z^2 \\
\sum M_Z &= 0
\end{aligned}$$

There are no moments about the $X$ or $Y$ axes and the inertia components $I_{XZ}$ and $I_{ZY}$ are both zero. This results in the forces in the X and Y planes being equal to each other similar to the particle problems in Chapter 4. The main focus for the moment equations is on the torque component about the Z axes. The summation of moments shows that, as expected, the greater the radius of the cylinders and their distance from the axis of rotation the more torque is required for a given angular acceleration $\dot{\omega}_Z$.

$$\begin{aligned}
0 &= F_5(\frac{L}{2}) - F_4(\frac{L}{2}) \\
0 &= F_1(\frac{L}{2}) - F_2(\frac{L}{2}) \\
T &= 0
\end{aligned}$$

Combining these equations with the summation of the forces equations below fully define the system and any sensitivity analysis can be performed similar to Chapter 3.

$$\begin{aligned}
F_1 &= \frac{m_1}{2}(g\cos\theta - (R_1 + r)\omega^2) + \frac{m_2}{2}(g\cos(\theta + \phi) - (R_2 + r)\omega^2\cos\phi) \\
F_2 &= \frac{m_1}{2}(g\cos\theta - (R_1 + r)\omega^2) + \frac{m_2}{2}(g\cos(\theta + \phi) - (R_2 + r)\omega^2\cos\phi) \\
F_3 &= 0 \\
F_4 &= -\frac{m_1}{2}g\sin\theta - \frac{m_2}{2}(g\sin(\theta + \phi) + (R_2 + r)\omega^2\sin\phi) \\
F_5 &= -\frac{m_1}{2}g\sin\theta - \frac{m_2}{2}(g\sin(\theta + \phi) + (R_2 + r)\omega^2\sin\phi)
\end{aligned}$$

The above results mirror nearly identically to the results shown in Eq. 4.6 to 4.10 from the two particle problem. The only difference is that the radius of the cylinder, $r$, is added into the problem. If the problem was set up so that the $R_1$ was from the axis of rotation to the center of mass for the cylinder, the above equations would be exactly identical to the equations for the rigid body cylinders.

These equations show an interesting conclusion. When studying the dynamic balance of a system rotating about a **single axis**, if the rotating bodies can be assumed to be identical in shape and principal axes can be applied, the bodies can be treated as point masses and will result in the same equations as if they were treated as rigid bodies. This greatly simplifies the problem in that the extra calculations of inertia need not be performed.

## 5.2 Examination of the Balance Force

Taking the same approach as in Sec. 4.1.1, the forces can be combined into a single resultant balance force $F_{res}$ that is in the same form. Specifically,

$$F_{res} = \sqrt{{F_R}^2 + {F_\theta}^2}. \tag{5.3}$$

Combining $F_1$ and $F_2$ to be $F_R$ and $F_4$ and $F_5$ to be $F_\theta$ results in the following.

$$F_{res} = \sqrt{\left(m_1\, w^2\ (R_1 + r) + m_2\, w^2\ \cos(\phi)\ (R_2 + r)\right)^2 + {m_2}^2\, w^4 \sin(\phi)^2\, (R_2 + r)^2} \tag{5.4}$$

As was mentioned in the previous section, the only difference between this form of the equation and Eq. 4.13 in Sec. 4.1.1 is the difference in the definition of the radii. In this formulation the radii is presented as the summation of the radius from the axis of rotation to the edge of the cylinder ($R_1$ and $R_2$) and the radius of the cylinders themselves ($r$). If this summation was substituted for the radius from the axis of rotation to the center of gravity for the cylinder, the formulation would be exactly the same. This formulation was chosen to show how the radius of the cylinder itself can directly affect the dynamic balance of the problem.

At this point, it is easy to see the the results following this step will be consistent with the results in Chapter 4. If similar parameters are chosen and substituted into Eq 5.4 the 3-D plot for the resultant balance force can be plotted. One difference is that the radius value of 2 will be equal to $R_2 + r$ instead of just $R_1$. This is consistent with the particle version because it is the distance to the center of gravity.

$$m_1 = 40kg \quad , \quad m_2 = 40kg$$

$$(R_1 + r) = 1m \quad , \quad \omega = 12rad/s$$

Fig. 5.3 is identical to the version from Chapter 4 with two particles, Fig 4.2. This is a consistent result and is expected since the equations for the balance force are identical.

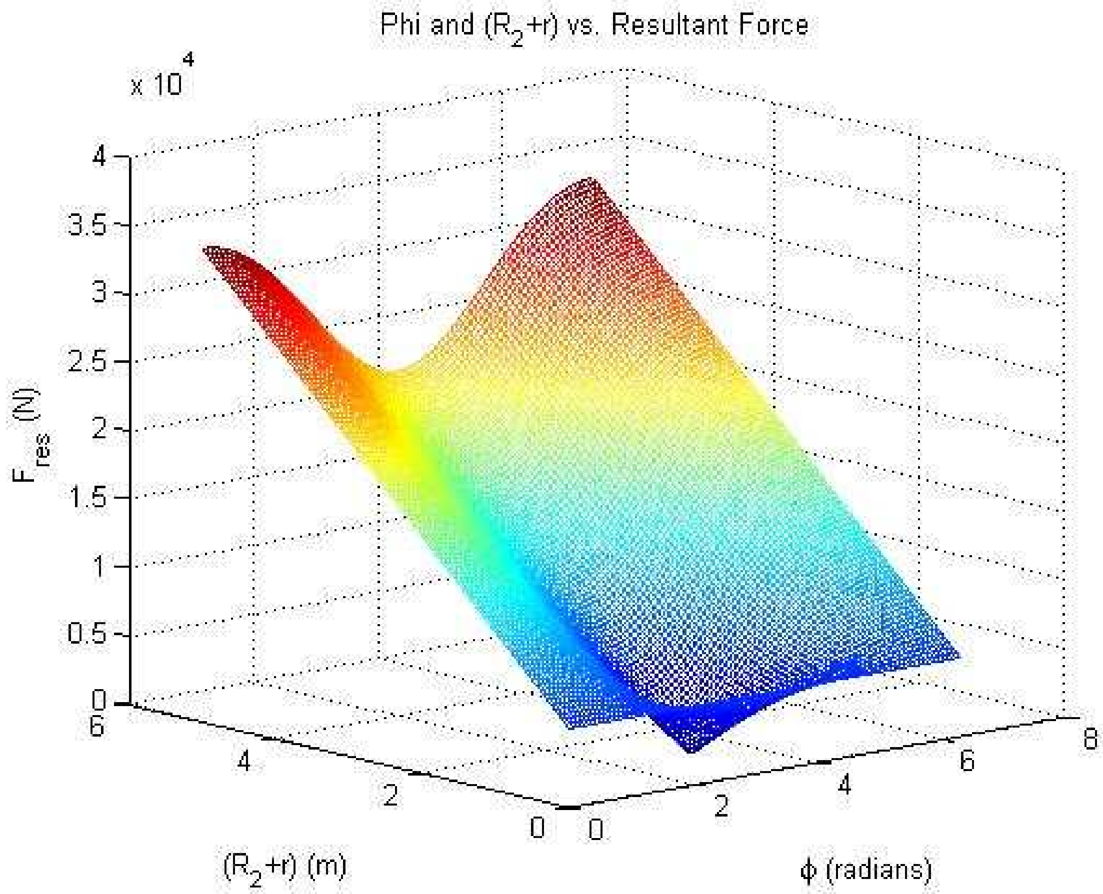


Figure 5.3: Simple Two Rigid Body System Balance Force

Further analysis of the derivatives for this would be redundant due to the similarity between this equation and the particle case. As stated in the previous section, if a problem has the unique set of parameters as in this rigid body problem it can be assumed to only contain particles. This will be a great benefit to reduce unnecessary calculations and allow for a more in depth analysis of dynamic balance.

## 5.3 General Form of Individual Forces and Moments for Rigid Body Problem with $n$ Cylinders

Similar to Sec. 4.2.3, there is a pattern in the equations that can be utilized to create a general form for the inertia matrix, individual forces and sum of moments for this type of problem. The assumptions for this problem are that each of the rigid bodies must be cylinders of equal radius, length and mass and that they are not rotating with respect to each other. Any number of bodies can be added to this problem at any given angle or radius and the following equations will be valid.

As with in Sec. 4.2.3, this is a great tool for a computer simulation or even utilizing hand calculations by eliminating most of the setup of the problem. This gives a quick shortcut to the analysis of the problem which will allow for a large number of parameters to be simply explored. Using the same convention as in Sec. 4.2.3, variable $\Theta_i$ is the corresponding angle from the reference coordinate system to that body's coordinate system. To better present this variable for clarity, it is written below in a vector format for the three body problem.

$$\Theta = [0, \phi, \gamma] \tag{5.5}$$

Substituting that angle variable into the sum of the forces equations from Sec. 5.1 results in the following equations.

$$\begin{aligned}
\sum F_X &= F_1 + F_2 + \sum_{i=1}^{n} -mg\cos(\theta + \Theta_i) \\
\sum F_Y &= F_4 + F_5 + \sum_{i=1}^{n} -mg\sin(\theta + \Theta_i) \\
\sum F_Z &= F_3
\end{aligned}$$

The inertia matrix is where the multiple bodies are captured in the problem. Instead of adding two inertia matrices together, there will be a summation of $n$ inertia matrices. One body must be the reference body so the first inertia matrix, $i = 1$, will not have the similarity transform for a rotated angle. This will be called matrix A. This also means that the summation will begin at body 2 so $i = 2$. The rest of the matrices will have a similar format and only differ by the angle rotated from the coordinate system and the radius from the axis of rotation.

$$[I]_{total} = [I]'_A + \sum_{i=2}^{n} [I]''_i$$

The above equation can be expanded into the individual non-zero components of the inertia matrix. In the $XX$, $YY$ and $XY$ components, the reference body will have only the regular cylinder inertia value and has no angle. This causes the summation to again begin at $i = 2$ for these values. In the $ZZ$ component each of the bodies adds an additional term to the component so the summation begins at $i = 1$.

$$\begin{aligned}
I_{XX_{total}} &= \frac{n}{12} m[3r^2 + l^2] + \sum_{i=2}^{n} m(r + R_i)^2 \sin^2(\Theta_i) \\
I_{YY_{total}} &= \frac{n}{12} m[3r^2 + l^2] + \sum_{i=2}^{n} m(r + R_i)^2 \cos^2(\Theta_i) \\
I_{ZZ_{total}} &= \frac{n}{2} mr^2 + \sum_{i=1}^{n} m(r + R_i)^2 \\
I_{XY_{total}} &= I_{YX_{total}} = \sum_{i=2}^{n} -\frac{1}{2} m(r + R_i)^2 \sin(2\Theta_i)
\end{aligned} \tag{5.6}$$

Using the above terms for the inertia matrix, the sum of the moments for the problem can be quickly and easily calculated using the equations below.

$$\begin{aligned}
\sum M_X &= F_5(\frac{L}{2}) - F_4(\frac{L}{2}) = -I_{XZ}\alpha_Z + I_{ZY}\omega_Z^2 \\
\sum M_Y &= F_1(\frac{L}{2}) - F_2(\frac{L}{2}) = -I_{YZ}\alpha_Z + I_{XZ}\omega_Z^2 \\
\sum M_Z &= 0
\end{aligned}$$

As in Sec. 4.2.3 these equations can be used by computer simulation models as the governing equation for a highly complex system involving a theoretical infinite number of cylinders rotating about a single axis. The equation's flexibility in the number of bodies, size of and orientation of those bodies with respect to each other is extremely beneficial in a design scenario. A number of iterative approaches can be solved quickly and the ideal design produced with a small amount of effort and time. This can lead to a thorough optimization analysis and mathematically prove that a certain set of parameters creates the best possible design with respect to the system's dynamic balance.

## 5.4 Summary of Results from Rigid Body Problem

This chapter was an example of the application of the Euler's equations of motion derived using cylindrical vector components. The example problem presented in this chapter shows the ease of use for this formulation when applied to systems involving single axis rotation. It provides a much more intuitive means of analysis by preserving the natural use of cylindrical coordinates throughout the problem and presents the final results in terms of design parameters such as angles, radii, and rigid body dimensions. Although the solution does not reveal any concrete answers for optimizing dynamic balance for this problem, it does give a new avenue to begin exploring different systems and problems for that type of output.

# CHAPTER 6

# Comparison and Discussion

This chapter explores the similarities and differences between the cylindrical vector component notation and other rotational focused notations. The previous chapters have illustrated the derivation and use of the cylindrical vector notation used in conjunction with Euler's equations. All of the previous problems could have been solved with any other number of methods or equations but the point of this particular notation is to create a more expedient and elegant way to get to the result while trying to preserve the connection between the solution and the problem parameters or inputs. The following sections are going to explore the advantages and disadvantages of this approach as compared with some of the approaches mentioned in the introduction.

## 6.1 Comparison Between Standard Cartesian Notation Euler's Equations and Cylindrical Vector Component Notation

For a classically trained engineer, the standard notation Euler's equations would be the obvious, and perhaps only, choice for evaluating the example problems presented. In standard Cartesian notation approach, all parameters would be expressed in an allowable body fixed or a space fixed $\hat{\imath}, \hat{\jmath}, \hat{k}$ coordinate system and would have to be related together by some transformation equation(s). This transformation equation would most likely be in some form of a rotation transformation that was applied at some point in the analysis. The major advantage that the Cylindrical Vector Components (CVC) approach is that the problem definition and parameters are explicitly identified and that definition flows very easily into the Euler's equation notation and subsequently the analysis.

In CVC, the link from the problem definition and input parameters to the result is very clear and this opens doors to find deeper ways to evaluate how these input parameters affect some of the outcomes of the problem. This can be seen in the resultant balance force analysis that is done in Sec. 4.1.1. With dynamic balance being of such critical importance to many rotational problems, this type of notation easily flows into and facilitates a very deep and complex look into dynamic balance. The further understanding of the relationship between the inputs and the outputs are the major benefit of the CVC approach.

One aspect that could be either an advantage or disadvantage depending upon the problem is that it works very well with problems whose parameters are naturally consistent with cylindrical coordinates. Generally for single axis rotation this would be an advantage since most problems will be defined using classic parameters such as angles and radii. However, in the case where the parameters are not already set up into these variables, some kind of trigonometry or other mathematical operations need to be performed to get it in the correct form.

One major drawback that of the CVC notation compared with the standard Cartesian notation is its applicability is limited to single axis rotation. While this give it the added advantage of being very easy and straightforward for single axis rotation problems, it is unable, or at least unclear, as to how it could be applied to additional axes of rotation. Perhaps it can be expanded to facilitate the multiple axis of rotation, however, this added functionally might take away from the beauty of this notation being simple and concise.

## 6.2 Comparison Between Euler Angles and Cylindrical Vector Component Notation.

The use of Euler angles and subsequent is one of the base lessons taught in dynamics. It is so powerful and widely known for a couple reasons. One of the main reasons is that they are so easily associated with a physical gimbal. Also, since most dynamics problems require the transformation from a body fixed coordinate system to a fixed space coordinate system, Euler angles are a great way to easily capture every transformation and are relatively easily applied and understood. The wide use of Euler angles along with its own strict guidelines has led it to become a standardized method in performing coordinate transformation.

However, the power and the robustness of Euler angles can also have some negative side effects. One side effect is that to completely apply the Euler angles correctly, a lot of information is needed. The angles of rotation, which convention of elemental rotation is being used, and the orientation of the fixed coordinate system all need to be known for the transformation to work properly. Also, there is a special case in which the Euler angles will lead to a phenomenon called "gimbal lock" or cause a singularity. This occurs when a special case of generally orthogonal angles cause a loss of degree of freedom where a change in either of two Euler angles results in the same change in orientation about a certain axis.

This problem occurs because of the trigonometric functions that are used in the Euler angle matrices which can cause a matrix to become an identity matrix, effectively removing an intermediate rotation. The following equations shows a how this situation can occur.

$$[R] = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.1}$$

Assuming that the above equation is a series of rotations and that each variable $(\alpha, \beta, \gamma)$ is an angle of rotation. Next, assume that $\beta = 0$ which will cause the cosine terms to be 1 and the sine terms to be 0 in the middle matrix. By inspection it can be seen that the middle matrix becomes the identity matrix. If the matrices are multiplied out and simplified they can be presented in the following equation.

$$[R] = \begin{bmatrix} \cos(\alpha+\gamma) & -\sin(\alpha+\gamma) & 0 \\ \sin(\alpha+\gamma) & \cos(\alpha+\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.2}$$

This rotation matrix has lost a degree of freedom because the last row and last column cannot be changed thus the rotation axis will always be around the Z axis. An advantage to using the CVC notation is that it does not require the use of matrices to multiply the coordinates by to transform them into the fixed coordinate system. The main coordinate system is attached to a single entity and all others are related back by the use of a known positional angle. Mainly, this advantage is due to the fact that no accommodations are made in this notation to facilitate multiple axis rotation.

### 6.3 Comparison Between Axis-Angle and Cylindrical Vector Component Notation.

The main advantage to the axis-angle approach to transforming a coordinate system is to reduce some of the calculations needed. Instead of having to multiply three different matrices to get to the final matrix used to perform the transformation, a few inputs into an equation which generates the full rotation matrix is done in one step. It is always very useful in real world applications because most objects in the real world do not rotate about three fixed axes but rather about random axes. If that random axis has a known orientation and the angle to be rotated is known, it is very easy to enter into an axis-angle equation and calculate the new orientation quickly.

However, the axis-angle approach has many of the same pitfalls as the Euler angle and rotation matrix approach. These can fall victim to singularities and rounding errors due to the fact that there are still sine and cosine terms involved. Also, while obvious, it is necessary to know the rotation vector orientation to employ this method. The axis-angle approach is generally a more robust version of the cylindrical vector component notation in that they both deal with rotations about a single axis; however, the CVC notation is better suited for rotation about an axis coincident with a coordinate axis.

## 6.4 Comparison Between Rodrigues' Rotation Formula and Cylindrical Vector Component Notation

Perhaps the most similar of the other notations to the CVC notation is the Rodrigues' rotation formula. The Rodrigues' rotation formula uses the same basic concept of a vector being broken into two components, one being a projection onto the orthogonal plane to the Z axis and a vector that is parallel or coincident with the Z axis [8]. Rodrigues' formula is as follows, where $\vec{v}$ is a vector which is being rotated $\theta$ about the unit vector $\vec{k}$.

$$\vec{v}_{rot} = \vec{v}\cos\theta + (\vec{k}\times\vec{v})\sin\theta + \vec{k}(\vec{k}\cdot\vec{v})(1-\cos\theta) \tag{6.3}$$

It specializes in the calculation of orientation for a vector after a single axis rotation.

The beauty of the formula is that it can calculate the new orientation independently from the orientation in space of the reference frame in which the new vector and rotation are about. Basically, with this formula, any vector can be easily rotated about any unit vector in space regardless of its orientation.

The issue with Rodrigues' formula is that for each rotation about an axis a new calculation needs to be performed. It is not obvious how this notation could easily show how several particles rotating about a single axis could be easily described in reference to one another. Also, once the transformation is complete and the formula calculated, the information of where the particle came from is lost. In the CVC notation, the particles location in space relative to the main coordinate system is still preserved by the fact that those parameters are easily discerned from the position vector. This keeps the clarity in the link between the results and the input parameters and leads to a greater understanding of the effect on the results that each have.

## 6.5 Comparison Between Quaternion Rotations and Cylindrical Vector Component Notation

Quaternion rotations are basically a way to neatly package a axis-angle rotation, which utilizes a vector unit axis and a scalar angle. The format for a quaternion would be as follows.

$$z = a + b\hat{i} + c\hat{j} + d\hat{k} \tag{6.4}$$

The formulation eliminates the need for sines and cosines to be used in the transformation which leads to removing the round off error as well as the issue of gimbal lock. Also, the representation of a quaternion is much more compact than any rotation matrix and is easily transferred to an axis-angle representation. A benefit to the more compact representation is directly applicable to computer systems as it takes much less memory to store a quaternion versus a rotation matrix.

The major downfall for quaternions is really not the fault of quaternions at all and that is that they just are not well known or used by the common engineer. It is very difficult to find a classically trained engineer with the working knowledge of a quaternion to be able to apply it to a problem they are working on. That being said, a quaternion is really only a way to represent a rotation transformation elegantly, but still requires some math to actually carry out the transformation. Again, with the simplicity of the CVC notation, it is able to handle a simple rotation in a straightforward easy to understand way which any classically trained engineer would understand.

## 6.6 Summary of Comparisons

The main point that is being made is echoing what was stated in the introduction chapter. The CVC notation does not allow for any problems to be solved that previously could not, it merely is a simplified version of some very robust methods of performing rotation transformations that can be easily integrated into Euler's equations of motion to elegantly describe a system with single axis rotation. The alternative methods add unnecessary complication when it comes to a very specific type of problem like single axis rotation. This notation could not be used on any where near the number of problems that any of the other methods described here could; however, seeing as there are so many applications which only deal with a single axis of rotation, it would still have a wide appeal in practical problems and can make a big difference.

# CHAPTER 7

# Conclusions

## 7.1 General Discussion

This thesis sought to provide a more elegant way for the engineer or dynamist evaluating a system with a single rotational axis. This tool would be a revamping of the notation used in the derivation of Euler's equations of motion to provide a direct link from the setup and definition of a problem to the solution and analysis. The benefit of this idea of unification of notion from definition through analysis was highlighted using some simple particle problems, and the full use of this new formulation of Euler's equations of motion was shown using a rigid body example. This is not the only notation that is used for performing rotations about a single or multiple axes and a few other methods were compared and contrasted with the method defined in this thesis. It is believed that the approach presented in this thesis provides a justifiable advantage in evaluating problems with a single axis of rotation and can be greatly beneficial to the engineer or dynamist when applied appropriately.

## 7.2 Summary of Contributions

This research presented a way to fully define and analyze a system in cylindrical vector components that had not existed before. The basis of nearly all dynamic models, Euler's equations of motion, was re-derived in a purely cylindrical component vector system. This gives a system designer another tool in the design process where a problem can be maintained in the more intuitive cylindrical vector components used to describe most rotational systems.

It was shown how this approach was a more elegant way to describe a single axis rotation system. This elegance can lead to more advanced methods of system design where the robustness of a system can be quantified and studied in-depth. A more complex system with higher degrees of freedom opens the possibility for smarter design decisions that will lead to a more robust design and understanding.

In a simple particle problem, the derivative of the resultant balance force with respect to the location of the particle in space around the balance force cannot be manipulated without fundamentally chancing the problem. A more interesting result is that the derivative is actually discontinuous at the balance point.

A simple rigid body problem was analyzed to show the benefit of the cylindrical vector component method. This notation is another beneficial tool for the dynamist or engineer looking at a problem or design with single axis rotation. The more elegant solution can provide a greater understanding of the relationship between the design parameters and the resultant reaction forces and dynamic balance of the system.

It was shown that under a certain set of parameters, rigid bodies in a single axis rotational problem can be treated as particles. This greatly reduces the number of calculations, variables and time it takes to perform a full analysis. With the greater simplicity, a more detailed approach can be conducted.

This notation was compared with a variety of other alternative way of describing a rotating system. In certain scenarios where the rotation is about a single axis, this notation was shown as a very useful tool that was easily integrated in the analysis of the system by the integration of the notation into Euler's equations of motion.

## 7.3 Prospect of Future Work

Some of the possibilities for future work in the field of designing for dynamic balance robustness include the following topics.

Explore the particle problems from Chapter 4 with some out of plane particles. Every example that has been presented has dealt with particles that all rotate in the same plane. Some out of plane interactions could provide some excellent insight into the real world dynamic balance of rotating machinery where everything may not be spinning in one plane.

Employ the use of bifurcation analysis in the simple particle problems. The current method of plotting the derivative of the resultant balance for is similar to bifurcation but this field is growing rapidly and new methods are being developed. Examining the sensitivity of the input parameters with this method may give some new insight into the balance of the system.

Consider trying to expand the CVC model to multi axis rotation and see if it can be integrated into Euler's equations of motion. The multiple axis version of the CVC could be extremely powerful in analysis rotational motion because of its ability to simplify complex problems and to provide a link between the system parameters and the output of the equations.

Consider higher order systems that have more particles or more rigid bodies. The problem that occurred with the simple particle or rigid body problems was a lack of design freedom. With there being only a few design parameters there is less of an opportunity to getting a robust design. These results showed a propensity for the design to be very unstable when it came to dynamic balance. Experience says that there should be more flexibility in a real life design that a simple particle or rigid body problem fails to capture.

Examine in a computer model the general form of the particle problem with more and more discrete particles. These models can try to discover behavior of a system that is in the grey area between a large number of particles and a rigid body and determine how this transition affects the dynamic balance of a system. A more continuous type system which may more closely represent the physical CT scanner model could result in a modified approach to solving and accounting for dynamic imbalance.

Re-evaluate the CT scanner problem with a larger focus on the upfront design.

Using this approach after a design is set to be dynamically and statically balanced gives very little if any opportunity to find what contributes the most to the dynamic balance without getting into the variation approach used by Rogers. The choice of certain design parameters in the beginning of the design process can rigidly determine the system's dynamic balance and leave little room for adjustment. With the ability to create a general equation using CVC, the design parameters might be more easily chosen to create a robustly design system with respect to dynamic balance.

Completely duplicate Rogers' approach using a purely CVC approach and see if the sensitivity analysis is clearer as predicted. This research was focused primarily on the dynamics and mathematical model of a system and the variation analysis was deemed out of scope. However, the variation analysis done completely in cylindrical coordinates may be give the biggest gain in understanding from this work because the direct design parameters will be present in the variation analysis equations.

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# APPENDIX A

## Detailed Calculation of Inertia Matrix for a Cylinder Using Cylindrical Vector Components

The calculation of the inertia matrix using CVC is performed by evaluating the triple integrals from Eq. 2.8 to Eq. 2.13. An example of these calculations is presented in this appendix. As stated in Chapter 3, the inertia matrix calculated using CVC will come out identically to the inertia matrix calculated by the conventional Cartesian coordinates if the coordinate systems are attached to the body in the same standard orientation. The standard orientation for a cylinder is for the coordinate to be centered at the mass center of the cylinder G, and for the Z axis to be aligned with the center of the cross-sectional circle of the cylinder.

A simple cylinder of radius $R$ and length $l$ is shown below in Fig. A.1 with a coordinate system $XYZ$ attached in the manner described above. Any standard example of inertia calculations for a cylinder will use this general model for its calculations.

The differential mass term at the end of Eq. 2.8 to Eq. 2.13 is defined using the variables needed to be integrated over. Since each equation uses the variables $R$, $\phi$ and $Z$, the differential mass term is defined in terms of those variables and a density term as shown.

$$\begin{aligned} dm &= \rho R \, dR \, d\phi \, dZ \\ \rho &= \frac{m}{\pi r^2 l} \end{aligned} \tag{A.1}$$

Notice that the differential mass term, when calculated, has the correct units of mass. This is the reason for the extra $R$ term in the beginning since the differential of $\phi$ has units of radians. To balance the units, the extra $R$ needs to be added.

For the calculations to be performed, the equations require bounds to be set up for each integral based on the variables they are being integrated over in the

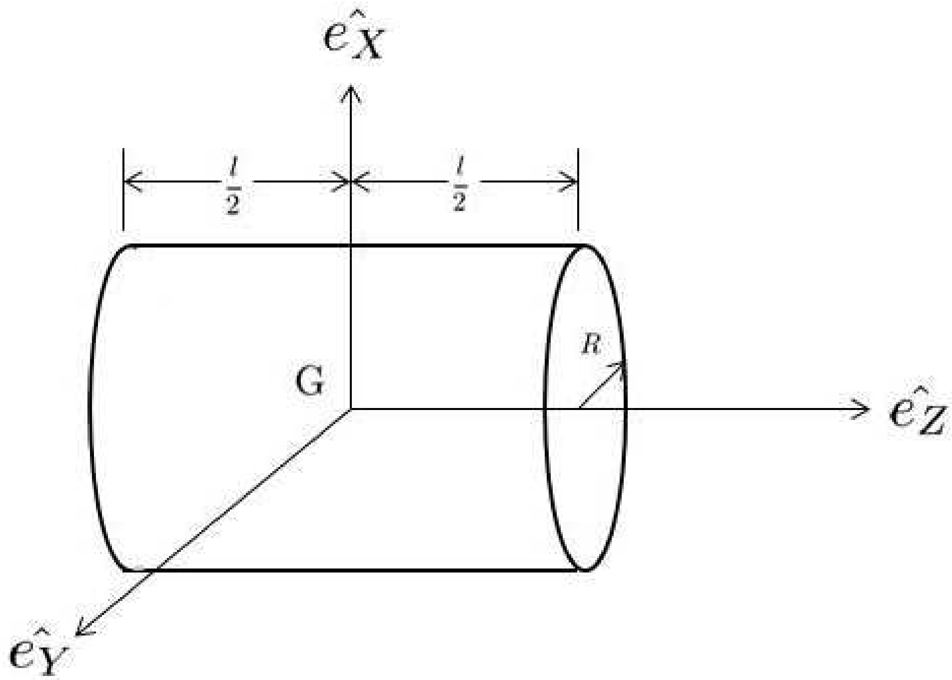


Figure A.1: Inertia Calculation Example

differential mass term. These are obtained by finding the two extremes for the body about each variable. The $R$ and $Z$ axes are simple with the $R$ term going from 0 to $r$ and the $Z$ going from $-\frac{l}{2}$ to $\frac{l}{2}$. The $\phi$ term is treated like and angle and since the body is round, the bounds on that variable is from 0 to $2\pi$.

$$
\begin{aligned}
0 \leq \quad R \quad &\leq r \\
0 \leq \quad \phi \quad &\leq 2\pi \\
-\frac{l}{2} \leq \quad Z \quad &\leq \frac{l}{2}
\end{aligned}
$$

.

The next step is to take the first moment of inertia, $I_{XX}$, and substitute in the value for $dm$ and the bounds on the integrals as defined above. Pulling out the constant density of the cylinder and performing each integration and evaluating it between the bounds is shown below.

$$I_{XX} = \int_0^r \int_0^{2\pi} \int_{-\frac{l}{2}}^{\frac{l}{2}} (R^2 \sin^2 \phi + Z^2)\rho R \, d\, R d\phi \, dZ \tag{A.2}$$

$$I_{XX} = \rho \int_0^r \int_0^{2\pi} \left[ zR^3 \sin^2 \phi + \frac{z^3 R}{3} \right]_{-\frac{l}{2}}^{\frac{l}{2}} dR \, d\phi$$

$$I_{XX} = \rho \int_0^r \int_0^{2\pi} \left[ lR^3 \sin^2 \phi + \frac{l^3 R}{12} \right] dR \, d\phi$$

$$I_{XX} = \rho \int_0^r \left[ \frac{lR^3 \phi}{2} - \frac{lR^3 \sin \phi \cos \phi}{2} + \frac{l^3 R \phi}{12} \right]_0^{2\pi} dR$$

$$I_{XX} = \rho \int_0^r \left[ lR^3 \pi + \frac{l^3 R \pi}{6} \right] dR$$

$$I_{XX} = \rho \left[ \frac{lR^4 \pi}{4} + \frac{l^3 R^2 \pi}{12} \right]_0^r$$

$$I_{XX} = \rho \frac{lr^4 \pi}{4} + \rho \frac{l^3 r^2 \pi}{12}$$

Once all of the integrations are complete, $\rho$ is substituted in from Eq. A.1 and the equation can be simplified.

$$I_{XX} = \frac{1}{12} m [3r^2 + l^2] \tag{A.3}$$

As expected, this equation is exactly the same as the inertia for $I_{XX}$ when calculated using the conventional Cartesian coordinate version of the triple integral. As stated in Chapter 3, this new means of calculating the inertia is very beneficial. It is never guaranteed that a triple integral will be easy or even possible to be calculated and having another formulation of the integral gives an additional route to the solution. Of course, it is always possible to create an approximation to an integral, but an analytical solution is always preferable.

The $I_{XX}$ is only the first of six of these triple integrals. Therefore, the exact same procedure for the $I_{YY}$ must be performed and will also lead to the same end equation.

$$I_{YY} = \int_0^r \int_0^{2\pi} \int_{-\frac{l}{2}}^{\frac{l}{2}} (Z^2 + R^2 \cos^2 \phi)\rho R \, dR \, d\phi \, dZ \tag{A.4}$$

$$I_{YY} = \rho \int_0^r \int_0^{2\pi} \left[ lR^3 \cos^2 \phi + \frac{l^3 R}{12} \right] dR \, d\phi \tag{A.5}$$

$$I_{YY} = \rho \int_0^r \left[ lR^3 \pi + \frac{l^3 R\pi}{6} \right] dR$$

$$I_{YY} = \rho \frac{lr^4\pi}{4} + \rho \frac{l^3 r^2 \pi}{12}$$

$$I_{YY} = \frac{1}{12} m[3r^2 + l^2] \tag{A.6}$$

This result is expected because the cylinder is symmetrical around the $Z$ axis. This means that the $X$ and $Y$ axis inertias should be exactly the same and would be the same for any orientation of the coordinate system if the $Z$ axis is aligned with the center of the cylinder.

Again, performing the triple integral for the remaining moment of inertia term, $I_{ZZ}$. This also follows the exact same procedure as with $I_{XX}$ and $I_{YY}$ only this time it will result in a different equation.

$$I_{ZZ} = \int_0^r \int_0^{2\pi} \int_{-\frac{l}{2}}^{\frac{l}{2}} (R^3)\rho R \, dR \, d\phi \, dZ \tag{A.7}$$

$$I_{ZZ} = \rho \int_0^r \int_0^{2\pi} (R^3 l) dR \, d\phi \tag{A.8}$$

$$I_{ZZ} = \rho \int_0^r (2\pi R^3 l) dR \tag{A.9}$$

$$I_{ZZ} = \rho \frac{\pi r^4 l}{2} \tag{A.10}$$

$$I_{ZZ} = \frac{1}{2} m r^2 \tag{A.11}$$

As with the $I_{XX}$ and $I_{YY}$, this is identical to the equation for the $I_{ZZ}$ moment of inertia when calculated using the conventional Cartesian coordinates.

The final pieces of the inertia matrix that needs to be calculated for this to be complete are the products of inertia. For a cylinder having a coordinate system aligned with its center, the products of inertia should be all zero because it is symmetrical about two axes. However, as an exercise, a product of inertia will be calculated to show that it does result in a zero. The equation is again set up using the bounds that were established above and is shown below. Each integral is performed and evaluated at those bounds and the density term can be substituted in at the end.

$$I_{XZ} = \int_0^r \int_0^{2\pi} \int_{-\frac{l}{2}}^{\frac{l}{2}} (R^2 Z \cos\phi) \rho R dR d\phi dZ \tag{A.12}$$

$$I_{XZ} = \rho \int_0^r \int_0^{2\pi} \left[ \frac{R^2 l^2}{4} \cos\phi \right] dR d\phi$$

$$I_{XZ} = \rho \int_0^r \left[ \frac{R^2 l^2}{4} \sin\phi \right]_0^{2\pi} dR$$

$$I_{XZ} = \rho \int_0^r [0] \, dR$$

$$I_{XZ} = I_{ZX} = 0 \tag{A.13}$$

As expected, the product of inertia term $I_{XZ}$ results in a 0. By inspection it can be seen that $I_{XY}, I_{YX}, I_{ZY}$ and $I_{YZ}$ will also be zero since $\int_0^{2\pi} (\cos\phi) d\phi$ and $\int_0^{2\pi} (\cos\phi \sin\phi) d\phi$ will be zero. Therefore:

$$I_{XZ} = I_{ZX} = I_{XY} = I_{YX} = I_{ZY} = I_{YZ} = 0 \tag{A.14}$$

To summarize, this appendix showed example calculations of the inertia matrix for a standard cylinder using CVC. These equations resulted in the same matrix as the standard Cartesian coordinate formulations of the triple integrals, but this is inherently a good thing. Had these formulations resulted in a different inertia matrix, the inertias for all of the standard bodies used in a CVC dynamic analysis would have to be recalculated and tabulated. The new triple integral formulations using CVC instead give a new means of calculating the inertia matrix for rigid bodies that are unknown, yet allows for the simple substitution of standard rigid body inertias into dynamic analysis.

# APPENDIX B

## Detailed Derivation of the Parallel Axis Theorem Using Cylindrical Vector Components

In this appendix, the parallel axis theorem will be derived using CVC. The main theme of using CVC is to preserve the link between the design parameters and the dynamic balance analysis, therefore each step in the analysis of a system must be performed using CVC. A major step in many dynamics problems is to employ the use of the parallel axis theorem and it is crucial that it is able to be performed using CVC.

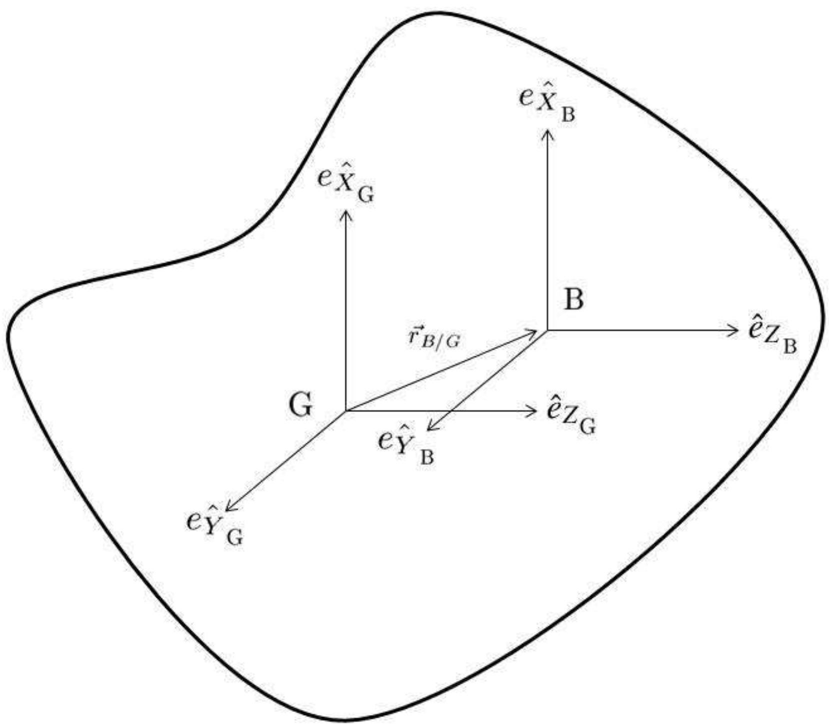


Figure B.1: Parallel Axis Theorem Diagram

Fig. B.1 shows the typical starting figure for the parallel axis theorem derivation with two coordinate systems attached to a rigid body. Coordinate system G is attached at the body's center of mass and coordinate system B is shown at a random yet known vector $r_{B/G}$ from coordinate system G. In the standard Cartesian coordinate derivation this vector would be defined simply as follows

$$\vec{r}_{B/G} = x_o e\hat{_{X_G}} + y_o e\hat{_{Y_G}} + z_o e\hat{_{Z_G}} \tag{B.1}$$

However, like before, the vector from B to G is split into the components of the projected vector $r_o$ along with the angle $\phi_o$ from the X axis on the XY plane and the vector $z_o$ along the Z axis shown in Fig. B.2. This new formulation of the vector $r_{B/G}$ is

$$\vec{r}_{B/G} = r_o \cos\phi_o e\hat{_{X_G}} + r_o \sin\phi_o e\hat{_{Y_G}} + z_o e\hat{_{Z_G}} \tag{B.2}$$

The angular momentum of coordinate system G can be transferred to coordinate system B by adding on the vector between the coordinate systems crossed with the cross product of the vector between the systems and the angular momentum of the body. Defining the angular momentum for coordinate system B with $\vec{H}_B$ and the angular momentum of coordinate system G as $\vec{H}_G$, this equation can be written as

$$\vec{H}_B = \vec{H}_G + m(\vec{r}_{B/G} \times [\vec{\omega} \times \vec{r}_{B/G}]) \tag{B.3}$$

Substituting the vector from Eq. B.2 and splitting the angular velocity $\omega$ into components $\omega_X$, $\omega_Y$, $\omega_Z$ will result in the following.

$$\begin{aligned}
\vec{r}_{B/G} \times [\vec{\omega} \times \vec{r}_{B/G}] \;=& \\
& [\omega_X(r_o^2 \sin^2\phi_o + z_o^2) - \omega_Y(r_o^2 \cos\phi_o \sin\phi_o) - \omega_Z(r_o z_o \cos\phi_o)]\hat{e_X} \\
+& [\omega_Y(z_o^2 + r_o^2 \cos^2\phi_o) - \omega_X(r_o^2 \cos\phi_o \sin\phi_o) - \omega_Z(r_o z_o \sin\phi_o)]\hat{e_Y} \\
+& [\omega_Z(r_o^2) - \omega_R(r_o z_o \cos\phi_o) - \omega_Y(r_o z_o \sin\phi_o)]\hat{e_Z}
\end{aligned} \tag{B.4}$$

Using this equation and also changing the notation for the angular momentum from $\vec{H}_B$ and $\vec{H}_G$ to $[I_B]\{\omega\}$ and $[I_G]\{\omega\}$ results in the following equation.

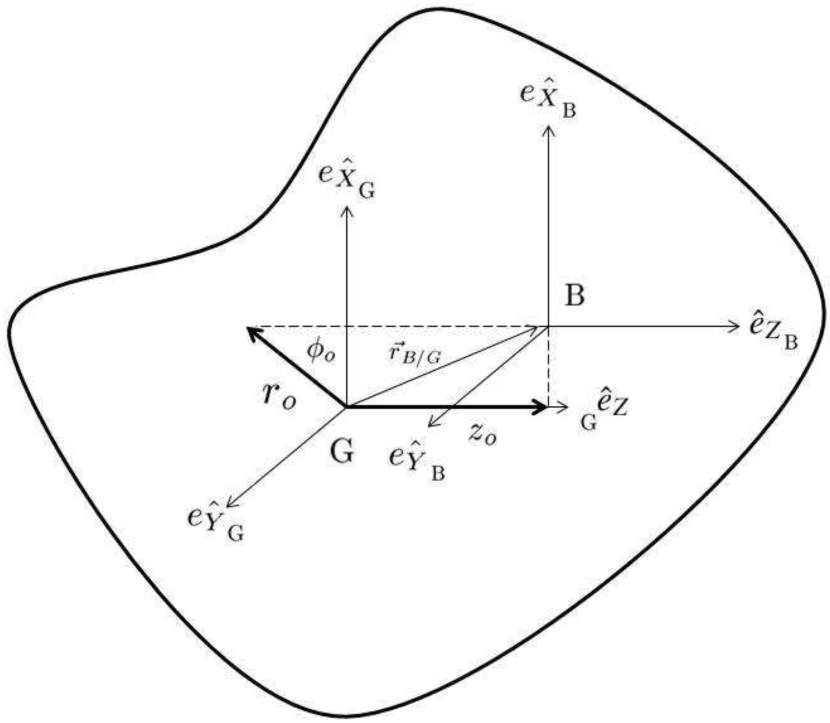


Figure B.2: Parallel Axis Theorem Diagram with Cylindrical Vector Components

$$
\begin{aligned}
[I_B]\{\omega\} &= \\
&[I_G]\{\omega\} + m \begin{bmatrix} r_o^2 \sin^2\phi_o + z_o^2 & -r_o^2 \cos\phi_o \sin\phi_o & -(r_o z_o \cos\phi_o) \\ -r_o^2 \cos\phi_o \sin\phi_o & z_o^2 + r_o^2 \cos^2\phi_o & -r_o z_o \sin\phi_o \\ -r_o z_o \cos\phi_o & -r_o z_o \sin\phi_o & r_o^2 \end{bmatrix} \{\omega\}
\end{aligned}
\tag{B.5}
$$

Splitting up the inertia matrices and witting out the result explicitly will result in the following equations. These equations provide an easy way to employ the parallel axis theorem using terms that are consistent with CVC.

$$\begin{aligned}
I_{XX_B} &= I_{XX_G} + m(r_o^2 \sin^2 \phi_o + z_o^2) \\
I_{YY_B} &= I_{YY_G} + m(r_o^2 \cos^2 \phi_o + z_o^2) \\
I_{ZZ_B} &= I_{ZZ_G} + mr_o^2 \\
\\
I_{XY_B} &= I_{YX_B} = I_{XY_G} + mr_o^2 \cos\phi_o \sin\phi_o \\
I_{XZ_B} &= I_{ZX_B} = I_{XZ_G} + mr_o z_o \cos\phi_o \\
I_{ZY_B} &= I_{YZ_B} = I_{XY_G} + mr_o z_o \sin\phi_o
\end{aligned} \tag{B.6}$$

Summarizing, this appendix fills a gap in the analysis of a dynamic problem using CVC by deriving the parallel axis theorem in CVC. To ensure that the entire problem can be defined using CVC and stays in CVC throughout the problem is a major goal of this thesis. Many applications require the use of the parallel axis theorem to comply with rules of dynamic analysis and where coordinate systems are allowed to be placed, such as at the center of mass or a point that is not accelerating. If this portion of the analysis was not in CVC, it could possibly compromise the benefit of using CVC which is to keep the problem input parameters intact all the way through the analysis to create a link between the two. The parallel axis theorem in CVC is a small yet important link in that process.