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Recurrence Relations for Moments of Dual Generalized Order Statistics from Weibull Gamma Distribution and Its Characterizations

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Abstract: In this paper, we establish explicit forms and new recurrence relations satisfied by the single and product moments of dual generalized order statistics from Weibull gamma distribution (WGD). The results include as particular cases the relations for moments of reversed order statistics and lower records. We present characterizations of WGD based on (i) recurrence relation for single moments, (ii) truncated moments of certain function of the variable and (iii) hazrad function.

Keywords: Dual generalized order statistics, lower records, single moments, product moments, recurrence relations, Weibull gamma distribution and characterization.

1 Introduction

Generalized order statistics (gos) includes imporatant concepts that have been used in statistical reseach such as ordinary order statistics and record values, see [6, Kamps (1995)]. [3, Burkschat et al. (2003)] introduced the concept of dual generalized order statistics (dgos) as a model of descendingly ordered random variables.

Let $n \in N$, $k \geqslant 1$, $m \in \mathbb{R}$, be the parameters such that

$$\gamma_r = k + (n-r)(m+1) > 0, \text{ for all } 1 \leqslant r \leqslant n.$$

Let $X'(1,n,m,k)$, $X'(2,n,m,k), \cdots, X'(n,n,m,k)$ be the dgos from an absolutely continuous distribution function $F(-)$ with density function $f(-)$, so, the joint probability density function (pdf) is

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^m f(x_i)\right)[F(x_n)]^{k-1} f(x_n), \tag{1}$$

for

$$F^{-1}(1) > x_1 \geq x_2 \geq \cdots \geq x_n > F^{-1}(0).$$

The marginal pdf of r-th dgos, $X'(r,n,m,k)$ is

$$f_{X'(r,n,m,k)}(x) = \frac{C_{r-1}}{\Gamma(r)}[F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)). \tag{2}$$

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The joint pdf of $X^{'}(r,n,m,k)$ and $X^{'}(s,n,m,k)$, $1 \le r \le s \le n$, $x > y$ is expressed from (1) as

$$f_{X^{'}(r,n,m,k),X^{'}(s,n,m,k)}(x,y)$$
$$= \frac{C_{s-1}}{\Gamma(r)\Gamma(s-r)}[F(x)]^{m} f(x) g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y), \quad (3)$$

where

$$\Gamma(r) = \int_0^\infty u^{r-1}\exp(-u)du,$$

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i,$$

$$h_m(x) = \begin{cases} \frac{-1}{m+1}x^{m+1}, & m \neq -1 \\ -\ln x, & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1),\ x \in [0,1).$$

Since $X^{'}(0,n,m,k) \to 0$ then $X^{'}(n+1,n,m,k) = 0$. If $m = 0$, $k = 1$, then $X^{'}(r,n,m,k)$ reduces to the $(n-r+1)-th$ reversed order statistics, $(X_{n-r+1:n})$ from the sample $X_1, X_2, \cdots, X_n$ and when $m = -1$, then $X^{'}(r,n,m,k)$ reduces to the k-lower record value [14, Pawlas and Szynal (2001)].

[12, Kumer (2011)] established moments of lower generalized order statistics from Frechet-type extreme value distribution and its characterization. Recurrence relations for single and product moments of dual generalized order statistics from the inverse Weibull distribution are derived by [14, Pawlas and Szynal (2001)]. [11, Khan et al. (2008)], [8, Khan and Kumer (2010)], [9, Khan and Kumer (2011a)] and [10, Khan and Kumer (2011b)] have established recurrence relations for moments of dual generlized order statistics from exponentiated Weibull, Pareto, gamma and generalized exponential distributions. [1, Ahsanullah (2004)] and [2, Athar and Faizan (2011)] characterized the uniform and power function distributions based on distributional properties of dual generalized order statistics, respectively. [7, Kamps (1998)] investigated the importance of recurrence relations of order statistics in characterization.

In this paper, explicit expressions for single and product moments of dgos from Weibull gamma distribution are established. Results for reversed order statistics and lower record values can be deduced as special cases. We present characterizations of WGD based on (i) recurrence relation for single moments, (ii) truncated moments of certain function of the variable and (iii) hazrad function.

A random variable X is said to have Weibull gamma distribution if its pdf given by:

$$f(x) = \frac{c\beta}{\delta}x^{c-1}\left[1+\frac{1}{\delta}x^{c}\right]^{-(\beta+1)}, x > 0,\ c,\ \beta,\ \delta > 0, \quad (4)$$

and corresponding cummulative distribution function (cdf):

$$F(x) = 1 - \left[1+\frac{1}{\delta}x^{c}\right]^{-\beta}, x \ge 0. \quad (5)$$

See [13, Molenberghs and Verbeke (2011)].

Therefore, from (4) and (5), we have

$$F(x) = \frac{1}{c\beta}\left[\beta x + \sum_{u=2}^{\beta+1}\binom{\beta+1}{u}\frac{1}{\delta^{u-1}}x^{(u-1)c+1}\right] f(x), \quad (6)$$

where β is a positive integer .

Remark: For $c = 1$ and $\delta = 1$, the distribution reduced to standard Pareto distribution.

2 Explicit Expression for Single Moments of dgos for WGD

We establish the explicit expression for $E[X^j(r,n,m,k)]$. Using (2), we have when $m \neq -1$

$$E\left[X^j(r,n,m,k)\right] = \frac{C_{r-1}}{\Gamma(r)} \int_0^\infty x^j\left(F(x)\right)^{\gamma_r-1} f(x) g_m^{r-1}\left(F(x)\right) dx. \tag{7}$$

By using binomial expansion, we can rewrite (7) as

$$E\left[X^j(r,n,m,k)\right] = \frac{c\beta C_{r-1}}{(m+1)^{r-1}\delta\Gamma(r)} \sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i \int_0^\infty x^{j+c-1} \times\left[1-\left(1+\frac{1}{\delta}x^c\right)^{-\beta}\right]^{\gamma_r+(m+1)i-1}\left[1+\frac{1}{\delta}x^c\right]^{-(\beta+1)} dx, \tag{8}$$

using transformation $z = 1-(1+\frac{1}{\delta}x^c)^{-\beta}$, we get

$$E\left[X^j(r,n,m,k)\right] = \frac{\delta^{\frac{j}{c}} C_{r-1}}{(m+1)^{r-1}\Gamma(r)} \sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i \int_0^1 z^{\gamma_r+(m+1)i-1} \times\left[(1-z)^{\frac{-1}{\beta}}-1\right]^{\frac{j}{c}} dz, \tag{9}$$

and, after some simplifications, we arrive at

$$\begin{aligned} &E\left[X^j(r,n,m,k)\right] \\ &= \frac{\delta^{\frac{j}{c}} C_{r-1}}{(m+1)^{r-1}\Gamma(r)} \sum_{i=0}^{r-1}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\binom{r-1}{i}\frac{\left(\frac{j-ca}{c\beta}\right)_{(b)}}{b!\left[\gamma_r+(m+1)i+b\right]} C\left(\frac{j}{c},a\right)(-1)^{i+a} \\ &= \frac{\delta^{\frac{j}{c}} C_{r-1}}{(m+1)^{r-1}\Gamma(r)} \sum_{i=0}^{r-1}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\frac{C(i,a,b)}{\left[\gamma_r+(m+1)i+b\right]}, \end{aligned} \tag{10}$$

where

$$\left(\frac{j-ca}{c\beta}\right)_{(b)} = \begin{cases} \left(\frac{j-ca}{c\beta}\right)\left(\frac{j-ca}{c\beta}+1\right)\ldots\left(b-1+\frac{j-ca}{c\beta}\right), & b>0 \\ 1, & b=0 \end{cases}$$

and

$$C(i,a,b) = \frac{1}{b!}\left(\frac{j-ca}{c\beta}\right)_{(b)}\binom{r-1}{i} C\left(\frac{j}{c},a\right)(-1)^{i+a}.$$

When $m=-1$,

$$E\left[X^{\prime}(r,n,-1,k)\right] = \delta^{\frac{j}{c}} k^r \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} C\left(\frac{j}{c},a\right)(-1)^a \frac{\left(\frac{j-ca}{c\beta}\right)_{(b)}}{b!\left[k+b\right]^r}. \tag{11}$$

3 Recurrence relations for single moments of dgos from WGD

By using the following theorms ((3.1)-(3.3) below) given in [11, Khan et al. (2008)], we obtain new recurrence relations for single moments of dgos from WGD as follows:

For $2 \leq r \leq n, k = 1,2,\cdots$

$$\begin{aligned} &E\left[X^j(r,n,m,k)\right] - E\left[X^j(r-1,n,m,k)\right] \\ &= \frac{-jC_{r-1}}{\gamma_r\Gamma(r)} \int_0^\infty x^{j-1}\left(F(x)\right)^{\gamma_r} g_m^{r-1}\left(F(x)\right) dx. \end{aligned} \tag{12}$$

For $2 \le r \le n$, $k = 1, 2, \cdots$

$$E\left[X^{j}(r-1,n,m,k)\right] - E\left[X^{j}(r-1,n-1,m,k)\right]$$

$$= \frac{j(m+1)C_{r-2}}{\gamma_1 \Gamma(r-1)} \int_0^{\infty} x^{j-1} (F(x))^{\gamma_r} g_m^{r-1}(F(x))\, dx. \tag{13}$$

For $2 \le r \le n$, $k = 1, 2, \cdots$

$$E\left[X^{j}(r,n,m,k)\right] - E\left[X^{j}(r-1,n-1,m,k)\right]$$

$$= \frac{-jC_{r-1}}{\gamma_1 \Gamma(r)} \int_0^{\infty} x^{j-1} (F(x))^{\gamma_r} g_m^{r-1}(F(x))\, dx. \tag{14}$$

Relation 3.1 For WGD, $2 \le r \le n$, $\beta \in \mathbb{N}$ and $k = 1, 2, \cdots$

$$E\left[X^{j}(r-1,n,m,k)\right]$$

$$= \frac{j}{c\beta\gamma_r} \sum_{u=2}^{\beta+1} \binom{\beta+1}{u} \frac{1}{\delta^{u-1}} E\left[X^{j+(u-1)c}(r,n,m,k)\right]$$

$$+ \left[1 + \frac{j}{c\gamma_r}\right] E\left[X^{l_j}(r,n,m,k)\right]. \tag{15}$$

Proof: From (6) and (12), we have

$$E\left[X^{j}(r,n,m,k)\right] - E\left[X^{j}(r-1,n,m,k)\right]$$

$$= \frac{-jC_{r-1}}{c\beta\gamma_r\Gamma(r)} \int_0^{\infty} x^{j-1} [F(x)]^{\gamma_r - 1} \left[\beta x + \sum_{u=2}^{\beta+1} \binom{\beta+1}{u} \frac{1}{\delta^{u-1}} x^{(u-1)c+1}\right]$$

$$\times f(x) g_m^{r-1}(F(x))\, dx, \tag{16}$$

$$= \frac{-jC_{r-1}}{c\gamma_r\Gamma(r)} \left[\int_0^{\infty} x^{j} [F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))\, dx\right.$$

$$\left. + \frac{1}{\beta} \sum_{u=2}^{\beta+1} \binom{\beta+1}{u} \frac{1}{\delta^{u-1}} \int_0^{\infty} x^{j+(u-1)c} [F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))\, dx, \right], \tag{17}$$

and hence the result.

Relation 3.2 For WGD, $2 \le r \le n$, $\beta \in \mathbb{N}$ and $k = 1, 2, \cdots$

$$E\left[X^{j}(r-1,n,m,k)\right] - E\left[X^{j}(r-1,n-1,m,k)\right]$$

$$= \frac{j(m+1)}{c\gamma_1\gamma_r} \left[E(X^{j}(r,n,m,k))\right.$$

$$\left. + \frac{1}{\beta} \sum_{u=2}^{\beta+1} \binom{\beta+1}{u} \frac{1}{\delta^{u-1}} E(X^{j+(u-1)c}(r,n,m,k))\right]. \tag{18}$$

Proof: Follows from (6) and (13).

Relation 3.3 For WGD, $2 \le r \le n$, $\beta \in \mathbb{N}$ and $k = 1, 2, \cdots$

$$E\left[X^{j}(r-1,n-1,m,k)\right] - \left[1 + \frac{j}{c\gamma_1}\right] E\left[X^{j}(r,n,m,k)\right]$$

$$= \frac{j}{c\beta\gamma_1} \sum_{u=2}^{\beta+1} \binom{\beta+1}{u} \frac{1}{\delta^{u-1}} E\left[X^{j+(u-1)c+1}(r,n,m,k)\right] \tag{19}$$

Proof : Follows from (6) and (14).

3.1 Remarks

(1) For $m = 0$, $k = 1$ in (15), we obtain a recurrence relation for single moment of reversed order statistics of the Weibull gamma distribution in the form:

$$E\left(X^{j}_{n-r+2:n}\right) = \frac{j}{c\beta(n-r+1)}\sum_{u=2}^{\beta+1}\binom{\beta+1}{u}\frac{1}{\delta^{u-1}}E\left(X^{j+(u-1)c}_{n-r+1:n}\right) + \left[1+\frac{j}{c(n-r+1)}\right]E\left(X^{j}_{n-r+1:n}\right).$$

(2) For $m = -1$, $k = 1$ in (15), we get recurrence relation for single moment of lower record values from Weibull gamma dustribution in the form:

$$E\left[X^{j}_{L(r-1)}(n,-1,1)\right] = \frac{j}{c\beta}\sum_{u=2}^{\beta+1}\binom{\beta+1}{u}\frac{1}{\delta^{u-1}}E\left[X^{j+(u-1)c}_{L(r)}(n,-1,1)\right] + \left[1+\frac{j}{c}\right]E\left[X^{j}_{L(r)}(n,-1,1)\right].$$

4 Explicit Expression for Product Moments of dgos of WGD

Employing (3) and the binomial expression, explicit expressions for the product moments of dgos $X^{l_i}(r,n,m,k)$ and $X^{l_j}(s,n,m,k), 1 \le r < s \le n$ can be obtained when $m \ne -1$ as

$$E\left[X^{i}(r,n,m,k)X^{j}(s,n,m,k)\right] = \frac{c\beta C_{s-1}}{\delta(m+1)^{s-2}\Gamma(r)\Gamma(s-r)}\sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}\binom{r-1}{a}\binom{s-r-1}{b}(-1)^{a+b} \times \int_0^\infty x^{i+c-1}\left[1-(1+\frac{1}{\delta}x^c)^{-\beta}\right]^{(m+1)(s-r+a-b)-1}\left(1+\frac{1}{\delta}x^c\right)^{-(\beta+1)} I_1(x)dx, \quad (20)$$

where

$$I_1(x) = \frac{c\beta}{\delta}\int_0^x y^{j+c-1}\left[1-\left(1+\frac{1}{\delta}y^c\right)^{-\beta}\right]^{\gamma_s+(m+1)b-1}\left[1+\frac{1}{\delta}y^c\right]^{-(\beta+1)}dy. \quad (21)$$

Setting $z = 1-(1+\frac{1}{\delta}y^c)^{-\beta}$ and using binomial expansion for real numbers, we get

$$I_1(x) = \delta^{\frac{j}{c}}\sum_{d=0}^{\infty}\sum_{l=0}^{\infty}C(\frac{j}{c},d)\frac{\left(\frac{j-cd}{c\beta}\right)_{(l)}}{l!}(-1)^d\left[\frac{\left(1-\left[1+\frac{1}{\delta}x^c\right]^{-\beta}\right)^{\gamma_s+(m+1)b+l}}{\gamma_s+(m+1)b+l}\right]. \quad (22)$$

Substituating $I_1(x)$ above in (20), we obtain

$$E\left[X^{i}(r,n,m,k)X^{j}(s,n,m,k)\right] = \frac{c\beta C_{s-1}\delta^{\frac{j}{c}}}{\delta(m+1)^{s-2}\Gamma(r)\Gamma(s-r)}\sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}\sum_{d=0}^{\infty}\binom{r-1}{a}\binom{s-r-1}{b}C(\frac{j}{c},d) \times (-1)^{a+b+d}\sum_{l=0}^{\infty}\frac{\left(\frac{j-cd}{c\beta}\right)_{(l)}}{l!\left[\gamma_s+(m+1)b+l\right]} \times \int_0^\infty x^{i+c-1}\left[1-\left(1+\frac{1}{\delta}x^c\right)^{-\beta}\right]^{\gamma_s+(m+1)(s-r+a)+l-1}\left[1+\frac{1}{\delta}x^c\right]^{-(\beta+1)}dx. \quad (23)$$

Letting $Q = 1-(1+\frac{1}{\delta}x^c)^{-\beta}$, and after some simplifications, we arrive at

$$E\left[X^i(r,n,m,k)X^j(s,n,m,k)\right]$$
$$= \frac{C_{s-1}\delta^{\frac{1}{c}(i+j)}}{(m+1)^{s-2}\Gamma(r)\Gamma(s-r)} \sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}\sum_{d=0}^{\infty}\sum_{l=0}^{\infty}\sum_{h=0}^{\infty}\sum_{t=0}^{\infty} \binom{r-1}{a}\binom{s-r-1}{b}$$
$$\times(-1)^{a+b+d+l+h+t} C(\frac{j}{c},d)C(\frac{i}{c},h) \frac{\left(\frac{j-cd}{c\beta}\right)_{(l)}}{t!l!\left[\gamma_s+(m+1)b+l\right]}$$
$$\times \frac{\left(\frac{j-ch}{c\beta}\right)_{(t)}}{\left[\gamma_s+(m+1)(s-r+a)+l+t\right]}, \tag{24}$$

where

$$\left(\frac{j-ca}{c\beta}\right)_{(l)} = \begin{cases} (\frac{j-ca}{c\beta})(\frac{j-ca}{c\beta}+1)\ldots(l-1+\frac{j-ca}{c\beta}), & l>0 \\ 1, & l=0, \end{cases}$$

and

$$\left(\frac{j-ca}{c\beta}\right)_{(t)} = \begin{cases} (\frac{j-ca}{c\beta})(\frac{j-ca}{c\beta}+1)\ldots(t-1+\frac{j-ca}{c\beta}), & t>0 \\ 1, & t=0, \end{cases}$$

When $m=-1$,

$$E\left[X^i(r,n,-1,k)X^j(s,n,-1,k)\right]$$
$$= \frac{k^s}{\Gamma(r)\Gamma(s-r)} \int_0^\infty x^i \left[-\ln\left[1-(1+\frac{1}{\delta}x^c)^{-\beta}\right]\right]^{r-1} \frac{f(x)}{F(x)} I_2(x)dx, \tag{25}$$

where

$$I_2(x) = \int_0^x y^j \left[\ln F(x) - \ln F(y)\right]^{s-r-1} \left[F(y)\right]^{k-1} f(y)dy. \tag{26}$$

Letting $z = \ln F(x) - \ln F(y)$, we have

$$I_2(x) = \delta^{\frac{j}{c}} \sum_{a=0}^{\infty}\sum_{l=0}^{\infty} C(\frac{j}{c},a)(-1)^a \left(\frac{a}{\beta}\right)_{(l)}$$
$$\times \frac{\Gamma(s-r)}{l!(k+l)^{s-r}}(F(x))^{k+l}, \tag{27}$$

where

$$\left(\frac{a}{\beta}\right)_{(l)} = \begin{cases} (\frac{a}{\beta})(\frac{a}{\beta}+1)\ldots(l-1+\frac{a}{\beta}), & l>0 \\ 1, & l=0. \end{cases}$$

Substituating $I_2(x)$ above in (25) and simplifying, we arrive at

$$E\left[X^i(r,n,-1,k)X^j(s,n,-1,k)\right]$$
$$= \delta^{\frac{1}{c}(i+j)} k^s \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{l=0}^{\infty}\sum_{d=0}^{\infty} C(\frac{j}{c},a)C(\frac{i}{c},b)(-1)^{a+b}$$
$$\times \frac{\left(\frac{a}{\beta}\right)_{(l)}\left(\frac{i-cb}{c\beta}\right)_{(d)}}{l!d!(k+l+d)^r(k+l)^{s-r}}. \tag{28}$$

where

$$\left(\frac{i-bc}{c\beta}\right)_{(d)} = \begin{cases} (\frac{i-bc}{c\beta})(\frac{i-bc}{c\beta}+1)\ldots(d-1+\frac{i-bc}{c\beta}), & d>0 \\ 1, & d=0. \end{cases}$$

5 Recurrence relations for product moments of dgos from WGD

By using Theorem 3 given by [11, Khan et al. (2008)], we obtain new recurrence relations for product moments of dgos from WGD as follows:

Relation 5.1 For WGD, $1 \leq r < s \leq n-1, \beta \in \mathbb{N}$ and $k = 1, 2, \cdots$

$$E\left[X^{i}(r,n,m,k)X^{j}(s-1,n,m,k)\right]$$
$$= \frac{1}{\beta}\sum_{u=2}^{\beta+1}\binom{\beta+1}{u}\frac{1}{\delta^{u-1}}E\left[X^{i}(r,n,m,k)X^{j+(u-1)c}(s,n,m,k)\right]$$
$$+\left[1+\frac{j}{c\gamma_s}\right]E\left[X^{i}(r,n,m,k)X^{j}(s,n,m,k)\right]. \tag{29}$$

Proof From the following relation [11, Khan et al. (2008)],

$$E\left[X^{i}(r,n,m,k)X^{j}(s,n,m,k)\right]-E\left[X^{i}(r,n,m,k)X^{j}(s-1,n,m,k)\right]$$
$$= -\frac{jC_{s-1}}{\gamma_s\Gamma(r)\Gamma(s-r)}\int_{\alpha}^{\beta}\int_{\alpha}^{x}x^{i}y^{j-1}\left[F(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right] \tag{30}$$
$$\times\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1}\left[F(y)\right]^{\gamma_s}dydx,$$

and (6), Equation (29) is obtained.

6 Characterization

Characterizations of distributions are important to many researchers in the applied fields. An investigators will be vitally interested to know if their model fits the requirements of a particular distribution. To this end, one will depend on the characterizations of this distribution which provide conditions under which the underlying distribution is indeed that particular distribution. Various characterizations of distributions have been established in many different directions. In this work, several characterizations of WGD are presented. These characterizations are based on: (i) recurrence relation for single moments, (ii) truncated moments of certain function of the variable and (iii) hazard function.

6.1 Characterization of WGD based on a recurrence relation for single moments

Theorem 6.1.1. Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > 0$, then for $\beta \in \mathbb{N}$

$$E\left[X^{j}(r-1,n,m,k)\right] = \frac{j}{c\beta\gamma_r}\sum_{u=2}^{\beta+1}\binom{\beta+1}{u}\frac{1}{\delta^{u-1}}E\left[X^{j+(u-1)c}(r,n,m,k)\right]$$
$$+\left[1+\frac{j}{c\gamma_r}\right]E\left[X^{j}(r,n,m,k)\right] \tag{31}$$

if and only if

$$F(x) = 1-\left[1+\frac{1}{\delta}x^{c}\right]^{-\beta}, x \geqslant 0.$$

Proof The necessary part follows immediately from Equations (6) and (12). On the other hand if the recurrence relation in (31) is satisfied, then from (6), we have

$$\frac{C_{r-2}}{\Gamma(r-1)}\int_{0}^{\infty}x^{j}\left[F(x)\right]^{\gamma_r+m}f(x)g_{m}^{r-2}\left[F(x)\right]dx$$
$$= \frac{jC_{r-1}}{c\beta\gamma_r\Gamma(r)}\sum_{u=2}^{\beta+1}\binom{\beta+1}{u}\frac{1}{\delta^{u-1}}\int_{0}^{\infty}x^{j+(u-1)c}\left[F(x)\right]^{\gamma_r-1}f(x)g_{m}^{r-1}\left[F(x)\right]dx$$
$$+\left[1+\frac{j}{c\gamma_r}\right]\frac{C_{r-1}}{\Gamma(r)}\int_{0}^{\infty}x^{j}\left[F(x)\right]^{\gamma_r-1}f(x)g_{m}^{r-1}\left[F(x)\right]dx. \tag{32}$$

Integrating the left hand side of Equation (32) by parts, we have

$$I=\frac{C_{r-2}}{\Gamma(r-1)}\int_0^\infty x^j\,[F(x)]^{\gamma_r+m}\,f(x)g_m^{r-2}\,[F(x)]\,dx$$

$$=\frac{jC_{r-1}}{\gamma_r\Gamma(r)}\int_0^\infty x^{j-1}\,[F(x)]^{\gamma_r}\,g_m^{r-1}\,[F(x)]\,dx$$

$$+\frac{C_{r-1}}{\Gamma(r)}\int_0^\infty x^{j-1}\,[F(x)]^{\gamma_r-1}\,f(x)g_m^{r-1}\,[F(x)]\,dx, \tag{33}$$

$$=\frac{jC_{r-1}}{c\beta\gamma_r\Gamma(r)}\sum_{u=2}^{\beta+1}\binom{\beta+1}{u}\frac{1}{\delta^{u-1}}\int_0^\infty x^{j+(u-1)c}\,[F(x)]^{\gamma_r-1}\,f(x)g_m^{r-1}\,[F(x)]\,dx$$

$$+\left[1+\frac{j}{c\gamma_r}\right]\frac{C_{r-1}}{\lceil(r)}\int_0^\infty x^j\,[F(x)]^{\gamma_r-1}\,f(x)g_m^{r-1}\,[F(x)]\,dx,$$

which implies that

$$\frac{jC_{r-1}}{\gamma_r\Gamma(r)}\int_0^\infty x^{j-1}\,[F(x)]^{\gamma_r-1}\,g_m^{r-1}\,[F(x)] \tag{34}$$

$$\times\left[F(x)-\frac{x}{c}f(x)-\frac{1}{c\beta}\sum_{u=2}^{\beta+1}\binom{\beta+1}{u}\frac{1}{\delta^{u-1}}x^{(u-1)c+1}f(x)\right]dx=0.$$

Now applying a generalization of the Muntz-Szasz theorm [5, Hwang and Lin (1984)]

$$\frac{F(x)}{f(x)}=\left[\frac{x}{c}+\frac{1}{c\beta}\sum_{u=2}^{\beta+1}\binom{\beta+1}{u}\frac{1}{\delta^{u-1}}x^{(u-1)c+1}\right],$$

from which we obtain

$$F(x)=1-\left[1+\frac{1}{\delta}x^c\right]^{-\beta}.$$

Remark 6.1.1. The above characterization is restricted to $\beta\in\mathbb{N}$. This restriction, however, will be removed in the following two subsections.

6.2 Characterization based on truncated moments of functions of the random variable

In this subsection we employ a single function ψ of the random variable X and characterize its distribution in terms of the truncated moment of $\psi(X)$. The following propositions have already appeared in [4, Hamedani (2013)] (Technical Report, MSCS), so we will just state them here for the sake of completeness.

Proposition 6.2.1. Let $X:\Omega\longrightarrow(a,b)$ be a continuous random variable with *cdf* F . Let $\psi(x)$ be a differentiable function on (a,b) with $\lim_{x\longrightarrow a}\psi(x)=1$. Then for $\xi\neq1$,

$$E\left[\psi(X)\mid X>x\right]=\xi\,\psi(x),\; x\in(a,b),$$

if and only if

$$\psi(x)=(1-F(x))^{\frac{1}{\xi}-1},\; x\in(a,b)$$

Proposition 6.2.2. Let $X:\Omega\longrightarrow(a,b)$ be a continuous random variable with *cdf* F . Let $\psi_1(x)$ be a differentiable function on (a,b) with $\lim_{x\longrightarrow b}\psi_1(x)=1$. Then for $\xi_1\neq1$,

$$E\left[\psi(X)\mid X<x\right]=\xi_1\psi_1(x),\; x\in(a,b)$$

if and only if

$$\psi(x) = (F(x))^{\frac{1}{\xi_1}-1}, \ x \in (a,b)$$

Remarks 6.2.1. (a) For $\psi(x) = 1 + \frac{1}{\delta}x^c$, $x \in (0,\infty)$, Proposition 6.2.1 will give a *cdf F(x)* given by (5) for $\xi = \frac{\beta}{\beta-1}, \beta \neq 1$.

(b) For $\psi_1(x) = \left\{1 - \left[1+\frac{1}{\delta}x^c\right]^{-\beta}\right\}^{\frac{1-\xi_1}{\xi_1}}$,$x \in (a,b)$,Proposition 6.2.2 will give a *cdf F(x)* given by (5).

We state below a more general characterization of WGD and give a brief proof for the sake of completeness.

Proposition 6.2.3. Let $X : \Omega \longrightarrow (a,b)$ be a continuous random variable with *cdf* F . Let $\psi(x)$ and $q(x)$ be two differentiable functions on $(0,\infty)$ such that $\int_a^b \frac{q^{'}(t)}{[q(t)-\psi(t)]}dt = \infty$.Then

$$E[\psi(X) \mid X > x] = q(x), \ a < x < b, \tag{35}$$

implies

$$F(x) = 1 - \exp\left\{-\int_a^x \frac{q^{'}(t)}{[q(t)-\psi(t)]}dt\right\}, \ a \leqslant x \leqslant b. \tag{36}$$

Proof. If (35) holds, then

$$\int_x^b \psi(t)f(t)dt = q(x)(1-F(x)).$$

Differentiating both sides of the above equation with respect to x, we arrive at

$$\frac{f(x)}{1-F(x)} = \frac{q^{'}(x)}{[q(x)-\psi(x)]}, a < x < b. \tag{37}$$

Now, integrating (37) from a to x , we have, in view of $\int_a^b \frac{q^{'}(t)}{[q(t)-\psi(t)]}dt = \infty$, a *cdf* F given by (36).

Remarks 6.2.2. Taking, e.g., $\psi(x) = \left[1+\frac{1}{\delta}x^c\right]^{\frac{\beta}{2}}$ and $q(x) = 2\psi(x)$ for $x \in (0,\infty)$, (36) gives *cdf* of WGD. Clearly, there are other appropriate pair of functions ψ and q .

6.3 Characterization of WGD based on hazard function

For the sake of completeness, we state the following definition. In what follows, we assume, when necessary, that our *cdf* is twice differentiable.

Definition 6.3.1. Let F be absolutely continuous distribution with the corresponding pdf f. The hazard function corresponding to F is denoted by h_F and is defined by

$$h_F(x) = \frac{f(x)}{1-F(x)}, \ x \in Supp\, F \ , \tag{38}$$

where $Supp\, F$ is the support of F.

It is obvious that the hazard function of twice differentiable function satisfies the first order differential equation

$$\frac{h_F^{'}(x)}{h_F(x)} - h_F(x) = q(x),$$

where $q(x)$ is an appropriate integrable function. Although this differential equation has an obvious form since

$$\frac{f^{'}(x)}{f(x)} = \frac{h_F^{'}(x)}{h_F(x)} - h_F(x), \tag{39}$$

for many univariate continuous distributions (39) seems to be the only differential equation in terms of the hazard function. The goal of the characterization based on hazard function is to establish a differential equation in terms of hazard function, which has as simple form as possible and is not of the trivial form (39). For some general families of distributions this may not be possible. Here, we present a characterization of WGD based on a nontrivial differential equation in terms of the hazard function.

Proposition 6.3.1. Let $X:\Omega\longrightarrow(0,\infty)$ be a continuous random variable. The *pdf* of X is (4) if and only if its $h_F(x)$ satisfies the differential equation

$$h_F'(x)-(c-1)x^{-1}h_F(x)=-\frac{c^2\beta}{\delta^2}x^{2(c-1)}\left[1+\frac{1}{\delta}x^c\right]^{-2},\ 0<x<\infty, \tag{40}$$

with initial condition

$$h_F(x_o)=\frac{c\beta}{\delta}x_o^{c-1}\left[1+\frac{1}{\delta}x^c\right]^{-1}.$$

Proof: If X has *pdf* (4), then clearly (40) holds. Now, if (40) holds, then

$$x^{-(c-1)}h_F'(x)-(c-1)x^{-c}h_F(x)=-\frac{c^2\beta}{\delta^2}x^{c-1}\left[1+\frac{1}{\delta}x^c\right]^{-2},$$

from which we have

$$\frac{d}{dx}\left\{x^{-(c-1)}h_F(x)\right\}=-\frac{c^2\beta}{\delta^2}x^{c-1}\left[1+\frac{1}{\delta}x^c\right]^{-2}. \tag{41}$$

Integrating both sides of (41) from x_o to x , and using the initial condition on $h_F(x)$, we arrive at

$$h_F(x)=\frac{f(x)}{1-F(x)}=\frac{c\beta}{\delta}x^{c-1}\left[1+\frac{1}{\delta}x^c\right]^{-1}.$$

Now, integrating both sides of the last equation from 0 to x , we obtain

$$1-F(x)=\left[1+\frac{1}{\delta}x^c\right]^{-\beta},\ x\geqslant 0.$$

7 Special cases

(1) For $m=0,\ k=1$ in (10), the explicit formula for single moments of reversed order statistics of the Weibull gamma distribution can be obtained as:

$$E(X^j_{n-r+1:n})=\frac{\delta^{\frac{j}{c}}C_{r:n}}{\Gamma(r)}\sum_{i=0}^{r-1}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\frac{\left(\frac{j-ca}{c\beta}\right)_{(b)}}{b!\,[n-r+b+i+1]}\binom{r-1}{i}C(\frac{j}{c},a)(-1)^{i+a}, \tag{42}$$

where

$$C_{r:n}=\frac{n!}{(r-1)!(n-r)!}.$$

(2) For $m=-1,\ k=1$ in (11), we deduce the explicit expression for the moments of lower record values for the Weibull gamma distribution as:

$$E\left[X^j_{L(r)}(n,-1,1)\right]=\delta^{\frac{j}{c}}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}C(\frac{j}{c},a)(-1)^a\frac{\left(\frac{j-ca}{c\beta}\right)_{(b)}}{b!\,[b+1]^r}. \tag{43}$$

(3) The product moments of reversed order statistics can be obtained by taking $m=0,\ k=1$ in (24) as follows:

$$E\left[X^{i}_{n-r+1:n}X^{j}_{n-s+1:n}\right]$$

$$= \delta^{\frac{1}{c}(i+j)}C_{r,s:n}\sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}\sum_{d=0}^{\infty}\sum_{l=0}^{\infty}\sum_{h=0}^{\infty}\sum_{t=0}^{\infty}\binom{r-1}{a}\binom{s-r-1}{b}\times(-1)^{a+b+d+l+h+t}C(\frac{j}{c},d)C(\frac{i}{c},h)\frac{\left(\frac{j-cd}{c\beta}\right)_{(l)}\left(\frac{i-ch}{c\beta}\right)_{(t)}}{t!l!}$$

$$\times\frac{1}{[n-s+b+l+1][n-r+a+l+t+1]}, \quad (44)$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r+1)!(n-s)!}.$$

(4) The product moments of lower record values can be obtained by taking $k=1, m=-1$ in (28) as follows:

$$E\left[X^{i}_{L(r)}(n,-1,1)X^{j}_{L(s)}(n,-1,1)\right]$$

$$= \delta^{\frac{1}{c}(i+j)}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{l=0}^{\infty}\sum_{d=0}^{\infty}C(\frac{j}{c},a)C(\frac{i}{c},b)(-1)^{a+b}$$

$$\times\frac{\left(\frac{a}{\beta}\right)_{(l)}\left(\frac{i-cb}{c\beta}\right)_{(d)}}{l!d!(k+l+d)^{r}(k+l)^{s-r}}. \quad (45)$$

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