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Existence of Disjoint Weakly Mixing Operators That Fail to Satisfy the Disjoint Hypercyclicity Criterion

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Abstract

Recently, Bès, Martin, and Sanders [11] provided examples of disjoint hypercyclic operators which fail to satisfy the Disjoint Hypercyclicity Criterion. However, their operators also fail to be disjoint weakly mixing. We show that every separable, infinite dimensional Banach space admits operators $T_1, T_2, \ldots, T_N$ with $N \geqslant 2$ which are disjoint weakly mixing, and still fail to satisfy the Disjoint Hypercyclicity Criterion, answering a question posed in [11]. Moreover, we provide examples of disjoint hypercyclic operators T_1, T_2 whose corresponding set of disjoint hypercyclic vectors is nowhere dense, answering another question posed in [11]. In fact, we explicitly describe their set of disjoint hypercyclic vectors. Those same disjoint hypercyclic operators fail to be disjoint

topologically transitive. Lastly, we create examples of two families of d-hypercyclic operators which fail to have any d-hypercyclic vectors in common.

Keywords

Hypercyclic operator, Hypercyclic vector, Disjoint hypercyclicity

1. Introduction

Let *X* be a separable, infinite dimensional Banach space over the real or complex scalar field $\mathbb{F}$, and let $B(X)$ be the algebra of bounded, linear operators on *X*. An operator *T* in $B(X)$ is *hypercyclic* if there exists a vector *x* in *X* for which its orbit $\mathrm{Orb}(T,x)=\{T^n x: n \geqslant 0\}$ is dense in *X*. Any such vector *x* is called a *hypercyclic vector* for the operator *T*, and we use the notation $\mathcal{HC}(T)$ to denote its set of hypercyclic vectors. It is well known that an operator *T* in $B(X)$ is hypercyclic if and only if its set $\mathcal{HC}(T)$ of hypercyclic vectors is a dense G_δ set if and only if it is topologically transitive; that is, for any nonempty open sets V_0, V_1 in *X*, there is a positive integer *m* for which $V_0 \cap T^{-m}(V_1) \neq \emptyset$, see Kitai [20].

Disjoint hypercyclicity, or d-hypercyclicity for short, which was independently introduced by Bernal-González [6] and Bès and Peris [13], examines the dynamics of a direct sum of a finite family of operators. Formally, the operators $T_1, T_2, \dots, T_N$ in $B(X)$ with $N \geqslant 2$ are *disjoint*, or *d-hypercyclic*, if there is a vector *x* in *X* for which the *N*-tuple vector $(x, x, \dots, x)$ is a hypercyclic vector for the direct sum operator $\oplus_{i=1}^{N} T_i$ on the space $\oplus_{i=1}^{N} X$ with respect to the product topology. Any such vector *x* is called a *d-hypercyclic vector* for the operators $T_1, T_2, \dots, T_N$, and we use the notation d-$\mathcal{HC}(T_1, T_2, \dots, T_N)$ to represent their set of d-hypercyclic vectors. If this set d-$\mathcal{HC}(T_1, T_2, \dots, T_N)$ of d-hypercyclic vectors is dense in *X*, we say the operators $T_1, T_2, \dots, T_N$ are *densely d-hypercyclic*.

In the present note, we show that several well know dynamical properties of a single hypercyclic operator fail to hold true in the disjoint setting. We first turn our attention to the Hypercyclicity Criterion and the Blow-up/Collapse Property. The Hypercyclicity Criterion is a sufficient condition for an operator to be hypercyclic. It was established independently by Kitai [20] and by Gethner and Shapiro [17] in a more general setting. Over the years, the Hypercyclicity Criterion has played a significant role in linear dynamics. For many classes of operators, hypercyclicity is equivalent to satisfying the Hypercyclicity Criterion. Moreover, many results in linear dynamics involve or depend on the criterion.

Bès and Peris connected the Hypercyclicity Criterion to the direct sum operator $T \oplus T$. They proved an operator *T* is hypercyclic if and only if it is *weakly mixing*; that is, the direct sum operator $T \oplus T$ is topologically transitive; see [12]. Combining their result together with an older result from Furstenberg [16] about the topological transitivity of direct sums of single operator gives us the following theorem:

Theorem 1.1

(See [7, Theorem 2.1].) Let T be an operator in $B(X)$. The following statements are equivalent:

(i) *The operator T satisfies the Hypercyclicity Criterion.*
(ii) *The operator T is weakly mixing.*
(iii) *For every integer $M \geqslant 1$, the direct sum operator $\oplus_{i=1}^{M} T$ is topologically transitive.*

It is important to mention that there do exist examples of hypercyclic operators that fail to be weakly mixing and hence fail to satisfy the Hypercyclicity Criterion. The first example was from De la Rosa and Reed [15], who constructed a Banach space which supported such a hypercyclic operator. Bayart and Matheron [2] showed such hypercyclic operators exist on our more common Banach spaces like ℓ^p with $1 \leqslant p < \infty$.

As for the Blow-up/Collapse Property, it was originally established Godefroy and Shapiro [18] as a sufficient condition for hypercyclicity. Later, Bernal-González and Grosse-Erdmann [7] and León-Saavedra [21] independently proved an operator satisfies the Blow-up/Collapse Property if and only if it satisfies the Hypercyclicity Criterion.

In Section 2, we examine the Hypercyclicity Criterion and the Blow-up/Collapse Property in the disjoint setting. The d-Hypercyclicity Criterion, which was established by Bès and Peris [13], is a sufficient condition for a finite family of operators to be densely d-hypercyclic, and it is a natural extension of the Hypercyclicity Criterion for a single operator. Keeping Theorem 1.1 in mind, they proved that operators $T_1, T_2, \dots, T_N$ in $B(X)$ with $N \geqslant 2$ satisfy the d-Hypercyclicity Criterion if and only if for each integer $M \geqslant 1$, the direct sum operators $\oplus_{i=1}^{M} T_1, \oplus_{i=1}^{M} T_2, \dots, \oplus_{i=1}^{M} T_N$ are d-topologically transitive; see Definition 1.2, Definition 1.4 and Theorem 1.5 below. We show that unlike the single operator situation, the d-Hypercyclicity Criterion cannot be weakened. In particular, for every integer $M \geqslant 1$, every separable, infinite dimensional Banach space supports operators $T_1, T_2, \dots, T_N$ for which the direct sum operators $\oplus_{i=1}^{M} T_1, \oplus_{i=1}^{M} T_2, \dots, \oplus_{i=1}^{M} T_N$ are d-topologically transitive, but the direct sum operators $\oplus_{i=1}^{M+1} T_1, \oplus_{i=1}^{M+1} T_2, \dots, \oplus_{i=1}^{M+1} T_N$ fail to be d-hypercyclic; see Theorem 2.3 below.

Following the definition for a single operator to be weakly mixing, the operators $T_1, T_2, \dots, T_N$ in $B(X)$ with $N \geqslant 2$ are *d-weakly mixing* if the direct sum operators $T_1 \oplus T_1, T_2 \oplus T_2, \dots, T_N \oplus T_N$ are d-topologically transitive. As a consequence of Theorem 2.3, every separable, infinite dimensional Banach space admits d-weakly mixing operators $T_1, T_2, \dots, T_N$ with $N \geqslant 2$ which fail to satisfy the d-Hypercyclicity Criterion, answering a question posed by Bès, Martin, and Sanders in [11]; see Corollary 2.5 below. Also from Theorem 2.3, we get that every separable, infinite dimensional Banach space admits densely d-hypercyclic operators $T_1, T_2, \dots, T_N$ which fail to be d-weakly mixing; see Corollary 2.7 below. It is important to note that even though there are examples of single hypercyclic operators which fail to be weakly mixing, they are nontrivial to construct and we do not even know if every separable, infinite dimensional Banach space admits such operators.

The Disjoint Blow-up/Collapse Property, also a natural extension of the original Blow-up/Collapse Property, is a sufficient condition for a finite family of operators to be densely d-hypercyclic; see Definition 1.6 and Proposition 1.7 below. Again, unlike the single operator case, it is no longer equivalent to the d-Hypercyclicity Criterion. Bès, Martin, and Sanders [11] recently showed that any d-hypercyclic weighted shifts $B_1, B_2, \dots, B_N$ with $N \geqslant 2$ on ℓ^p for $1 \leqslant p < \infty$, must satisfy the Disjoint Blow-up/Collapse Property, but they fail to satisfy the d-Hypercyclicity Criterion. In their situation, the weighted shifts $B_1, B_2, \dots, B_N$ fail to be d-weakly mixing. Since d-weakly mixing operators satisfy the Disjoint Blow-up/Collapse Property, see Proposition 1.8 below, it immediately follows from our Corollary 2.5 below that a separable, infinite dimensional Banach space always admits operators $T_1, T_2, \dots, T_N$ which satisfy the Disjoint Blow-up/Collapse Property, but fail to satisfy the d-Hypercyclicity Criterion; see Corollary 2.6 below.

In Section 3, we focus our attention on the set of d-hypercyclic for the finite family of operators. For a single operator *T*, the existence of just one hypercyclic vector *x* implies that the corresponding set $\mathcal{HC}(T)$ of hypercyclic vectors is a dense G_δ, and so there is no need to make a distinction between hypercyclic and densely hypercyclic and topologically transitive. Moreover, from the Baire Category Theorem, any countable collection of hypercyclic operators has a dense G_δ set of hypercyclic vectors in common. However, the same cannot be said in the disjoint setting. Given an integer $M \geqslant 1$, we show that every separable, infinite dimensional Banach space admits operators $T_1,\ T_2$ for which the direct sum operators $\oplus_{i=1}^{M} T_1, \oplus_{i=1}^{M} T_2$ are d-hypercyclic, but the corresponding set d-$\mathcal{HC}\left(\oplus_{i=1}^{M} T_1, \oplus_{i=1}^{M} T_2\right)$ of d-hypercyclic vectors is nowhere dense because it is contained within a finite dimensional subspace; see Theorem 3.4 and Corollary 3.5 below. Thus, there exist d-hypercyclic operators $T_1,\ T_2$ which fail to be densely d-hypercyclic, answering a question posed by Bès, Martin, and Sanders in [11]. Even more, dense d-hypercyclicity is equivalent to d-topological transitivity, see Proposition 1.3 below, and so those same d-hypercyclic operators T_1, T_2 fail to be d-topologically transitive; see Corollary 3.8 below. Lastly, using those operators again, we create an example of two families of d-hypercyclic operators which fail to have a single d-hypercyclic vector in common; see Corollary 3.6 below.

Since we do not need to make a distinction between hypercyclicity and topological transitivity, an operator T is weakly mixing if and only if the direct sum operator $T \oplus T$ is hypercyclic. Again using Theorem 3.4 below, we provide an example of operators T_1, T_2 for which the direct sum operators $T_1 \oplus T_1$, T2⊕T2 are d-hypercyclic, but operators T_1, T_2 fail to be d-weakly mixing; see Corollary 3.9 below.

Lastly, a single invertible operator is hypercyclic if and only if its inverse is hypercyclic; see Kitai [20]. Bès, Martin, and Peris [9] showed this property does not necessarily hold in the disjoint setting by constructing two invertible, d-topologically transitive composition operators on the Hardy space $\mathcal{H}^2(\mathbb{D})$ whose inverses fail to be d-hypercyclic. We conclude Section 3 by showing every separable, infinite dimensional Banach space admits such invertible, densely d-hypercyclic operators; see Corollary 3.11 below.

We finish our introduction with some important definitions and results in disjoint hypercyclicity which are used throughout the note. For more on hypercyclicity in the disjoint setting, we refer the reader to [6], [8], [9], [10], [11], [13], [23], [24]. For more on general linear dynamics, we refer the reader to the book by Bayart and Matheron [3] and the book by Grosse-Erdmann and Peris [19].

Definition 1.2

We say that the operators $T_1, T_2, \dots, T_N$ in $B(X)$ with $N \geqslant 2$ are *d-topologically transitive* if for any non-empty open subsets $V_0, V_1, \dots, V_N$ in *X*, there exists a positive integer *m* so that

$$\emptyset \neq V_0 \cap T_1^{-m}(V_1) \cap T_2^{-m}(V_2) \cap \cdots \cap T_N^{-m}(V_N).$$

Disjoint topological transitivity is equivalent to dense disjoint hypercyclicity.

Proposition 1.3

(See [13, Proposition 2.3].) Let $T_1, T_2, \dots, T_N$ be operators in $B(X)$ with $N \geqslant 2$. Then the following statements are equivalent:

a) *The operators $T_1, T_2, \dots, T_N$ are d-topologically transitive.*
b) *The operators $T_1, T_2, \dots, T_N$ are densely d-hypercyclic.*
c) *The set d-$\mathcal{HC}(T_1, T_2, \dots, T_N)$ of d-hypercyclic vectors for the operators $T_1, T_2, \dots, T_N$ is a dense G_δ.*

Now, we state the d-Hypercyclicity Criterion. Note that if we let N=1 in Definition 1.4 below, we get the single operator version of the Hypercyclicity Criterion established by Bès and Peris; see [12].

Definition 1.4

We say the operators $T_1, T_2, \dots, T_N$ in $B(X)$ with $N \geqslant 2$ satisfy the *d-Hypercyclicity Criterion* provided there exist a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers, dense subsets $X_0, X_1, \dots, X_N$ of *X* and mappings $S_{m,k}: X_m \longrightarrow X$ with $1 \leqslant m \leqslant N$, $k \in N$ for which

$$T_m^{n_k} \underset{k\to\infty}{\to} 0 \text{pointwise on} X_0,$$

$$S_{m,k} \underset{k\to\infty}{\to} 0 \text{pointwise on} X_m \text{,and}$$

$$\left(T_m^{n_k} S_{i,k} - \delta_{i,m} Id_{X_m}\right) \underset{k\to\infty}{\to} 0 \text{ pointwise on} X_m \text{for} 1 \leqslant i \leqslant N.$$

In Theorem 1.5 below, we see the connection between the d-Hypercyclicity Criterion and direct sums of operators.

Theorem 1.5

(See [13, Theorem 2.7].) Let $T_1, T_2, \dots, T_N$ be operators in $B(X)$ with $N \geqslant 2$. The following statements are equivalent:

a) *The operators $T_1, T_2, \dots, T_N$ satisfy the d-Hypercyclicity Criterion.*
b) *For each integer $M \geqslant 1$, the direct sum operators $\oplus_{i=1}^{M} T_1, \oplus_{i=1}^{M} T_2, \dots, \oplus_{i=1}^{M} T_N$ are d-topologically transitive on $\oplus_{i=1}^{M} X$.*

Next, we have the Disjoint Blow-up/Collapse Property. Again by letting $N = 1$ in Definition 1.6 below, we get the standard Blow-up/Collapse Property for a single operator *T*; see Godefroy and Shapiro [18, Corollary 1.3].

Definition 1.6

We say that the operators $T_1, T_2, \dots, T_N$ in $B(X)$ with $N \geqslant 2$ satisfy the *Disjoint Blow-up/Collapse Property* if for any non-empty open subsets $W, V_0, V_1, \dots, V_N$ in *X* with $0 \in W$, there exists a positive integer *m* so that

$W \cap T_1^{-m}(V_1) \cap \cdots \cap T_N^{-m}(V_N) \neq \emptyset$ and

$V_0 \cap T_1^{-m}(W) \cap \cdots \cap T_N^{-m}(W) \neq \emptyset$.

A finite family of operators that satisfies the Disjoint Blow-up/Collapse Property is, in fact, d-topologically transitive.

Proposition 1.7

(See [13, Proposition 2.4].) If the operators $T_1, T_2, \dots, T_N$ in $B(X)$ with $N \geqslant 2$ satisfy the Disjoint Blow-up/Collapse Property, then they are d-topologically transitive and hence densely d-hypercyclic.

We conclude this section with one last proposition involving d-weakly mixing operators.

Proposition 1.8

(See [11, Proposition 1.11].) If the operators $T_1, T_2, \dots, T_N$ in $B(X)$ with $N \geqslant 2$ are d-weakly mixing, then they satisfy the Disjoint Blow-up/Collapse Property.

2. d-Hypercyclicity Criterion and d-weakly mixing

Shkarin [24] provided a short and elegant proof of the existence of d-hypercyclic operators $T_1, T_2, \dots, T_N$ with $N \geqslant 2$ on any separable, infinite dimensional Fréchet space. He did this by characterizing the d-hypercyclic vectors for any finite family of operators consisting entirely of conjugates of a single hypercyclic operator; see [24, Lemma 2.1]. We need this result to establish the existence of d-weakly mixing operators $T_1, T_2, \dots, T_N$ which fail to satisfy the d-Hypercyclicity Criterion, as well as several other results throughout the note.

Lemma 2.1

Let M, N be positive integers, and let $T, L_1, L_2, \dots, L_N$ be operators in $B(X)$ with the operators $L_1, L_2, \dots, L_N$ invertible. For each integer $1 \leqslant m \leqslant N$, set $T_m = L_m^{-1} T L_m$. The vector $(x_1, x_2, \dots, x_M)$ is a d-hypercyclic vector for the direct sum operators $\oplus_{i=1}^{M} T_1, \oplus_{i=1}^{M} T_2, \dots, \oplus_{i=1}^{M} T_N$ if and only if the vector

$(L_1x_1, L_1x_2, \dots, L_1x_M, L_2x_1, L_2x_2, \dots, L_2x_M, \dots, L_Nx_1, L_Nx_2, \dots, L_Nx_M)$

is a hypercyclic vector for the direct sum operator $\oplus_{i=1}^{MN} T$.

Proof

Observe that for any operators $T, L \in B(X)$ with the operator L invertible and for any vectors $x, y \in X$, we have $T^{n_k}(Lx) \longrightarrow y$ as $k \to \infty$ if and only if $(L^{-1}TL)^{n_k}x = L^{-1}T^{n_k}(Lx) \longrightarrow L^{-1}y$ as $k \to \infty$. The result now follows from the definition of d-hypercyclic vector and the definition of hypercyclic vector. □

Note that for $N = 1$ in Lemma 2.1, the d-hypercyclic vector $(x_1, x_2, \dots, x_M)$ is just a traditional hypercyclic vector for the single direct sum operator $\oplus_{i=1}^{M} T_1$.

The next lemma is a technical result needed within the proof of Theorem 2.3 below. It involves sequences of vectors whose limits are linearly independent.

Lemma 2.2

For integers i, j, k, M with M⩾2, $1 \leqslant i, j \leqslant M$ *and* $k \geqslant 1$, *let* $(h_{i,k})_{k=1}^{\infty}$ *be a sequence of vectors in X, let* $\alpha_{i,j}$ *be scalars in* $\mathbb{F}$, *and set*

$$y_{j,k} = \sum_{i=1}^{M} \alpha_{i,j} h_{i,k}.$$

If there exists a linearly independent set $\{y_1, y_2, \dots, y_M\}$ *in X such that for integers* $1 \leqslant j \leqslant M$, *we have* $y_{j,k} \longrightarrow y_j$ *as* $k \to \infty$, *then there exist vectors* $h_1, h_2, \dots, h_M \in \operatorname{span}\{y_1, y_2, \dots, y_M\}$ *such that for integers* $1 \leqslant i \leqslant M$, *we have* $h_{i,k} \longrightarrow h_i$ *as* $k \to \infty$.

Proof

Let $A = [\alpha_{i,j}]_{i,j=1,\dots,M}$ be the $M \times M$ matrix. To prove our result, it suffices to show the column vectors of the matrix *A* are linearly independent. If this is the case, then matrix *A* is invertible. By letting $A^{-1} = [\beta_{i,j}]_{i,j=1,\dots,M}$, note that for any integers *i*, *k* with $1 \leqslant i \leqslant M$ and $k \geqslant 1$, we then have $h_{i,k} = \sum_{j=1}^{M} \beta_{i,j} y_{j,k}$, and so $h_{i,k} \longrightarrow \sum_{i=1}^{M} \beta_{i,j} y_j$ as $k \to \infty$.

To this end, let $c_1, c_2, \dots, c_M$ be the column vectors of the matrix *A*, and suppose $\gamma_1 c_1 + \gamma_2 c_2 + \cdots + \gamma_M c_M = 0$ for some scalars $\gamma_1, \gamma_2, \dots, \gamma_M \in \mathbb{F}$. That is, for each integer *i* with $1 \leqslant i \leqslant M$, we have

(2.1)

$$\sum_{j=1}^{M} \gamma_j \alpha_{i,j} = 0.$$

For each integer $k \geqslant 1$, we have

$$\begin{aligned}\gamma_1 y_{1,k} + \gamma_2 y_{2,k} + \cdots + \gamma_M y_{M,k} &= \left(\sum_{i=1}^{M} \gamma_1 \alpha_{i,1} h_{i,k}\right) + \left(\sum_{i=1}^{M} \gamma_2 \alpha_{i,2} h_{i,k}\right) + \cdots + \left(\sum_{i=1}^{M} \gamma_M \alpha_{i,M} h_{i,k}\right) \\ &= \left(\sum_{j=1}^{M} \gamma_j \alpha_{1,j}\right) h_{1,k} + \left(\sum_{j=1}^{M} \gamma_j \alpha_{2,j}\right) h_{2,k} + \cdots + \left(\sum_{j=1}^{M} \gamma_j \alpha_{M,j}\right) h_{M,k} = 0, \text{by}(2.1),\end{aligned}$$

and so, $\gamma_1 y_{1,k} + \gamma_2 y_{2,k} + \cdots + \gamma_M y_{M,k} \longrightarrow 0$ as $k \to \infty$. However, by our assumptions, we also have $\gamma_1 y_{1,k} + \gamma_2 y_{2,k} + \cdots + \gamma_M y_{M,k} \longrightarrow \gamma_1 y_1 + \gamma_2 y_2 + \cdots + \gamma_M y_M$ as $k \to \infty$. Hence, $\gamma_1 y_1 + \gamma_2 y_2 + \cdots + \gamma_M y_M = 0$. It follows that $\gamma_1 = \gamma_2 = \cdots = \gamma_M = 0$ because the set $\{y_1, y_2, \dots, y_M\}$ is linearly independent. □

From Theorem 1.1, we see that if the direct sum operator $\oplus_{i=1}^{M} T$ is topologically transitive for an integer M⩾2, then the direct sum operator $\oplus_{i=1}^{M+1} T$ must also be topologically transitive. Using Lemma 2.1 and Lemma 2.2, one can show this dynamical property does not translate to the disjoint setting.

Theorem 2.3

Let X be a separable, infinite dimensional *Banach space*. For any integers M, N with $M \geqslant 1$ *and* $N \geqslant 2$*, there exist operators* $T_1, T_2, \dots, T_N$ *in* $B(X)$ *for which the direct sum operators*

$$\bigoplus_{i=1}^{M} T_1, \bigoplus_{i=1}^{M} T_2, \dots, \bigoplus_{i=1}^{M} T_N$$

are densely d-hypercyclic and hence d-topologically transitive, but the direct sum operators

$$\bigoplus_{i=1}^{M+1} T_1, \bigoplus_{i=1}^{M+1} T_2, \dots, \bigoplus_{i=1}^{M+1} T_N$$

fail to be d-hypercyclic.

Proof

Every separable, infinite dimensional Banach space admits a hypercyclic operator which satisfies the Hypercyclicity Criterion; see Ansari [1], Bernal-González [5] or [3, Corollary 29]. Let *T* be any such hypercyclic operator in $B(X)$. To construct the desired operators $T_1, T_2, \dots, T_N$ from the operator *T*, first note by Theorem 1.1, the direct sum operator $\oplus_{i=1}^{MN} T$ is hypercyclic. Select any vector

(2.2)

$$\left(f_1, \dots, f_M, g_{2,1}, \dots, g_{2,M}, g_{3,1}, \dots, g_{3,M}, \dots, g_{N,1}, \dots, g_{N,M}\right) \in \mathcal{HC}\left(\bigoplus_{i=1}^{MN} T\right).$$

Due to the density of the orbit of this vector, the set

$$\{f_1, \dots, f_M\} \cup \{g_{m,j} : 1 \leqslant j \leqslant M, 2 \leqslant m \leqslant N\}$$

is linearly independent, and so by a corollary of the Hahn–Banach Theorem [14, Corollary 6.6, p. 79], there exist linear functionals $\lambda_1, \lambda_2, \dots, \lambda_M$ in the dual space X^{*} satisfying

(2.3)

$$\lambda_i(f_j) = \begin{cases} 1, & \text{if} j = i \\ 0, & \text{if} j \neq i, \end{cases}$$ and

(2.4)

$$\lambda_i(g_{m,j}) = 0$$

for all integers $1 \leqslant i, j \leqslant M$ and $2 \leqslant m \leqslant N$. Let $L_1 = I$, where *I* is the identity map on *X*, and for integers *m* with $2 \leqslant m \leqslant N$, define the operator $L_m : X \longrightarrow X$ by

(2.5)

$$L_m x = x + \sum_{i=1}^{M} \lambda_i(x) g_{m,i}.$$

Clearly, the operator L_1 is invertible. Moreover, the operators $L_2, L_3, \dots, L_N$ are invertible. Using (2.3) and (2.4), one can easily verify the inverse operator is given by $L_m^{-1}x = x - \sum_{j=1}^{M} \lambda_j(x) g_{m,j}$. Set $T_m = L_m^{-1} T L_m$ for each integer $1 \leqslant m \leqslant N$.

To show the operators $\oplus_{i=1}^{M} T_1, \oplus_{i=1}^{M} T_2, \dots, \oplus_{i=1}^{M} T_N$ are densely d-hypercyclic, let $U_1 \oplus U_2 \oplus \cdots \oplus U_M$ be a nonempty basic open set in $\oplus_{i=1}^{M} X$, and we need to show

$$(U_1 \oplus U_2 \oplus \cdots \oplus U_M) \cap \text{d-}\mathcal{HC}\left(\bigoplus_{i=1}^{M} T_1, \bigoplus_{i=1}^{M} T_2, \dots, \bigoplus_{i=1}^{M} T_N\right) \neq \emptyset.$$

To this end, consider the bounded linear operator $A: \oplus_{i=1}^{M} X \longrightarrow B(\mathbb{F}^M)$ given by

$$A(x_1, x_2, \dots, x_M) = \begin{bmatrix} \lambda_1(x_1) & \lambda_1(x_2) & \cdots & \lambda_1(x_M) \\ \lambda_2(x_1) & \lambda_2(x_2) & \cdots & \lambda_2(x_M) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_M(x_1) & \lambda_M(x_2) & \cdots & \lambda_M(x_M) \end{bmatrix}$$

Using (2.3), one can easily verify the operator A is an onto map. Note that the set $\mathcal{O}$ of all invertible $M \times M$ matrices is dense $B(\mathbb{F}^M)$. Also, the set $\mathcal{O}$ of all invertible $M \times M$ matrices is open in $B(\mathbb{F}^M)$; see [14, Theorem 2.2, p. 192]. It follows from the Open Mapping Theorem [14, Theorem 12.1, p. 90] that the set $A^{-1}(\mathcal{O})$ is also open and dense in $\oplus_{i=1}^{M} X$. Therefore, $A^{-1}(\mathcal{O}) \cap (U_1 \oplus U_2 \oplus \cdots \oplus U_M)$ is a nonempty open set. Now, by (2.2), the vector $(f_1, f_2, \dots, f_M)$ is a hypercyclic vector for the direct sum operator $\oplus_{i=1}^{M} T$, and so there exists an integer $r \geqslant 1$ such that

$$(T^r f_1, T^r f_2, \dots, T^r f_M) = \left(\bigoplus_{i=1}^{M} T\right)^r (f_1, f_2, \dots, f_M) \in A^{-1}(\mathcal{O}) \cap (U_1 \oplus U_2 \oplus \cdots \oplus U_M).$$

To complete this portion of the proof, it remains to establish that the vector $(T^r f_1, T^r f_2, \dots, T^r f_M)$ is also in the set $\mathcal{HC}\big(\oplus_{i=1}^{M} T_1, \oplus_{i=1}^{M} T_2, \dots, \oplus_{i=1}^{M} T_N\big)$. From Lemma 2.1, it suffices to show the vector

$$\big(L_1(T^r f_1), \dots, L_1(T^r f_M), L_2(T^r f_1), \dots, L_2(T^r f_M), \dots, L_N(T^r f_1), \dots, L_N(T^r f_M)\big)$$

is in $\mathcal{HC}\big(\oplus_{i=1}^{MN} T\big)$. For this, let

$\big(x_1, \dots, x_M, y_{2,1}, \dots, y_{2,M}, \dots, y_{N,1}, \dots, y_{N,M}\big)$ be any vector in $\oplus_{i=1}^{MN} X$. From (2.2), one can easily verify that

$$\big(T^r f_1, \dots, T^r f_M, g_{2,1}, \dots, g_{2,M}, \dots, g_{N,1}, \dots, g_{N,M}\big) \in \mathcal{HC}\left(\bigoplus_{i=1}^{MN} T\right),$$

and so there is a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that for the integers *i*, *m* with $1 \leqslant i \leqslant M$ and $2 \leqslant m \leqslant N$,

(2.6)

$T^{n_k}(T^r f_i) \longrightarrow x_i$ as $k \to \infty$,and

(2.7)

$$T^{n_k}(g_{m,i}) \longrightarrow \sum_{j=1}^{M} b_{j,i}(y_{m,j} - x_j) \text{as} k \to \infty,$$

where $[b_{j,i}]_{j,i=1,\dots,M}$ is the inverse of the invertible matrix $A(T^r f_1, T^r f_2, \dots, T^r f_M) = [\lambda_i(T^r f_j)]_{i,j=1,\dots,M}$. Clearly, for integers $1 \leqslant l \leqslant M$, we have $T^{n_k}(L_1 T^r f_l) = T^{n_k}(T^r f_l) \longrightarrow x_l$. Furthermore, for integers m, l with $2 \leqslant m \leqslant N$ and $1 \leqslant l \leqslant M$,

$$\begin{aligned}
T^{n_k}(L_m T^r f_l) &= T^{n_k}(T^r f_l) + \sum_{i=1}^{M} \lambda_i(T^r f_l) T^{n_k}(g_{m,i}), \text{by (2.5)} \\
&\longrightarrow x_l + \sum_{i=1}^{M} \lambda_i(T^r f_l) \left[\sum_{j=1}^{M} b_{j,i}(y_{m,j} - x_j)\right], \text{by(2.6)and(2.7)} \\
&= x_l + \sum_{j=1}^{M} \left[\sum_{i=1}^{M} b_{j,i} \lambda_i(T^r f_l)\right](y_{m,j} - x_j) \\
&= x_l + (y_{m,l} - x_l), \text{because} A(T^r f_1, T^r f_2, \dots, T^r f_n)^{-1} = [b_{j,i}]_{j,i=1,\dots,M} \\
&= y_{m,l} \text{as} k \to \infty.
\end{aligned}$$

Lastly, to prove the direct sum operators $\oplus_{i=1}^{M+1} T_1, \oplus_{i=1}^{M+1} T_2, \dots, \oplus_{i=1}^{M+1} T_N$ fail to be d-hypercyclic, suppose $(x_1, x_2, \dots, x_{M+1})$ is a d-hypercyclic vector for these operators, and we use Lemma 2.2 to derive a contradiction. To this end, let $\{y_1, y_2, \dots, y_{M+1}\}$ be any linearly independent set in *X*. Select a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers for which

(2.8)

$$\begin{aligned}
\left(\bigoplus_{i=1}^{M+1} T_1\right)^{n_k} (x_1, x_2, \dots, x_{M+1}) &= (T_1^{n_k} x_1, T_1^{n_k} x_2, \dots, T_1^{n_k} x_{M+1}) = (T^{n_k} x_1, T^{n_k} x_2, \dots, T^{n_k} x_{M+1}) \\
&\longrightarrow (0,0, \dots, 0) \text{as} k \to \infty,
\end{aligned}$$

and

(2.9)

$$\left(\bigoplus_{i=1}^{n+1} T_2\right)^{n_k} (x_1, x_2, \dots, x_{M+1}) = (T_2^{n_k} x_1, T_2^{n_k} x_2, \dots, T_2^{n_k} x_{M+1}) \longrightarrow (y_1, y_2, \dots, y_{M+1}) \text{as} k \to \infty.$$

For the integers *i*, *j*, *k* with $1 \leqslant i, j \leqslant M$ and $k \geqslant 1$, let

(2.10)

$$y_{j,k} = T_2^{n_k} x_j - L_2^{-1} T^{n_k} x_j,$$

(2.11)

$h_{i,k} = L_2^{-1} T^{n_k} g_{2,i}$,

(2.12)

$\alpha_{i,j} = \lambda_i(x_j)$.

Observe that

$$
\begin{aligned}
y_{j,k} &= T_2^{n_k} x_j - L_2^{-1} T^{n_k} x_j, \text{by(2.10)}\\
&= L_2^{-1} T^{n_k} L_2 x_j - L_2^{-1} T^{n_k} x_j\\
&= L_2^{-1} T^{n_k} x_j + \sum_{i=1}^{M} \lambda_i(x_j) L_2^{-1} T^{n_k} g_{2,i} - L_2^{-1} T^{n_k} x_j \text{ ,by(2.5)}\\
&= \sum_{i=1}^{M} \lambda_i(x_j) L_2^{-1} T^{n_k} g_{2,i}\\
&= \sum_{i=1}^{M} \alpha_{i,j} h_{i,k}, \text{by(2.11)and(2.12)}.
\end{aligned}
$$

For each integer *j* with $1 \leqslant j \leqslant M$, by (2.8), (2.9) we have

$y_{j,k} = T_2^{n_k} x_j - L_2^{-1} T^{n_k} x_j \longrightarrow y_j - L_2^{-1}(0) = y_j \text{as} k \to \infty$.

Moreover, the set $\{y_1, y_2, \ldots, y_M\}$ is linearly independent in *X*. Thus by Lemma 2.2, there exist vectors $h_1, h_2, \ldots, h_M \in \operatorname{span}\{y_1, y_2, \ldots, y_M\}$ such that for integers $1 \leqslant i \leqslant M$, we have $h_{i,k} \longrightarrow h_i$ as $k \to \infty$.

Now, let's examine the sequence $\left(T_2^{n_k} x_{M+1}\right)_{k=1}^{\infty}$. By (2.9), we have $T_2^{n_k} x_{M+1} \longrightarrow y_{M+1}$ as $k \to \infty$. However, we also have

$$
\begin{aligned}
T_2^{n_k} x_{M+1} &= L_2^{-1} T^{n_k} L_2 x_{M+1}\\
&= L_2^{-1} T^{n_k} x_{M+1} + \sum_{i=1}^{M} \lambda_i(x_{M+1}) L_2^{-1} T^{n_k} g_{2,i}, \text{by(2.5)}\\
&= L_2^{-1} T^{n_k} x_{M+1} + \sum_{i=1}^{M} \lambda_i(x_{M+1}) h_{i,k} \text{ ,by(2.11)}\\
&\longrightarrow L_2^{-1}(0) + \sum_{i=1}^{M} \lambda_i(x_{M+1}) h_i \text{ as} k \to \infty\\
&= \sum_{i=1}^{M} \lambda_i(x_{M+1}) h_j \text{ as} k \to \infty.
\end{aligned}
$$

Hence, $y_{M+1} = \sum_{i=1}^{M} \lambda_i(x_{M+1}) h_i \in \operatorname{span}\{y_1, y_2, \ldots, y_M\}$, which contradictions the assumption that the set $\{y_1, y_2, \ldots, y_{M+1}\}$ is linearly independent. □

Remark 2.4

In the construction of the operators $T_1, T_2, \dots, T_N$ in Theorem 2.3, the operator T_1 is any operator which satisfies the Hypercyclicity Criterion. Thus, one may control the operator T_1 in Theorem 2.3. It is also interesting to note that the other operators $T_2, \dots, T_N$ are all conjugates of that original hypercyclic operator T_1.

As a consequence of Theorem 2.3 and Theorem 1.5, the operators $T_1, T_2, \dots, T_N$ given in Theorem 2.3 fail to satisfy the d-Hypercyclicity Criterion. In particular, taking M=2 in Theorem 2.3, we get an example of d-weakly mixing operators $T_1, T_2, \dots, T_N$ which fail to satisfy the d-Hypercyclicity Criterion. Even more, since the operator T_1 satisfies the Hypercyclicity Criterion, using Remark 2.4 one can easily verify that each of the operators $T_1, T_2, \dots, T_N$ satisfies the Hypercyclicity Criterion with respect to the same strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers.

Corollary 2.5

For any integer $N \geqslant 2$, every separable, infinite dimensional Banach space X admits operators $T_1, T_2, \dots, T_N$ which are d-weakly mixing and yet fail to satisfy the d-Hypercyclicity Criterion. Even more, we may assume all the operators $T_1, T_2, \dots, T_N$ individually satisfy the Hypercyclicity Criterion with respect to the same strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers.

From Proposition 1.8, we see that d-weakly mixing implies the Disjoint Blow-up/Collapse Property. Combining this result together with Corollary 2.5 and the equivalence of Blow-Up/Collapse Property and the Hypercyclicity Criterion for a single operator [7], [21] yields the following corollary, which is in sharp contrast to the single operator situation.

Corollary 2.6

For any integer $N \geqslant 2$, every separable, infinite dimensional Banach space X admits operators $T_1, T_2, \dots, T_N$ which satisfy the Disjoint Blow-up/Collapse Property, but they fail to satisfy the d-Hypercyclicity Criterion. Even more, we may assume all the operators $T_1, T_2, \dots, T_N$ individually satisfy the Blow-up/Collapse Property.

Bès, Martin, and Sanders [11] showed that the Banach space ℓ^p with $1 \leqslant p < \infty$ admits weighted shift operators $B_1, B_2, \dots, B_N$ with $N \geqslant 2$ which are densely d-hypercyclic, but they fail to satisfy the d-Hypercyclicity Criterion because they are not d-weakly mixing. By taking $M = 1$ in Theorem 2.3, we get that every separable, infinite dimensional Banach space admits such operators.

Corollary 2.7

For any integer $N \geqslant 2$, every separable, infinite dimensional Banach space X admits densely d-hypercyclic operators $T_1, T_2, \dots, T_N$ which fail to be d-weakly mixing.

From the statement before Corollary 2.5 together with Theorem 1.1, we can assume each of the operators $T_1, T_2, \dots, T_N$ given in Corollary 2.7 is individually weakly mixing, and yet when all together they fail to be d-weakly mixing.

Lastly, it is interesting to mention that Bès, Martin, and Sanders [11] in fact showed weighted shift operators $B_1, B_2, \dots, B_N$ with $N \geqslant 2$ are d-hypercyclic if and only if they satisfy the Disjoint Blow-up/Collapse Property, and yet they always fail to be d-weakly mixing. However, even with Theorem 2.3, we cannot say for certain whether the operators $T_1, T_2, \dots, T_N$ given in Corollary 2.7 must also satisfy the Disjoint Blow-up/Collapse Property. This observation leads us to the following question.

Question 2.8

Does every separable, infinite dimensional Banach space *X* admit operators $T_1, T_2, \dots, T_N$ with $N \geqslant 2$ which satisfy the Disjoint Blow-up/Collapse Property, and yet they fail to be d-weakly mixing?

We conclude this section with one final remark.

Remark 2.9

Even though the results in this section are given for the Banach space setting, all the results in fact hold true in the more general Fréchet space setting.

3. Non-dense set of d-hypercyclic vectors

In Section 3, we provide examples which illustrate that some important dynamical properties for a single operator fail to hold true in the disjoint setting. To help simplify our computations, we restricted our examples of d-hypercyclic operators to just two operators. However, by following the ideas presented in this section, one may extend the examples to families of three or more d-hypercyclic operators.

Similar to the results in Section 2, the conjugates of a hypercyclic operator which satisfies the Hypercyclicity Criterion will play a vital role in the construction of our examples. The following lemma is a simple consequence on Lemma 2.1, and it will be used at different points in the section.

Lemma 3.1

Let T, L_1, L_2 be operators in $B(X)$ with the operators L_1, L_2 invertible, and set $T_m = L_m^{-1} T L_m$ for integers $m =$ 1,2. If $x \in$ d-$\mathcal{HC}(T_1, T_2)$, then $L_2 x - L_1 x \in \mathcal{HC}(T)$.

Proof

If $x \in$ d-$\mathcal{HC}(T_1, T_2)$, then from Lemma 2.1, we get that $(L_1 x, L_2 x) \in \mathcal{HC}(T \oplus T)$. Since the orbit $\mathrm{Orb}\big(T \oplus T, (L_1 x, L_2 x)\big)$ is dense in $X \oplus X$, one can easily verify the orbit $\mathrm{Orb}(T, L_2 x - L_1 x)$ is also dense *X*. □

As stated in the Introduction, a single operator *T* is hypercyclic if and only if its corresponding set $\mathcal{HC}(T)$ of hypercyclic vectors is a dense G_δ. Consequentially, we do not need to make a distinction between hypercyclic and densely hypercyclic. However, the same cannot be said in the disjoint setting. The set of d-hypercyclic vectors is not necessarily dense, and so d-hypercyclicity does not automatically imply dense d-hypercyclicity. Before we provide examples of such d-hypercyclic operators, we first need the following proposition. It gives a sufficient condition for the existence of d-hypercyclic direct sum operators whose corresponding set of d-hypercyclic vectors is contained within a finite dimensional subspace.

Proposition 3.2

Let $M \geqslant 1$ and let T_1 be an operator in $B(X)$. Suppose there exist an injective operator R in $B(X)$ and vectors $x_1, \dots, x_M, y_1, \dots, y_M$ in X such that

(i) $(x_1, \dots, x_M, y_1, \dots, y_M) \in \mathcal{HC}\big(\oplus_{i=1}^{2M} T_1\big)$
(ii) $R x_j = y_j$ *for integers* $1 \leqslant j \leqslant M$, *and*
(iii) $R(X) \cap \mathcal{HC}(T_1) \subseteq \mathrm{span}\{y_1, \dots, y_M\}$

There exists an operator T_2 in $B(X)$ for which the set d-$\mathcal{HC}\big(\oplus_{i=1}^{M} T_1, \oplus_{i=1}^{M} T_2\big)$ *of d-hypercyclic vectors is exactly*

$$\mathcal{A} = \left\{(h_1, \dots, h_M) \in \bigoplus_{i=1}^{M} \text{span}\{x_1, \dots, x_M\}: h_1, \dots, h_M \text{linearly independent}\right\}.$$

Proof

Suppose the vectors $x_1, \dots, x_M, y_1, \dots, y_M$ in *X* and the injective operator *R* satisfy conditions (i)–(iii). Select a scalar $c > \|R\|$, and set $L_1 = cI$ and $L_2 = cI + R$, where *I* is the identity map on *X*. Note that by the choice of the scalar *c*, the operator L_2 is invertible; for example, see [14, Lemma 2.1 on p. 192]. For integers $m = 1,2$, let $T_m = L_m^{-1} T_1 L_m$.

To establish $\mathcal{A} \subseteq$ d-$\mathcal{HC}(\oplus_{i=1}^{M} T_1, \oplus_{i=1}^{M} T_2)$, suppose the vectors $h_1, \dots, h_M$ in $\text{span}\{x_1, \dots, x_M\}$ are linearly independent. Thus, for integers $1 \leqslant l \leqslant M$, there exist scalars $\alpha_{1,l}, \dots, \alpha_{M,l}$ in F such that $h_l = \alpha_{1,l} x_1 + \cdots + \alpha_{M,l} x_M$ and the $M \times M$ matrix $A = [\alpha_{i,j}]_{i,j=1,\dots,M}$ is invertible. Let $A^{-1} = [\beta_{i,j}]_{i,j=1,\dots,M}$. Using the formal definition of a hypercyclic vector, one can easily show that the vector

$$(L_1 h_1, \dots, L_1 h_M, L_2 h_1, \dots, L_2 h_M) = (ch_1, \dots, ch_M, ch_1 + Rh_1, \dots, ch_M + Rh_M)$$

is a hypercyclic vector for direct sum operator $\oplus_{i=1}^{2M} T_1$ if and only if the vector $(h_1, \dots, h_M, Rh_1, \dots, Rh_M)$ is also a hypercyclic vector, and so by Lemma 2.1 it suffices to show the orbit of this second vector is dense. To this end, let $(w_1, \dots, w_M, z_1, \dots, z_M) \in \oplus_{i=1}^{2M} X$. From condition (i), there exists a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that for each integer *j* with $1 \leqslant j \leqslant M$, we have

$$T_1^{n_k} x_j \longrightarrow \sum_{i=1}^{M} \beta_{i,j} w_i \text{and} T_1^{n_k} y_j \longrightarrow \sum_{i=1}^{M} \beta_{i,j} z_i \text{as} k \to \infty,$$

and so for integers $1 \leqslant l \leqslant M$, we have

$$T_1^{n_k} h_l = \sum_{j=1}^{M} \alpha_{j,l} T_1^{n_k} x_j \longrightarrow \sum_{j=1}^{M} \alpha_{j,l} \left(\sum_{i=1}^{M} \beta_{i,j} w_i\right) = \sum_{i=1}^{M} \left(\sum_{j=1}^{M} \beta_{i,j} \alpha_{j,l}\right) w_i = w_l \text{as} k \to \infty \text{because} A^{-1}$$
$$= [\beta_{i,j}]_{i,j=1,\dots,M}.$$

From condition (ii), we have $Rh_l = \alpha_{1,l} y_1 + \cdots + \alpha_{M,l} y_M$, and so a similar argument yields $T_1^{n_k} Rh_l \longrightarrow z_l$ as $k \to \infty$ for integers $1 \leqslant l \leqslant M$.

To establish d-$\mathcal{HC}(\oplus_{i=1}^{M} T_1, \oplus_{i=1}^{M} T_2) \subseteq \mathcal{A}$, let $(h_1, \dots, h_M)$ be a d-hypercyclic vector for the operators $\oplus_{i=1}^{M} T_1, \oplus_{i=1}^{M} T_2$. First, note that the vectors $h_1, \dots, h_M$ must be linearly independent. Next by Lemma 3.1, for each integer *l* with $1 \leqslant l \leqslant M$, we have $Rh_l = L_2 h_l - L_1 h_l \in \mathcal{HC}(T_1)$. Condition (iii) then gives us that $Rh_l = \alpha_{1,l} y_1 + \cdots + \alpha_{M,l} y_M$ for some scalars $\alpha_{1,l}, \dots, \alpha_{M,l}$. From condition (ii), we also have $R(\alpha_{1,l} x_1 + \cdots + \alpha_{M,l} x_M) = \alpha_{1,l} y_1 + \cdots + \alpha_{M,l} y_M$. Since the operator *R* is injective, we have $h_l = \alpha_{1,l} x_1 + \cdots + \alpha_{M,l} x_M$ and this concludes our proof. □

Remark 3.3

As with the operators given in Theorem 2.3, the operator T_2 given within the proof of Proposition 3.2 is a conjugate of the operator T_1.

It turns out that every separable, infinite dimensional Banach space *X* admits operators which satisfy the hypothesis of Proposition 3.2.

Theorem 3.4

Let X be a separable, infinite dimensional Banach space. For any integer $M \geqslant 1$, there exist operators T_1, T_2 in $B(X)$ and vectors $x_1, \dots, x_M$ in X for which the set d-$\mathcal{HC}\left(\oplus_{i=1}^{M} T_1, \oplus_{i=1}^{M} T_2\right)$ *of d-hypercyclic vectors for the direct sum operators $\oplus_{i=1}^{M} T_1$, $\oplus_{i=1}^{M} T_2$ is exactly*

$$\left\{(h_1, \dots, h_M) \in \bigoplus_{i=1}^{M} \operatorname{span}\{x_1, \dots, x_M\}: h_1, \dots, h_M \text{linearly independent}\right\}.$$

Proof

First, we construct the operator T_1 for Proposition 3.2. Since X is a separable, infinite dimensional Banach space, there exists a sequence $(e_k)_{k=0}^{\infty}$ of vectors in X and a sequence $(\lambda_k)_{k=0}^{\infty}$ of linear functionals in the dual space $X^{\square}$ for which the following hold:

- (A1) $\overline{\operatorname{span}\{e_k: k \geqslant 0\}} = X$ and $\|e_k\| = 1$ for all integers $k \geqslant 0$,
- (A2) $\bigcap_{k=0}^{\infty} \operatorname{Ker}(\lambda_k) = \{0\}$ and $\sum_{k=0}^{\infty} \|\lambda_k\| < \infty$,
- (A3) $\lambda_k(e_j) = 0$ whenever $j \neq k$ and $\lambda_k(e_k) = c_k > 0$ for all integers $j, k \geqslant 0$.

One may, for instance, refer to Pelczynski [22] to get such a sequence with an extra assumption $c_k > (1 - \epsilon)\|\lambda_k\|$ for each $\epsilon > 0$.

Consider the operators S, T_1 in $B(X)$ given by

(3.1)

$$Sx = \sum_{k=0}^{\infty} \lambda_{k+1}(x) e_k \text{and} T_1 = I + S$$

where *I* is the identity operator on *X*. Note that the operator *S* is a nuclear operator. Based on the operator T_1, we establish an obstacle for a vector *x* in *X* from belonging to the set $\mathcal{HC}(T_1)$.

Claim 1

If x is a vector in X such that the real part $\operatorname{Re}(\lambda_k(x)) \geqslant 0$ for all sufficiently large integers k, then $x \notin \mathcal{HC}(T_1)$.

Proof

From the definition of the operator *S* in (3.1), it follows that for any vector *x* and any integer $j \geqslant 0$,

(3.2)

$$\lambda_0(S^j x) = \gamma_j \lambda_j(x)$$

where $\gamma_0 = 1$ and $\gamma_j = c_0 c_1 \cdots c_{j-1}$ for integers $j \geqslant 1$. Note that by condition (A3), each $\gamma_j > 0$. Combining Eq. (3.2) with the definition of the operator T_1 in (3.1) yields

(3.3)

$$\lambda_0(T_1^n x) = \sum_{j=0}^{n} \binom{n}{j} \lambda_0(S^j x) = \sum_{j=0}^{n} \binom{n}{j} \gamma_j \lambda_j(x)$$

for any vector *x* in *X* and integer $n \geqslant 0$.

Now, let's suppose x is a vector in X such that $\mathrm{Re}(\lambda_k(x)) \geqslant 0$ for every integer $k > K$. To show $x \notin \mathcal{HC}(T_1)$, it suffices to show the set $\{\lambda_0(T_1^n x): n \geqslant 0\}$ fails to be dense in the scalar field F. For this we have two cases.

Case 1.

Assume $\mathrm{Re}(\lambda_k(x)) = 0$ for every integer $k > K$. In this case, by Eq. (3.3), we get

$$\mathrm{Re}(\lambda_0(T_1^n x)) = \sum_{j=0}^{K} \binom{n}{j} \gamma_j \mathrm{Re}\left(\lambda_j(x)\right),$$

which is a polynomial in the variable n with degree at most K. This implies that $|\mathrm{Re}(\lambda_0(T_1^n x))| \longrightarrow \infty$ as $n \to \infty$, and so the set $\{\lambda_0(T_1^n x): n \geqslant 0\}$ is not dense in $\mathbb{F}$.

Case 2.

Assume $\mathrm{Re}\left(\lambda_{k_0}(x)\right) > 0$ for some integer $k_0 > K$. In this case, note that by Eq. (3.3) and the fact that $\mathrm{Re}(\lambda_k(x)) \geqslant 0$ for every integer $k > K$ gives us

$$\mathrm{Re}(\lambda_0(T_1^n x)) = \sum_{j=0}^{K} \binom{n}{j} \gamma_j \mathrm{Re}\left(\lambda_j(x)\right) + \sum_{j=K+1}^{n} \binom{n}{j} \gamma_j \mathrm{Re}\left(\lambda_j(x)\right) \geqslant \binom{n}{k_0} \gamma_{k_0} \mathrm{Re}\left(\lambda_{k_0}(x)\right) + \sum_{j=0}^{K} \binom{n}{j} \gamma_j \mathrm{Re}\left(\lambda_j(x)\right)$$

for any integer $n \geqslant k_0$. One may easily verify that

$$\frac{1}{n^{k_0}} \binom{n}{j} \longrightarrow \begin{cases} \frac{1}{k_0!}, & \text{if} j = k_0 \\ 0, & \text{if} 0 \leqslant j < k_0 \end{cases} \quad \text{as} n \to \infty.$$

Therefore,

$$\liminf_{n\to\infty} \frac{\mathrm{Re}(\lambda_0(T_1^n x))}{n^{k_0}} \geqslant \lim_{n\to\infty} \frac{1}{n^{k_0}} \left[\binom{n}{k_0} \gamma_{k_0} \mathrm{Re}\left(\lambda_{k_0}(x)\right) + \sum_{j=1}^{K} \binom{n}{j} \gamma_j \mathrm{Re}\left(\lambda_j(x)\right) \right] = \frac{\gamma_{k_0}}{k_0!} \mathrm{Re}\left(\gamma_{k_0}(x)\right) > 0.$$

Again, this implies $|\mathrm{Re}(\lambda_0(T^n x))| \longrightarrow \infty$ as $n \to \infty$, and so the set $\{\lambda_0(T^n x): n \geqslant 0\}$ fails to be dense in $\mathbb{F}$, which completes the proof of our claim. □

For each integer $j \geqslant 1$, consider the vector g_j in X given by

(3.4)

$$g_j = \sum_{i=1}^{\infty} 2^{-ij} e_i.$$

From condition (A1), we get $\|g_j\| \leqslant \sum_{i=1}^{\infty} 2^{-ij} = \frac{1}{2^j - 1}$. This estimate implies that the operator R_0 in $B(X)$ given by

(3.5)

$$R_0 x = \sum_{j=1}^{\infty} \lambda_j(x) g_j$$

is yet another nuclear operator. This operator plays a crucial role in the creation of the operator *R* for Proposition 3.2. Our next claim gives us some useful properties of this operator R_0.

Claim 2

The operator R_0 has the following properties:

(1) *If $x \in X$ and $m \geqslant 1$ satisfy $\lambda_m(x) \neq 0$ and $\lambda_j(x) = 0$ for integers $1 \leqslant j < m$, then $\lambda_k(R_0 x) = \lambda_m(x) c_k 2^{-mk}\big(1 + o(1)\big)$ as $k \to \infty$.*
(2) $\mathrm{Ker}(R_0) = \mathrm{span}\{e_0\}$.
(3) $R_0(X) \cap \mathcal{HC}(T_1) = \emptyset$.

Proof

For property (1), let *x* be a vector in *X* and $m \geqslant 1$ with $\lambda_m(x) \neq 0$ and $\lambda_j(x) = 0$ for integers $1 \leqslant j < m$. Observe that for each integer $n \geqslant 0$, we have

$$\lambda_k(R_0 x) = \sum_{j=1}^{\infty} \lambda_j(x) \lambda_k(g_j), \text{by}(3.5)$$

$$= \sum_{j=1}^{\infty} 2^{-jk} c_k \lambda_j(x), \text{by}(3.4)\text{and (A3)}$$

$$= \sum_{j=m}^{\infty} 2^{-jk} c_k \lambda_j(x)$$

$$= 2^{-mk} c_k \lambda_m(x)(1 + a_k), \text{where} a_k = \sum_{j=1}^{\infty} \frac{2^{-jk} \lambda_{j+m}(x)}{\lambda_m(x)}.$$

Clearly, we have

$$|a_k| \leqslant \sum_{j=1}^{\infty} 2^{-jk} \frac{|\lambda_{j+m}(x)|}{|\lambda_m(x)|} \leqslant 2^{-k} \frac{\|x\|}{|\lambda_m(x)|} \sum_{j=1}^{\infty} \|\lambda_{j+m}\| \longrightarrow 0 \text{as} k \to \infty$$

because of condition (A2) which completes the proof of property (1).

For property (2), note that $R_0 e_0 = \sum_{j=1}^{\infty} \lambda_j(e_0) g_j = 0$ by condition (A3), and so $\mathrm{span}\{e_0\} \subseteq \mathrm{Ker}(R_0)$. For the inclusion $\mathrm{Ker}(R_0) \subseteq \mathrm{span}\{e_0\}$, assume $x \notin \mathrm{span}\{e_0\}$. Then there exists an integer $m \geqslant 1$ such that $\lambda_m(x) \neq 0$ and $\lambda_j(x) = 0$ for integers $1 \leqslant j < m$. [Otherwise, $x - \lambda_0(x) e_0 \in \bigcap_{k=0}^{\infty} \mathrm{Ker}\,(\lambda_k) = \{0\}$ which is a contradiction.] Property (1) then gives us that $\lambda_k(R_0 x) \neq 0$ for all sufficiently large integers *k* which in turn implies that $x \notin \mathrm{Ker}(R_0)$.

Lastly for property (3), suppose that $R_0 x \in \mathcal{HC}(T_1)$. From above, we get $x \notin \mathrm{Ker}(R_0) = \mathrm{span}\{e_0\}$. Again, this implies that there is an integer $m \geqslant 1$ such that $\lambda_m(x) \neq 0$ and $\lambda_j(x) = 0$ for integers $1 \leqslant j < m$. From condition (1), it follows that $\lambda_k(R_0 x) = \lambda_m(x) c_k 2^{-mk}\big(1 + o(1)\big)$ as $k \to \infty$, and so

$$\mathrm{Re}\left(\lambda_k\left(\frac{1}{\lambda_m(x)} R_0 x\right)\right) \geqslant 0$$

for all sufficiently large integers k. By Claim 1, we get $\frac{1}{\lambda_m(x)} R_0 x \notin \mathcal{HC}(T_1)$, and so $R_0 x \notin \mathcal{HC}(T_1)$ as well which gives us a contradiction. Therefore, $R_0(X) \cap \mathcal{HC}(T_1) = \emptyset$, and the proof of Claim 2 is now complete. □

There is one more claim before we construct the desired operator R for Proposition 3.2. We want to select a hypercyclic vector $(x_1, x_2, \dots, x_M, y_1, y_2, \dots, y_M)$ for the direct sum operator $\oplus_{i=1}^{2M} T_1$ for which the sequences $\left(\lambda_k(y_j)\right)_{k=0}^{\infty}$ converge to 0 arbitrarily fast.

Claim 3

Let $(\epsilon_k)_{k=0}^{\infty}$ be a sequence of positive real numbers. There is a vector $(x_1, x_2, \dots, x_M, y_1, y_2, \dots, y_M)$ in $\mathcal{HC}(\oplus_{i=1}^{2M} T_1)$ such that $\lambda_0(x_1) \neq 0$, $\lambda_0(y_1 - R_0 x_1) \neq 0$ and $\lim_{k\to\infty} \frac{\lambda_k(y_j)}{\epsilon_k} = 0$ for integers $1 \leqslant j \leqslant M$.

Proof

To begin, select a decreasing sequence $(d_i)_{i=0}^{\infty}$ of positive real numbers such that for each integer $k \geqslant 0$,

(3.6)

$c_i d_i \leqslant \epsilon_i \text{and} c_{i+1} d_{i+1} \leqslant c_i d_i.$

Define the operators D, S_0 in $B(X)$ by the formulas

(3.7)

$$Dx = \sum_{i=0}^{\infty} d_i \lambda_i(x) e_i \text{and} S_0 x = \sum_{i=0}^{\infty} \frac{d_{i+1} c_{i+1}}{d_i c_i} \lambda_{i+1}(x) e_i.$$

By the density of the linear span $\{e_k : k \geqslant 0\}$ from condition (A1) and by condition (A3), the operator D has dense range. From the definition of the operators in (3.1) and (3.7), one can easily verify that

(3.8)

$SDe_0 = DS_0 e_0 = 0 \text{and} SDe_k = DS_0 e_k = c_k^2 d_k e_{k-1} \text{for integers} k \geqslant 1.$

From condition (A1) again, we get $SD = DS_0$ on X. Setting $T_0 = I + S_0$, it then follows that $T_1 D = DT_0$.

The direct sum operator $\oplus_{i=1}^{2M} T_0$ is hypercyclic. To see this, observe that for each integer $n \geqslant 1$, we have $S_0^n(X) \cap \operatorname{Ker}(S_0^n) = \operatorname{span}\{e_k : 0 \leqslant k \leqslant n-1\}$. Again, by condition (A1), the union $\bigcup_{n=1}^{\infty} S_0^n(X) \cap \operatorname{Ker}(S_0^n) = \operatorname{span}\{e_k : k \geqslant 0\}$ is dense in X. According to [3, Theorem 2.2], the operator T_0 must be topologically mixing, and hence satisfies the Hypercyclicity Criterion.

For the selection of the hypercyclic vector $(x_1, x_2, \dots, x_M, y_1, y_2, \dots, y_M)$, consider the linear functionals φ, ψ on $\oplus_{i=1}^{2M} X$ given by

$\varphi(u_1, u_2, \dots, u_M, v_1, v_2, \dots, v_M) = \lambda_0(Du_1),$

$\psi(u_1, u_2, \dots, u_M, v_1, v_2, \dots, v_M) = \lambda_0(Dv_1 - R_0 Du_1).$

Since the operator D has a dense range, the linear functionals φ, ψ are nonzero. Thus, the set W given by

$$W = \left\{(u_1, \dots, u_M, v_1, \dots, v_M) \in \bigoplus_{i=1}^{2M} X : \varphi\psi \neq 0\right\}$$

is a nonempty open set. Hence, we can select $(u_1, \dots, u_M, v_1, \dots, v_M) \in W \cap \mathcal{HC}(\oplus_{i=1}^{2M} T_0)$. Set $x_j = Du_j$ and $y_j = Dv_j$ for integers $1 \leqslant j \leqslant M$. Now, observe that

$$\begin{aligned}\operatorname{Orb}\left(\bigoplus_{i=1}^{2n} T_1, (x_1, \dots, x_M, y_1, \dots, y_M)\right) &= \{(T_1^n Du_1, \dots, T_1^n Du_M, T_1^n Dv_1, \dots, T_1^n Dv_M) : n \geqslant 0\} \\ &= \{(DT_0^n u_1, \dots, DT_0^n u_M, DT_0^n v_1, \dots, DT_0^n v_M) : n \geqslant 0\} \\ &= \bigoplus_{i=1}^{2M} D\left[\operatorname{Orb}\left(\bigoplus_{i=1}^{2M} T_0, (u_1, \dots, u_M, v_1, \dots, v_M)\right)\right].\end{aligned}$$

Again, since the operator *D* has a dense range, we get $(x_1, \dots, x_M, y_1, \dots, y_M) \in \mathcal{HC}(\oplus_{i=1}^{2M} T_1)$.

Lastly, note that by the definition of the linear functionals φ, ψ, we have $\lambda_0(x_1) \neq 0$ and $\lambda_0(y_1 - R_0 x_1) \neq 0$. Moreover, for integers *j*, *k* with $1 \leqslant j \leqslant M$ and $k \geqslant 0$, we have

$$|\lambda_k(y_j)| = |\lambda_k(Dv_j)| = c_k d_k |\lambda_k(v_j)| \leqslant c_k d_k \|\lambda_k\| \|v_j\|.$$

Recall that by condition (A2), we have $\sum_{k=0}^{\infty} \|\lambda_k\| < \infty$. This together with inequality (3.6) gives us

$$\frac{|\lambda_k(y_j)|}{\epsilon_k} \leqslant \frac{c_k d_k}{\epsilon_k} \|\lambda_k\| \|v_j\| \leqslant \|\lambda_k\| \|v_j\| \longrightarrow 0 \text{as} k \to \infty,$$

and this concludes the proof of Claim 3. □

To construct the operator *R* for Proposition 3.2, let $(x_1, \dots, x_M, y_1, \dots, y_M)$ be the hypercyclic vector for the direct sum operator $\oplus_{i=1}^{2M} T_1$ from Claim 3 associated with the sequence $(\epsilon_k)_{k=0}^{\infty}$ where $\epsilon_k = c_k 2^{-k^2}$. In particular, note that

(3.9)

$$\lambda_0(x_1) \neq 0 \text{and} \lambda_0(y_1 - R_0 x_1) \neq 0,$$

(3.10)

$$\lim_{k \to \infty} \frac{2^{k^2} \lambda_k(y_j)}{c_k} = 0 \text{for integers} 1 \leqslant j \leqslant M.$$

Next, note that the set $\{x_1, \dots, x_M\}$ must be linearly independent, and so by a corollary of the Hahn–Banach Theorem [14, Corollary 6.6, p. 79], there exist linear functionals $\tilde{\lambda}_2, \dots, \tilde{\lambda}_M$ in the dual space X^{*} such that

(3.11)

$$\tilde{\lambda}_i(x_j) = \begin{cases} 1, & \text{if} i = j \\ 0, & \text{if} i \neq j. \end{cases}$$

Define the operator $R : X \longrightarrow X$ by the formula

(3.12)

$$Rx = R_0x + \lambda_0(x)f_1 + \sum_{i=2}^{M} \tilde{\lambda}_i(x)f_i$$

with $f_1 = \frac{1}{\lambda_0(x_1)}(y_1 - R_0x_1)$ and $f_i = y_i - R_0x_i - \lambda_0(x_i)f_1$ for integers $2 \leqslant i \leqslant M$. Using the above definitions, observe that for each vector *x* in *X* we can write

(3.13)

$$Rx = R_0(x - ax_1 - \sum_{i=2}^{M} b_ix_i) + ay_1 + \sum_{i=2}^{M} b_iy_i$$

where $a = \frac{\lambda_0(x)}{\lambda_0(x_1)} + \sum_{i=2}^{M} \tilde{\lambda}_i(x)\frac{\lambda_0(x_i)}{\lambda_0(x_1)}$ and $b_i = \tilde{\lambda}_i(x)$ for integers $2 \leqslant i \leqslant M$.

Now, we are ready to prove T_1, *R* are the desired operators which satisfy the hypothesis of Proposition 3.2. First, note that condition (i) of Proposition 3.2 is clearly satisfied. To show the operator *R* is injective, suppose that Rx=0. From Eq. (3.13), this implies that

(3.14)

$$R_0\left(x - ax_1 - \sum_{i=2}^{M} b_ix_i\right) = -ay_1 - \sum_{i=2}^{M} b_iy_i.$$

Now, since $(x_1, \dots, x_M, y_1, \dots, y_M) \in \mathcal{HC}(\oplus_{i=1}^{2M} T_1)$, the set $\{y_1, y_2, \dots, y_M\}$ is linearly independent. Furthermore, one can easily verify that any nonzero vector in $\text{span}\{y_1, y_2, \dots, y_M\}$ is also in $\mathcal{HC}(T_1)$. However, from Claim 2, part (3), we have $R_0(X) \cap \mathcal{HC}(T_1) = \emptyset$. This observation together with Eq. (3.14) tells us that $a = b_2 = \cdots = b_M = 0$. Substituting these zero scalars back into Eq. (3.14) yields $R_0x = 0$; that is, $x \in \text{Ker}(R_0) = \text{span}\{e_0\}$ by Claim 2, part (2). Write $x = \alpha e_0$ for some scalar α∈F. Now, observe that

$$0 = a = \frac{\lambda_0(x)}{\lambda_0(x_1)} + \sum_{i=2}^{M} \tilde{\lambda}_i(x)\frac{\lambda_0(x_i)}{\lambda_0(x_1)}$$

$$= \frac{\alpha c_0}{\lambda_0(x_1)} + \sum_{i=2}^{M} b_i\frac{\lambda_0(x_i)}{\lambda_0(x_1)}, \text{because} x = \alpha e_0 \text{and condition (A3)}$$

$$= \frac{\alpha c_0}{\lambda_0(x_1)}, \text{since} b_2 = \cdots = b_M = 0.$$

Since $c_0 > 0$, it follows that $\alpha = 0$. Therefore, $x = \alpha e_0 = 0$.

Next, to establish condition (ii) in Proposition 3.2, observe that

$$Rx_1 = R_0x_1 + \lambda_0(x_1)f_1 + \sum_{i=2}^{M} \tilde{\lambda}_i(x_1)f_i$$

$$= R_0 x_1 + \frac{\lambda_0(x_1)}{\lambda_0(x_1)}(y_1 - R_0 x_1), \text{by}(3.11)$$

$$= y_1,$$

and for integer j with $2 \leqslant j \leqslant M$,

$$Rx_j = R_0 x_j + \lambda_0(x_j) f_1 + \sum_{i=2}^{M} \tilde{\lambda}_i(x_j) f_i$$

$$= R_0 x_j + \lambda_0(x_j) f_1 + \left(y_j - R_0 x_j - \lambda_0(x_j) f_1\right), \text{by}(3.11)$$

$$= y_j.$$

Lastly, to prove condition (iii) in Proposition 3.2 and complete our proof, assume $Rx \in \mathcal{HC}(T_1)$. By Eq. (3.13), we can write

(3.15)

$$Rx = R_0 z + a y_1 + \sum_{i=2}^{M} b_i y_i$$

where $z = x - a x_1 - \sum_{i=2}^{M} b_i x_i$. We have two cases based on whether the vector z is in span$\{e_0\}$.

Case 1.

Assume $z \notin$ span$\{e_0\}$. In this case, there exists an integer $m \geqslant 1$ such that $\lambda_m(z) \neq 0$ and $\lambda_j(z) = 0$ for integers $2 \leqslant j \leqslant M$. Claim 2, part (1), then implies that $\lambda_k(R_0 z) = \lambda_m(z) c_k 2^{-mk}\left(1 + o(1)\right)$ as $k \to \infty$. Combining this with the limits in (3.10) and Eq. (3.15) give us

$$\lambda_k(Rx) = \lambda_k\left(R_0 z + a y_1 + \sum_{i=2}^{M} b_i y_i\right) = \lambda_m(z) c_k 2^{-mk}\left(1 + o(1)\right) \text{as} k \to \infty.$$

Same as in the proof of Claim 2, part (3), this implies that $\frac{1}{\lambda_m(z)} Rx \notin \mathcal{HC}(T_1)$, and so $Rx \notin \mathcal{HC}(T_1)$ which gives us a contradiction. Thus, we must have Case 2.

Case 2.

Assume $z = \alpha e_0$ for some scalar α in F. In this case, we have

$$\alpha e_0 = x - a x_1 - \sum_{i=2}^{M} b_i x_i,$$

and by applying linear functional λ_0 yields

$$\alpha c_0 = \lambda_0(x) - a \lambda_0(x_1) - \sum_{i=2}^{M} b_i \lambda_0(x_i)$$

$$= \lambda_0(x) - \left(\frac{\lambda_0(x)}{\lambda_0(x_1)} + \sum_{i=2}^{M} \tilde{\lambda}_i(x) \frac{\lambda_0(x_i)}{\lambda_0(x_1)}\right) \lambda_0(x_1) - \sum_{i=2}^{M} \tilde{\lambda}_i(x) \lambda_0(x_i) = 0.$$

Since $c_0 > 0$, we must have $\alpha = 0$. Hence, $x = ax_1 + \sum_{i=2}^{M} b_i x_i \in \mathrm{span}\{x_1, \dots, x_M\}$. Therefore, $Rx \in \mathrm{span}\{y_1, \dots, y_M\}$ because $Rx_j = y_j$ for integers $1 \leqslant j \leqslant M$. □

Clearly, the set d-$\mathcal{HC}(\oplus_{i=1}^{M} T_1, \oplus_{i=1}^{M} T_2)$ of d-hypercyclic vectors given in Theorem 3.4 is nowhere dense because it is contained within a finite dimensional subspace, and so the associated d-hypercyclic operators fail to be densely d-hypercyclic.

Corollary 3.5

Let X be a separable, infinite dimensional Banach space. For any integer $M \geqslant 1$, there exist operators T_1, T_2 for which the direct sum operators $\oplus_{i=1}^{M} T_1$, $\oplus_{i=1}^{M} T_2$ are d-hypercyclic, but they fail to be densely d-hypercyclic. In fact, their corresponding set of d-$\mathcal{HC}(\oplus_{i=1}^{M} T_1, \oplus_{i=1}^{M} T_2)$ *of d-hypercyclic vectors is nowhere dense.*

Since the d-hypercyclic direct sum operators $\oplus_{i=1}^{M} T_1$, $\oplus_{i=1}^{M} T_2$ fail to be densely d-hypercyclic, the d-hypercyclic operators T_1, T_2 given in Theorem 3.4 or Corollary 3.5 are yet another example of d-hypercyclic operators which fail to satisfy the d-Hypercyclicity Criterion; see Proposition 1.3 and Theorem 1.5. It is interesting to note that these same operators satisfy a property similar to the operators given in Theorem 2.3; that is, even though the operators $\oplus_{i=1}^{M} T_1$, $\oplus_{i=1}^{M} T_2$ are d-hypercyclic, the operators $\oplus_{i=1}^{M+1} T_1$, $\oplus_{i=1}^{M+1} T_2$ fail to be d-hypercyclic. To see this, suppose that $(f_1, \dots, f_{M+1}) \in$ d-$\mathcal{HC}(\oplus_{i=1}^{M+1} T_1, \oplus_{i=1}^{M+1} T_2)$. It follows that both vectors $(f_1, f_2, \dots, f_M)$ and $(f_2, f_3, \dots, f_{M+1})$ are in

$$\text{d-}\mathcal{HC}\left(\bigoplus_{i=1}^{M} T_1, \bigoplus_{i=1}^{M} T_2\right) = \left\{(h_1, \dots, h_M) \in \bigoplus_{i=1}^{M} \mathrm{span}\{x_1, \dots, x_M\}: h_1, \dots, h_M \text{ linearly independent}\right\}.$$

However, this implies that the vectors $f_1, \dots, f_{M+1} \in \mathrm{span}\{x_1, \dots, x_M\}$, which is impossible because the vectors $f_1, \dots, f_{M+1}$ must be linearly independent.

Let us concentrate on the case when $M = 1$ in Theorem 3.4. Not only does there exist d-hypercyclic operators whose d-hypercyclic vectors consist only of nonzero scalar multiples of a single vector, but we have freedom to choose that vector. Moreover, we can find densely d-hypercyclic operators for which the two families of d-hypercyclic operators fail to have a d-hypercyclic vector in common. This is yet another difference between hypercyclicity and disjoint hypercyclicity. However, this is not surprising because the fact that any countable collection of hypercyclic operators has a dense G_δ set of hypercyclic vectors in common follows from an application of the Baire Category Theorem, which Corollary 3.5 shows we cannot necessarily apply in the disjoint setting.

Corollary 3.6

Let X be a separable, infinite dimensional Banach space and let g be any nonzero vector in X. There exist d-hypercyclic operators A_1, A_2 in $B(X)$ which fail to be densely d-hypercyclic because

$$\text{d-}\mathcal{HC}(A_1, A_2) = \mathrm{span}\{g\} \setminus \{0\}.$$

Moreover, there exist densely d-hypercyclic operators $\tilde{A}_1$, $\tilde{A}_2$ such that

$$\text{d-}\mathcal{HC}(A_1, A_2) \cap \text{d-}\mathcal{HC}\left(\tilde{A}_1, \tilde{A}_2\right) = \emptyset.$$

That is, the two families of operators fail to have any d-hypercyclic vectors in common.

Proof

Let $g \in X \setminus \{0\}$. Recall that we use Proposition 3.2 to establish Theorem 3.4 with $M = 1$. From that proof we get d-hypercyclic operators T_1, T_2, an injective operator *R*, and vectors x_1, y_1 in *X* for which

(3.16)

$$(x_1, y_1) \in \mathcal{HC}(T_1 \oplus T_1),$$

(3.17)

$$Rx_1 = y_1,\text{and}$$

(3.18)

$$R(X) \cap \mathcal{HC}(T_1) \subseteq \text{span}\{y_1\}.$$

Moreover, d-$\mathcal{HC}(T_1, T_2) = \text{span}\{x_1\} \setminus \{0\}$. Now, let *L* be any invertible operator in $B(X)$ with $Lg = x_1$. Set $R_1 = L^{-1}RL$ and $A_1 = L^{-1}T_1L$.

To show that A_1, R_1 are the operators that satisfy Proposition 3.2, first note that the operator R_1 is clearly injective. Next, observe that

$$\left(Lg, L(L^{-1}y_1)\right) = (x_1, y_1) \in \mathcal{HC}(T_1 \oplus T_1),\text{by(3.16)}$$

and so by Lemma 2.1 with $N = 1$ and $M = 2$, we get $(g, L^{-1}y_1) \in \mathcal{HC}(A_1 \oplus A_1)$. Also, Eq. (3.17) gives us $R_1g = L^{-1}RLg = L^{-1}Rx_1 = L^{-1}y_1$, establishing conditions (i) and (ii) in Proposition 3.2.

For condition (iii) in Proposition 3.2, suppose $R_1x \in \mathcal{HC}(A_1)$. From the definition of the operators R_1, A_1 and by Lemma 2.1 again, it follows that $RLx = LR_1x \in \mathcal{HC}(T_1)$. Thus, from (3.18), we get $RLx = \alpha y_1$ for some scalar α. Therefore, $R_1x = L^{-1}RLx = \alpha L^{-1}y_1$, and so $R_1(X) \cap \mathcal{HC}(A_1) \subseteq \text{span}\{L^{-1}y_1\}$. Hence, from Proposition 3.2 we get the desired operator A_2 such that d-$\mathcal{HC}(A_1, A_2) = \text{span}\{g\} \setminus \{0\}$.

For the construction of the operators A˜1, A˜2 in the second half of the proof, let *A* be any operator that satisfies the Hypercyclicity Criterion and that $g \notin \mathcal{HC}(A)$. Let $(z_1, z_2) \in \mathcal{HC}(A \oplus A)$. Note that the set $\{z_1, z_2\}$ is linearly independent, and so there exists a linear functional λ in the dual space X^{*} such that $\lambda(z_1) = 0$ and $\lambda(z_2) = 0$. Let $\tilde{L}_1 = I$, where *I* is the identity map on *X*, and define the operator $\tilde{L}_2 : X \longrightarrow X$ by $\tilde{L}_2 x = x + \lambda(x) z_2$. One can easily verify $\tilde{L}_2$ is invertible. In fact, $\tilde{L}_2^{-1} x = x - \lambda(x) z_2$. Set $\tilde{A}_m = \tilde{L}_m^{-1} A \tilde{L}_m$ for integers $m = 1,2$.

Now, one can apply the exact same argument found within the proof of Theorem 2.3 with $M = 1$ and $N = 2$ to show the operators $\tilde{A}_1$, $\tilde{A}_2$ are densely d-hypercyclic. However, observe that for any nonzero scalar α in $\mathbb{F}$,

$$\left(\tilde{L}_1(\alpha g), \tilde{L}_2(\alpha g)\right) = \left(\alpha g, \tilde{L}_2(\alpha g)\right) \notin \text{d-}\mathcal{HC}(A \oplus A)$$

because $g \notin \mathcal{HC}(A)$. By Lemma 2.1, we get $\alpha g \notin$ d-$\mathcal{HC}\left(\tilde{A}_1, \tilde{A}_2\right)$. Thus d-$\mathcal{HC}(A_1, A_2) \cap$ d-$\mathcal{HC}\left(\tilde{A}_1, \tilde{A}_2\right) = \emptyset$.

□

Remark 3.7

Following the same ideas given in the first half of the proof of Corollary 3.6, we can construct operators whose set of d-hypercyclic vectors for their direct sum operators is completely predetermined beforehand. That is, given any integer $M \geqslant 1$ and any linearly independent set $\{g_1, g_2, \dots, g_M\}$, there are operators A_1, A_2 in $B(X)$ for which the set of d-$\mathcal{HC}\left(\oplus_{i=1}^{M} A_1, \oplus_{i=1}^{M} T_2\right)$ is exactly

$$\left\{(h_1, \dots, h_M) \in \bigoplus_{i=1}^{M} \operatorname{span}\{g_1, \dots, g_M\}: h_1, \dots, h_M \text{are linearly independent}\right\}.$$

For a single operator, hypercyclicity is equivalent to topological transitivity. Likewise, for a finite family of operators, dense d-hypercyclicity is equivalent to d-topological transitivity; see Proposition 1.3. However, with Corollary 3.5 in mind, we see that we cannot drop the term "densely" from this disjoint characterization.

Corollary 3.8

Let X be a separable, infinite dimensional Banach space. There exist d-hypercyclic operators T_1, T_2 in $B(X)$ which fail to be d-topologically transitive.

Now, let us consider the case when $M = 2$. Again, due to the equivalence of hypercyclicity and topological transitivity, an operator *T* is weakly mixing if and only if the direct sum operator $T \oplus T$ is hypercyclic. Likewise from Proposition 1.3, the operators T_1, T_2 are d-weakly mixing if and only if the direct sum operators $T_1 \oplus T_1$, $T_2 \oplus T_2$ are densely d-hypercyclic. However from Corollary 3.5 with $M = 2$, the term "densely" cannot be dropped from this disjoint equivalency. That is, the direct sum operators $T_1 \oplus T_1$, $T_2 \oplus T_2$ being d-hypercyclic do not imply the operators T_1, T_2 are d-weakly mixing.

Corollary 3.9

Let X be a separable, infinite dimensional Banach space. There exist operators T_1, T_2 in $B(X)$ for which the direct sum operators $T_1 \oplus T_1$, $T_2 \oplus T_2$ are d-hypercyclic, but the operators T_1, T_2 fail to be d-weakly mixing.

We now turn our attention to the dynamics of invertible operators. Due to the equivalence of hypercyclicity and topological transitivity, an invertible operator *T* is hypercyclic if and only if its inverse $T - 1$ is hypercyclic; see Kitai [20]. Even though dense d-hypercyclicity and d-topological transitivity are equivalent, Bès, Martin, and Peris [9] proved this standard result about invertible operators does not translate to the disjoint setting. They did this by constructing an example of invertible composition operators C_{ϕ_1}, C_{ϕ_2} on the Hardy space $H^2(\mathbb{D})$ which are d-topologically transitive and so densely d-hypercyclic, but whose inverses $C_{\phi_1}^{-1}$, $C_{\phi_2}^{-1}$ fail to be d-hypercyclic. Using techniques from the proof of Corollary 3.6, we can show every separable, infinite dimensional Banach space admits similar invertible d-hypercyclic operators. To do this, we first need the following proposition.

Proposition 3.10

Let T_1 be an invertible operator in $B(X)$. Suppose there exist vectors x_1, y_1 in X such that

(i) $(x_1, y_1) \in \mathcal{HC}(T_1 \oplus T_1)$,

(ii) $y_1 \notin \mathcal{HC}(T_1^{-1})$.

Then there exists an invertible operator T_2 in $B(X)$ for which the operators T_1, T_2 are densely d-hypercyclic, but the *inverse operators T_1^{-1}, T_2^{-1} fail to be d-hypercyclic.*

Proof

Let $L_1 = I$, where *I* is the identity map on *X*, and define the operator $L_2: X \longrightarrow X$ by $L_2 x = x + \lambda(x) y_1$, where λ is a linear functional in X^{*} satisfying $\lambda(x_1) = 1$ and $\lambda(y_1) = 0$. One can easily verify the inverse operator $L_2^{-1}: X \longrightarrow X$ is given by $L_2^{-1} x = x - \lambda(x) y_1$. Set $T_m = L_m^{-1} T_1 L_m$ for integers $m = 1,2$. Using the exact same argument found in the proof of Theorem 2.3 with $M = 1$ and $N = 2$, it follows that the operators T_1, T_2 are densely d-hypercyclic.

To finish the proof, note that $T_m^{-1} = L_m T_1^{-1} L_m^{-1}$. Thus, if $x \in$ d-$\mathcal{HC}(T_1^{-1}, T_2^{-1})$, then by Lemma 3.1, we get $-\lambda(x)y_1 = L_2^{-1}x - L_1^{-1}x \in \mathcal{HC}(T_1^{-1})$, which contradicts condition (ii). Hence, d-$\mathcal{HC}(T_1^{-1}, T_2^{-1}) = \emptyset$. $\square$

Every separable, infinite dimensional Banach space *X* admits a hypercyclic operator which satisfies the hypothesis of Proposition 3.10, and so we have the following corollary.

Corollary 3.11

Every separable, infinite dimensional Banach space X admits invertible operators T_1, T_2 which are densely d-hypercyclic, but the inverse operators T_1^{-1}, T_2^{-1} fail to be d-hypercyclic.

Proof

Every separable, infinite dimensional Banach space admits an operator of the form $T = I + K$, where *K* is a nuclear operator with $\|K\| < 1$, that satisfies the Hypercyclicity Criterion and a vector e_0 in *X* for which $Te_0 = e_0$; see [3, Remark 2.12]. Note that the operator *T* is invertible because $\|T - I\| = \|K\| < 1$; for example, see [14, Lemma 2.1, p. 192]. Also, $e_0 \notin \mathcal{HC}(T^{-1})$ because $T^{-1}e_0 = e_0$.

To construct an operator T_1 which satisfies the hypothesis on Proposition 3.10, let $\{x_1, y_1, e_0, f, g\}$ be a linearly independent set in *X* such that $(f, g) \in \mathcal{HC}(T \oplus T)$. Let $T_1 = J^{-1}TJ$, where *J* is an invertible operator for which $Jx_1 = f$, $Jy_1 = g$, and $Je_0 = y_1$. Observe that $(Jx_1, Jx_2) = (f, g) \in \mathcal{HC}(T \oplus T)$, and so by Lemma 2.1, we get $(x_1, y_1) \in \mathcal{HC}(T_1 \oplus T_1)$. Moreover, $J^{-1}y_1 = e_0 \notin \mathcal{HC}(T^{-1})$, and so by Lemma 2.1 again, it follows that $y_1 \notin \mathcal{HC}(JTJ^{-1}) = \mathcal{HC}(T_1^{-1})$. $\square$

As stated in [9], Beauzamy [4] constructed an example of an invertible hypercyclic bilateral weighted shift *T* so that the sets $\mathcal{HC}(T)$ and $\mathcal{HC}(T^{-1})$ of hypercyclic vectors do not coincide. Bès, Martin, and Peris [9] noted that the invertible operator $T = C_{\phi_1} \oplus C_{\phi_2}$, where C_{ϕ_1}, C_{ϕ_2} are their example of invertible d-hypercyclic composition operators described above Proposition 3.10, satisfies this property. From the proof of Corollary 3.11, we see that every separable, infinite dimensional Banach space always admits an invertible hypercyclic operator for which the set of hypercyclic vectors for the operator and the set of hypercyclic vectors for its inverse fail to coincide.

Remark 3.12

Even though the results of Section 3 are stated for a separable, infinite dimensional Banach space, all the results in the section also hold true for a separable, infinite dimensional Fréchet space with a continuous norm.

We conclude this section with two open problems. Throughout the literature on disjoint hypercyclicity, we have examples of densely d-hypercyclic operators. In this section, we see that the set of d-hypercyclic set may also be nowhere dense. This leads us to the following question.

Question 3.13

Does there exist d-hypercyclic operators T_1, T_2 in $B(X)$ for which the set d-$\mathcal{HC}(T_1, T_2)$ of d-hypercyclic vectors is somewhere dense, but not dense?

Hereditary hypercyclicity, which was not mentioned in the Introduction, is another characterization of the Hypercyclicity Criterion. An operator *T* satisfies the Hypercyclicity Criterion if and only if it is hereditarily hypercyclic; for more on hereditary hypercyclicity see Bès and Peris [12]. In the disjoint setting, there are similar concepts called d-hereditary hypercyclicity and dense d-hereditary hypercyclicity, and as in the single operator case, a finite family of operators satisfies the d-Hypercyclicity Criterion if and only if the operators are densely d-hereditarily hypercyclic; see Bès and Peris [13]. Since there exist examples of d-hypercyclic operators which fail to be densely d-hypercyclic, we can ask the following question.

Question 3.14

Does there exist d-hereditarily hypercyclic operators T_1, T_2 in $B(X)$ which fail to be densely d-hereditarily hypercyclic?

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