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Abstract
An analytical expression for static stability of a rectangular slab with two simply supported and two elastically restrained edges is presented. The linear elastic isotropic slab can represent a rigid pavement resting on an elastic foundation and loaded by a uniform in-plane axial load per unit length along the edges. The partially restrained edges are connected to the ground by translational and rotational elastic springs; an appropriate magnitude of the springs can capture classical boundary conditions such as free, simply supported, and clamped edges. Results from classical boundary

conditions and a finite-element model were used to validate the proposed stability equation. The generalized boundary conditions were found to change the critical load by a factor of two and greatly affected the first buckling mode shape of a typical concrete pavement. The critical load was not sensitive to the slab's geometry if the length was four times longer than the width, but this was not the case for small aspect ratios. Finally, the translational spring was found to be a defining factor in determining the influence of the other variables on the critical load.

Introduction

Analytical models aimed at studying concrete pavement blowup lack a proper representation of the boundary conditions. Regardless of slab geometry, material properties, and loading conditions, existing approaches have focused on slabs with classical boundary conditions (i.e., simply supported, clamped, or free). For certain slab geometries, such as circular and rectangular, there are exact stability equations as long as the boundaries are either free or fully restrained (Wang et al. 2005).

The buckling of a long slab resting on an elastic foundation, considering the potential detachment between slab and foundation, has been studied (Seide 1958), but the stability equation was limited to simply supported slabs. Similarly, the stability of an infinitely long and wide slab has been addressed using the Fourier transform (Kim 2004). The model considered a moving load of constant amplitude, a stationary harmonic load, and a moving harmonic load. Even though static and dynamic stability was captured, the solution could not be applied to slabs with finite dimensions, such as concrete pavements. Yu and Wang also studied rectangular slabs on elastic foundations (Yu and Wang 2008); however, the stability equations were different for the various combinations of classical boundary conditions, which complicated their implementation. There are procedures to analyze the stability of beam-columns on elastic foundations with generalized boundary conditions (Areiza-Hurtado et al. 2005), but there are none for slabs.

Rigid pavement blowup can be understood by studying the stability of slabs on elastic foundations (also known as liquid or Winkler foundations). Rigid pavements contract in cold temperatures, increasing joint spacing. The space between slabs might be filled with incompressible debris, constraining pavement expansion in high temperatures. Restrained expansion translates into axial forces in the concrete slab, which might increase until reaching buckling load. Some attempts have been made to provide a theoretical explanation of pavement blowup, mainly by Kerr and coauthors (Kerr and Shade 1984; Kerr and Dallis 1985; Kerr 1994, 1997). This work revolved around determining safe temperature increments before rigid pavement buckling, and it assumed uniform temperature increments. Long pavement was considered, and the analysis was performed on a unit-width slab supported on an infinitely rigid base. In addition, nonlinear pavement-base shear interface forces were included (Kerr and Shade 1984; Kerr and Dallis 1985). The solution was used to determine the relevance of different variables on rigid pavement stability such as coefficient of thermal expansion, pavement thickness and stiffness, and pavement-base interface forces (Kerr and Dallis 1985). The methodology was extended to quantify the influence of an adjacent rigid structure on pavement blowup (Kerr 1994). Probably the main drawback of this work lays on assuming long pavement and infinitely rigid base support.

The solution presented here addresses the aforementioned limitation. On the one hand, using the proposed equation to calculate blowup load, concrete pavement could have any in-plane dimensions, in particular infinitely long or wide. On the other hand, two opposite edges of the slab were partially restrained to rotation and displacement by assigning translational and rotational elastic springs. Consequently, any combination of classical boundary conditions could be captured in a single stability equation, including interaction with a rigid structure. Furthermore, the obtained results were successfully compared with expressions assuming classical boundary conditions and values from a finite-element model.

Structural Model and Stability Equation

Consider a slab of length L and width $2bL$ ($\text{aspect ratio} = 2b$) made of a linear elastic material with elastic modulus E and Poisson's ratio ν. The slab, whose thickness h is small compared to the shortest plan dimension, is supported on an elastic foundation with modulus of subgrade reaction Λ. The origin of the coordinate system is located at the midpoint of the left-hand edge of the slab, with the x'- and y'-axes pointing along the length and width of the slab, respectively, as shown in Fig. 1.

The slab is loaded along the x' - and y'- directions by in-plane load per unit length N. The slab is assumed simply supported along edges parallel to the y'-axis. Along the edges parallel to the x'-axis, generalized boundary conditions are assumed; the vertical displacement and the rotation are partially restrained by translational and rotational springs of magnitude S_a, S_b, κ_a, and κ_b, respectively, as shown in Fig. 1. Traditional boundary conditions can be captured assigning appropriate values to the spring constants. For instance, if S_a and S_b are significantly large, the edge of the slab does not have any vertical displacement. If $\kappa_a = \kappa_b = 0$, the slab is free to rotate, which constitutes a simply supported condition. On the other hand, if both rotational springs approach infinite, the edge cannot rotate and a clamped boundary condition is obtained.

Based on thin slabs theory, the partial differential equation for the vertical deflection w of a slab resting on a elastic foundation is given by

(1)

$$D\left[\frac{\partial^4 w}{\partial x'^4} + \frac{\partial^4 w}{\partial x'^2 \partial y'^2} + \frac{\partial^4 w}{\partial y'^4}\right] + N\left[\frac{\partial^2 w}{\partial x'^2} + \frac{\partial^2 w}{\partial y'^2}\right] + \Lambda w = 0$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ = slab bending stiffness. Along the elastically restrained edges, the shear force and bending moment are

(2)

$$V(x', bL) = S_a w(x', bL)$$

(3)

$$V(x', -bL) = -S_b w(x', -bL)$$

(4)

$$M(x', bL) = \kappa_a \theta(x', bL)$$

(5)

$$M(x', -bL) = -\kappa_b \theta(x', -bL)$$

where V, M, and θ = shear force, bending moment, and rotation of the slab. Normalizing lengths with respect to L, the partial differential equation for w becomes

(6)

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + k\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] + \lambda^4 w = 0$$

where the dimensionless buckling load coefficient k and the dimensionless subgrade stiffness coefficient λ are

(7)

$$k = \frac{NL^2}{D}$$

(8)

$$\lambda^4 = \frac{\Lambda L^4}{D}$$

Because edges parallel to the y'-axis are simply supported, the solution of Eq. (6) can be assumed as

(9)

$$w(x, y) = f(y) \sin(\alpha x)$$

where $\alpha = n\pi$, with n being an integer. Replacing Eq. (9) in Eq. (6) and Eqs. (2)–(5), the differential equation for $f(y)$ is

(10)

$$\frac{d^4 f}{dy^4} + (k - 2\alpha^2)\frac{d^2 f}{dy^2} + (\alpha^4 - k\alpha^2 + \lambda^4) f = 0$$

and its boundary conditions are

(11)

$$f'''(b) - [\alpha^2(2 - \nu) - k] f'(b) = T_a f(b)$$

(12)

$$f'''(-b) - [\alpha^2(2 - \nu) - k] f'(-b) = -T_b f(-b)$$

(13)

$$f''(b) - \nu\alpha^2 f(b) = -R_a f'(b)$$

(14)

$$f''(-b) - \nu\alpha^2 f(-b) = R_b f'(-b)$$

where $T_a = \frac{S_a L^3}{D}$, $T_b = \frac{S_b L^3}{D}$, $R_a = \frac{\kappa_a L}{D}$, and $R_b = \frac{\kappa_b L}{D}$ = translational and rotational stiffness indexes along both edges of the slab, respectively. Eq. (10) is a fourth-order linear differential equation with constant coefficients and characteristic equation:

(15)

$$\beta^4 + (k - 2\alpha^2)\beta^2 + (\alpha^4 - k\alpha^2 + \lambda^4) = 0$$

then

(16)

$$\beta^2 = \frac{1}{2}\left[-(k - 2\alpha^2) \pm \sqrt{k^2 - 4\lambda^4}\right]$$

The solution $f(y)$ depends on the nature of the roots β, and three cases are identified:

- Case 1: if $\Delta > 0$ and $2\alpha^2 - k > \sqrt{\Delta}$, then the roots are real and the solution is
  (17)
  $$f(y) = c_1 e^{\beta_1 y} + c_2 e^{(-\beta_1 y)} + c_3 e^{\beta_2 y} + c_4 e^{-\beta_2 y}$$
  with
  (18)
  $$\beta_1^2 = \frac{2\alpha^2 - k + \sqrt{\Delta}}{2}$$
  (19)
  $$\beta_2^2 = \frac{2\alpha^2 - k - \sqrt{\Delta}}{2}$$
- Case 2: If $\Delta > 0$ and $2\alpha^2 - k < \sqrt{\Delta}$, then the roots are complex and the solution is
  (20)
  $$f(y) = c_1 e^{\beta_1 y} + c_2 e^{\beta_2 y} + c_3 \sin(\beta_2 y) + c_4 \cos(\beta_2 y)$$
  with
  (21)
  $$\beta_1^2 = \frac{2\alpha^2 - k + \sqrt{\Delta}}{2}$$
  (22)
  $$\beta_2^2 = -\frac{2\alpha^2 - k - \sqrt{\Delta}}{2}$$
- Case 3: if $\Delta < 0$, then the roots are complex conjugate and the solution is
  (23)
  $$f(y) = c_1 e^{sy}\cos(ty) + c_2 e^{sy}\sin(ty) + c_3 e^{-sy}\cos(ty) + c_4 e^{-sy}\sin(ty)$$
  with
  (24)
  $$\beta_1^2 = \frac{2\alpha^2 - k + \sqrt{4\lambda^4 - k^2}\,i}{2} = s + ti$$
  (25)

$$\beta_2^2 = \frac{2\alpha^2 - k - \sqrt{4\lambda^4 - k^2}i}{2} = -s + ti$$

Replacing $f(y)$ from Eqs. (17), (20), and (23) in the boundary conditions in Eqs. (11)–(14), a homogenous system of equations of the form $[A]_{4\times4} \cdot [C]_{4\times1} = [0]_{4\times1}$ can be built, where $[A]_{4\times4}$ stores the coefficients of c_1, c_2, c_3, and c_4, which are grouped in $[C]_{4\times1}$. The critical axial load N_{critc} is the value of N that makes the determinant of $[A]$ equal to zero. Equating the determinant $[A]$ to zero results in the characteristic stability equation, which, regardless of the form of $f(y)$, can be written as

(26)

$$[RT]_{1\times16}[COEF]_{16\times t}[TRIG]_{t\times1} + [RT]_{1\times16}[REM]_{16\times1} = 0$$

where t = integer whose value depends on the type of roots: $t = 4$ for real roots (Case 1), $t = 8$ for complex roots (Case 2), and $t = 11$ for complex conjugate roots (Case 3); $[RT]_{1\times16}$ = vector containing combinations of R_a, R_b, T_a, and T_b; $[TRIG]_{t\times1}$ = vector of trigonometric functions; $[COEF]_{16\times t}$ = matrix storing coefficients of trigonometric functions in $[TRIG]_{t\times1}$; $[REM]_{16\times1}$ = terms not multiplying trigonometric functions.

The values of $[RT]$, $[TRIG]$, $[COEF]$, and $[REM]$ for the three types of solution are provided in the Tables 5–11 and definitions in the Appendix. Corresponding to the first columns in the tables, $[RT]$ is the same for all cases; $[TRIG]$ depends on the solution for $f(y)$ and is given by the column headings in Tables 5–10; $[COEF]$ are the entries in the tables; and $[REM]$, which is different from zero only when the roots are real, is given in Table 10. The buckling mode shapes are obtained by replacing the critical load in the homogeneous system $[A] \cdot [C] = [0]$, solving for one of the four constants in $[C]$, and replacing in the corresponding $f(y)$ [Eqs. (17), (20), or (23)].

A finite-element model was developed using ABAQUS to verify the results of the proposed equations. Four-node full-integration shell elements were used to model the slab resting on an elastic foundation. Two-node three-dimensional (3D) connector elements with translational and rotational spring constants simulated the semirigid connections. A biaxial uniform load per unit length was applied along the edges of the slab, and an eigenvalue buckling analysis was performed. The vertical displacements of the simply supported edges, the ones parallel to the y'-axis, were fully restrained to vertical displacement. In addition, a kinematic constraint was created to guarantee that the two opposite edges would have negligible displacement in the y'-direction. The resulting critical loads were compared with results obtained using the stability equation in Eq. (26).

Slab with Classical Boundary Conditions

Consider a slab with the simply supported edges parallel to the y'-axis and the other two edges fully restrained to vertical displacement (i.e., $S_a = S_b$ tending to infinite). When $\Delta > 0$ and $2\alpha^2 - k > \sqrt{\Delta}$, the terms in Eq. (26) are obtained by finding the limit when $S_a = S_b \to \infty$. From Table 11 and replacing into Eq. (26)

(27)

$$[RT]_{1\times4}[COEF]_{4\times4}[TRIG]_{4\times1} + [RT]_{1\times4}[REM]_{4\times1} = 0$$

(28)

$$\begin{pmatrix} R_a R_b \\ R_a \\ R_b \\ 1 \end{pmatrix}^T \begin{pmatrix} -2B_p^2 & 2B_m^2 & 0 & 0 \\ 0 & 0 & -2B_m B_p^2 & 2B_m^2 B_p \\ 0 & 0 & -2B_m B_p^2 & 2B_m^2 B_p \\ -2B_m^2 B_p^2 & 2B_m^2 B_p^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh(2b\beta_1 - 2b\beta_2) \\ \cosh(2b\beta_1 + 2b\beta_2) \\ \sinh(2b\beta_1 - 2b\beta_2) \\ \sinh(2b\beta_1 + 2b\beta_2) \end{pmatrix} + \begin{pmatrix} R_a R_b \\ R_a \\ R_b \\ 1 \end{pmatrix}^T \begin{pmatrix} 8\beta_1\beta_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

where $B_p = \beta_1 + \beta_2$ and $B_m = \beta_1 - \beta_2$. After performing the matrix operations, Eq. (28) becomes

(29)

$$\begin{aligned} -2\cosh(2b\beta_1 - 2b\beta_2)\, B_m^2 B_p^2 + 2\cosh(2b\beta_1 + 2b\beta_2)\, B_m^2 B_p^2 \\ + R_a\left[2\sinh(2b\beta_1 + 2b\beta_2)\, B_m^2 B_p - 2\sinh(2b\beta_1 - 2b\beta_2)\, B_m B_p^2\right] \\ + R_b\left[2\sinh(2b\beta_1 + 2b\beta_2)\, B_m^2 B_p - 2\sinh(2b\beta_1 - 2b\beta_2)\, B_m B_p^2\right] \\ + R_a R_b\left[2\cosh(2b\beta_1 + 2b\beta_2)\, B_m^2 - 2\cosh(2b\beta_1 - 2b\beta_2)\, B_p^2 + 8\beta_1\beta_2\right] = 0 \end{aligned}$$

If the slab is simply supported along all edges, the stability equation when the roots are real is obtained making $R_a = R_b = 0$. Then Eq. (29) becomes

(30)

$$(\beta_1^2 - \beta_2^2)^2 \sinh(2b\beta_1)\sinh(2b\beta_2) = 0$$

In addition, $R_a = R_b \to \infty$ provides the equations for a clamped slab. A similar procedure can be followed for the other cases (i.e., complex and complex conjugate roots). Tables 1 and 2 summarize the stability equations for a slab with two edges simply supported and clamped, respectively.

Table 1. Stability equations for a slab with all edges simply supported

Case	Equation
$\Delta > 0$, $2\alpha^2 - k > \sqrt{\Delta}$	$(\beta_1^2 - \beta_2^2)^2 \sinh(2b\beta_1)\sinh(2b\beta_2) = 0$
$\Delta > 0$, $2\alpha^2 - k > \sqrt{\Delta}$	$4(\beta_1^2 + \beta_2^2)^2 \sin(b\beta_2)\cos(b\beta_2)\sinh(2b\beta_1) = 0$
$\Delta < 0$	$8s^2t^2[\cos(4bt) - \cosh(4bs)] = 0$

Table 2. Stability equations for a slab with two edges simply supported and two edges clamped

Case	Equation
$\Delta > 0$, $2\alpha^2 - k > \sqrt{\Delta}$	$2(\beta_1 + \beta_2)^2\cosh[2b(\beta_1 - \beta_2)] - 2(\beta_1 - \beta_2)^2\cosh[2b(\beta_1 + \beta_2)] = 0$
$\Delta > 0$, $2\alpha^2 - k > \sqrt{\Delta}$	$4\beta_1\beta_2\cos^2(b\beta_2)\cosh(2b\beta_1) - 4\beta_1\beta_2\sin^2(b\beta_2)\cosh(2b\beta_1) - 4(\beta_1^2 - \beta_2^2)\sin(b\beta_2)\cos(b\beta_2)\sinh(2b\beta_1) = 0$
$\Delta < 0$	$4\beta_1\beta_2\cos^2(b\beta_2)\cosh(2b\beta_1) - 4\beta_1\beta_2\sin^2(b\beta_2)\cosh(2b\beta_1) - 4(\beta_1^2 - \beta_2^2)\sin(b\beta_2)\cos(b\beta_2)\sinh(2b\beta_1) = 0$

Table 3 compares the critical load calculated using the proposed equations, the finite-element method, and the values reported by Yu and Wang (2008) for two boundary conditions: simply supported and clamped. The slab's length is twice its width ($b = 0.25$), and three support conditions are assumed

($\lambda = 0$, 2, and 5). The table also shows the percentage difference with respect to the proposed method. As can be observed, the agreement is excellent.

Table 3. Comparison between Proposed equation, Yu and Wang, and ABAQUS for b=0.25

Boundary condition	λ	Equation	Yu and Wang (2008)		ABAQUS	
		k	k	Difference (%)	k	Difference (%)
Simply Supported	0	49.3475	49.35	−0.005	49.2219	0.255
	2	49.6722	49.67	0.004	49.5546	0.237
	5	62.0136	62.02	−0.010	62.2191	−0.331
Clamped	0	150.9549	150.99	−0.023	149.7369	0.807
	2	151.2148	151.24	−0.017	150.0006	0.803
	5	158.1408	158.13	0.007	159.9969	−1.174

Critical Load of Concrete Pavement

Consider a square slab ($L = sbL = 4.0\ \text{m}$) of a rigid pavement whose longitudinal joints are assumed simply supported. The slab has a thickness of $h = 0.3\ \text{m}$, and it is resting on a elastic foundation with modulus of subgrade reaction $\Lambda = \frac{18.2\ \text{MN}}{\text{m}^3}$. The concrete has an elastic modulus of $E = 25{,}000\ \text{MPa}$ and a Poisson's ratio of $\nu = 0.15$. The transverse joint ahead of traffic is in good condition, meaning that there is a good transfer of shear force and bending moment to the next slab ($\kappa_a = 1.439 \times 10^6\ \text{kN} \times \frac{\text{m}}{\text{rad/m}}$ and $T_a = 8.991 \times \frac{\frac{10^5\ \text{N}}{\text{m}}}{\text{m}}$). Conversely, the other transverse joint is progressively deteriorating to the point that there is no load transfer to the adjacent slab. The objective is to find the effect of joint deterioration on the critical load and the first mode of buckling.

The bending stiffness of the slab is $D = \frac{\frac{Eh^3}{12}}{1-\nu^2} = 57.5\ \text{MN} \cdot \text{m}$. The translational and rotational stiffness indexes of the joints ahead of traffic are $T_a = \frac{S_a L^3}{D} = 100$ and $R_a = \frac{\kappa_a L}{D} = 1{,}000$. The parameter associated with the elastic foundation is $\lambda = \frac{\Lambda L^4}{D} = 3.0$. Six joint deterioration conditions are considered by assigning different values to the parameters T_b and R_b. The values range between $R_a = 0.001$ and $S_a = 0.01$, which represent no load transfer between slabs, to $R_a = 200$ and $S_a = 2{,}000$.

The blowup loads N_{critc} for the various joint conditions are calculated using Eq. (26) and are summarized in Table 4. As the stiffness of the joint is reduced, the slab's restriction to motion also decreases. This reduction in stiffness, as expected, decreases the magnitude of the critical load. It is also observed that for the selected values of R_a and S_a, the change in N_{critc} is almost linear, highlighting the relevance of accurate characterization of joint deterioration for the prediction of concrete pavement critical load.

Table 4. Effect of joint deterioration on critical load

Deterioration	k	N_{critc} (kN/m)
$R_a = 0.001, S_a = 0.01$	18.10	65,110.22

$R_a = 10, S_a = 1$	21.51	77,377.14
$R_a = 20, S_a = 10$	24.36	87,605.53
$R_a = 50, S_a = 100$	29.16	104,861.5
$R_a = 100, S_a = 1000$	34.24	123,156.9
$R_a = 200, S_a = 2000$	37.80	135,912.0

Fig. 2 shows the effect of joint deterioration on normalized buckling mode shapes. If the joint is in good condition ($S_b = 2{,}000$ and $T_b = 200$), the maximum deflection is located toward the center of the slab. As the transverse joint deteriorates, the point of maximum deflection shifts toward the weaker joint. For the weakest joint condition, not only is the maximum deflection located at the joint but the curvature of the slab has changed. The deformed shaped shown in Fig. 2 agrees with the deformation of concrete pavement when it fails by buckling.

It should be highlighted that the proposed method has the capability of considering rotational and translational spring stiffness independently. However, in the case of actual concrete pavements, joint deterioration causes stiffness reduction in a coupled fashion, indicating that a function relating R_a to S_a and R_b to S_b must be included. Consequently, some of the $R_b - S_b$ combinations in Fig. 2 are not likely to occur in real life. This observation also applies to the results in the following sections.

Effect of Joint Stiffness on Concrete Pavement Blowup

The influence of degree of vertical displacement and rotation restrain on the critical load of a square slab was studied. Two support cases were considered: for the first one, no elastic foundation was considered ($\lambda = 0$); for the second one, slab geometry, material properties, and modulus of subgrade reaction provided $\lambda = 2$. Slab edges not being simply supported were assumed to have the same degree of restrain ($R_a = R_b$ and $S_a = S_b$); R_a ranged between 10^{-3} and 10^4, whereas S_a varied between 10^{-2} and 10^4. The range of values for R_a and S_a could be physically interpreted as various joint deterioration degrees.

Figs. 3 and 4 show the variation in the dimensionless buckling load coefficient k with S_a and R_a for both values of λ. If the vertical restraint is low ($S_a < 20$), the rotational restraint has a negligible effect on the critical load. For instance, when $\lambda = 2$, k increases by only 0.02% after R_a increases from 10^{-3} to 10^4, indicating that vertical displacement of the edges is more important than rotation. On the other hand, the relevance of R_a is very high because the magnitude of S_a is higher, with k almost doubled as the magnitude of the rotational stiffness index changes between its extreme values when $S_a = 10^4$ (increments of 91% and 87% for $\lambda = 0$ and 2, respectively).

Similarly, the greatest influence of the translational restraint is seen for the highest magnitude of R_a. If $R_a = 10^{-3}$, the increment of k is 15% when there is no elastic foundation and S_a changes from 50 to 10^4. On the other hand, for the same change in S_a but $R_a = 10^4$, the dimensionless buckling load coefficient doubles from $k = 18.73$ to $k = 37.67$. In general, three zones can be identified as characterizing the effect of R_a and S_a on critical load. If $R_a < 0.1$ and $R_a > 100$, the rotational stiffness index has no effect on the critical load regardless the magnitude of S_a. Conversely, if $10 <$

$R_a < 100$, the change in k with R_a is almost linear in the semilogarithmic scale. These results indicate that preventive measures against concrete pavement blowup should include joint quality inspection.

Fig. 5 compares the results from the proposed equation and ABAQUS. The continuous line represents the values predicted by the finite-element model. The figure also shows the equation of a linear fit to the cloud of points, which identifies good agreement between the results obtained using Eq. (26) and ABAQUS. However, the proposed solution does not need any special-purpose software such as ABAQUS and can be easily implemented.

Effect of Slab Size on Concrete Pavement Blowup

The effect of a slab's aspect ratio on the critical load under various restraint conditions was analyzed. Fig. 6 shows such variation when bb changes between 0.3 and 2.0; $b = 0.3$ represents a slab whose width is 60% longer than its length. Variation in half the aspect ratio can be physically interpreted as a slab with a fixed length whose width continuously decreases. Aspect ratio values that are too small or too big represent structural behavior different from that of a slab, which is out of the scope of this work. The presented results were obtained assuming $S_a = S_b$ and $R_a = R_b$.

The dimensionless buckling load coefficient k is insensitive to aspect ratio if S_a is low, regardless of the rotational restraint for both values of λ considered. For instance, if $R_a = 10^4$, $S_a = 1$, and $\lambda = 2$, k changes 2.6% between the two extreme values of bb. As vertical restraint increases, critical load is augmented and the aspect ratio's influence becomes significant. For the same case ($\lambda = 2$ and $R_a = 10^4$) and $S_a = 50$, critical load decreases 42% when bb changes from 0.3 to 2.0. Similarly, the impact of R_a on the critical load increases as S_a increases. These results show that proper characterization of vertical displacement at the edges of the slab is crucial for accurate calculation of the critical load.

Most of the influence of the semirigid connections derives from b less than 1.2. As b increases beyond 1.2, the influence of the rotational and translational stiffness indexes decreases to the point that all lines become almost coincidental and parallel to each other. In other words, the boundary conditions of the edges that are not simply supported become irrelevant once the bending of the slab is predominantly in one direction—that is, when the slab's width is significantly greater than its length.

To summarize, a slab's aspect ratio is a relevant factor when determining critical load, and its influence is coupled with the boundary conditions. Consequently, care should be exercised when using solutions that consider slabs with infinite in-plane dimensions. In addition, in the design of concrete pavement, the aspect ratio should be kept small to minimize the likelihood of blowup.

Practical Implementation

Rigid pavement design methodologies such as found in the *Guide for Mechanistic-Empirical Pavement Design of New and Rehabilitated Pavement Structures* (*MEPDG*) account for joint deterioration using empirical equations (ARA 2004). In *MEPDG*, joint deterioration is related to load transfer efficiency *LTE*, which is defined as the ratio between percentage approach-slab deflection w_a and percentage leave-slab deflection w_l (Fig. 7). In other words

(31)

$$LTE = \frac{w_a}{w_l} \times 100$$

Load transfer efficiency is affected by aggregate interlock, the quality of the concrete slab support, and details of the dowel system connecting the two slabs. It can be calculated as (ARA 2004):

(32)

$$LTE = 100\left[1 - \left(1 - \frac{LTE_{\text{dowel}}}{100}\right)\left(1 - \frac{LTE_{\text{agg}}}{100}\right)\left(1 - \frac{LTE_{\text{base}}}{100}\right)\right]$$

where LTE_{dowel}, LTE_{agg}, and LTE_{base} = contribution to dowel system LTE, aggregate interlock, and supporting base, respectively.

The spring connecting the two slabs in the vertical direction develops a force V_s equals to

(33)

$$V_s = k_s \times \Delta = k_s \times (w_l - w_a) = k_s \times \left(1 - \frac{LTE}{100}\right) w_l$$

where k_s = spring stiffness; and Δ = spring deformation. The shear equilibrium at the joint between the approach and leave slabs requires

(34)

$$V_l - V_a = V_s = k_s \times \left(1 - \frac{LTE}{100}\right) w_l$$

where V_l and V_a = shear at the edges of the approach and leave slabs. In Eq. (34), V_l, V_a, and w_l can be calculated by setting the equilibrium equations for each slab and using the appropriate boundary conditions. Joint deterioration can be accounted for through changes in LTE as implemented in *MEPDG*. *MEPDG* does not take into account the transfer of rotation, but similar continuity conditions as in Eq. (34) can be established for rotation at slab joints.

Summary and Conclusions

The stability equation, derived for a linear elastic slab resting on an elastic foundation with two simply supported and two partially restrained edges, enables critical load calculation using a single expression for various boundary conditions, including classical cases (simply supported, free, and clamped). In addition, the presented solution was verified using published results for the classical boundary conditions and a finite-element model that considers edges elastically restrained to translation and rotation. An example of a slab with simply supported and clamped edges also demonstrated a step-by-step procedure to implement the derived stability equation.

The coupled effect of semirigid connections on critical load was showed; the rotational spring greatly influences critical load as long as the translational spring has relevant magnitude. Furthermore, when analyzing a typical concrete pavement, the high relevance of the boundary conditions is found, not only to the buckling load but also to the first buckling mode shape. As joint deficiency diminishes (i.e., reducing spring constants), the point of maximum deflection in buckling mode shape shifts from the slab's center to its edges. Finally, the influence of the slab's aspect ratio on static buckling increases as boundary conditions become stiffer. However, this influence significantly decreases for large aspect ratios. From a practical point view, it can be concluded that the likelihood of concrete pavement blowup can be reduced by reducing the slab's aspect ratio in the design phase and by keeping the joints in good condition in the maintenance stage.

Appendix.

Terms in the Characteristic Stability Equation

The following definitions were used in Tables 5–11:

(35)

$$F = k - \alpha^2(2 - \nu)$$

(36)

$$C = k - 2\alpha^2(1 - \nu)$$

(37)

$$G = \beta_1^2(\alpha^2\nu - \beta_2^2) - \beta_1\beta_2 C + \alpha^2\nu(\beta_2^2 + F)$$

(38)

$$H = \beta_1^2(\alpha^2\nu - \beta_2^2) + \beta_1\beta_2 C + \alpha^2\nu(\beta_2^2 + F)$$

(39)

$$J = (F + \beta_1^2)(\alpha^2\nu + \beta_2^2)$$

(40)

$$K = (F - \beta_2^2)(\alpha^2\nu - \beta_1^2)$$

(41)

$$L = 2\alpha^2 F\nu + \beta_2^2(k - 2\alpha^2) + \beta_1^2(2\alpha^2 + 2\beta_2^2 - k)$$

(42)

$$COEF_{CR,(5,6)} = -\beta_1^4(\alpha^2\nu + \beta_2^2) + \beta_1^2(-\beta_2^4 + \alpha^2 F\nu + 2\beta_2^2 F) + \alpha^2\beta_2^2\nu[-\alpha^2 - (\nu - 2) + \beta_2^2 - k]$$

(43)

$$COEF_{CR,(5,16)} = \beta_1^6(\alpha^2\nu + \beta_2^2)^2 - \alpha^4\beta_2^2\nu^2(F - \beta_2^2)^2 + \beta_1^2\{2\alpha^2\beta_2^6\nu + \alpha^4F^2\nu^2 + 4\alpha^2\beta_2^2F^2\nu + \beta_2^4F[k - \alpha^2(3\nu + 2)]\} + \beta_1^4\{-\beta_2^6 + 2\alpha^4F\nu^2 + 4\beta_2^4F + \beta_2^2F[\alpha^2(3\nu + 2) - k]\}$$

(44)

$$\mathrm{COEF}_{CR,(7,6)} = 2\beta_1^4 + \beta_2^4 - 2\alpha^2F\nu + (\beta_1^2 - \beta_2^2)(k - 2\alpha^2)$$

(45)

$$P = 2\alpha^2 - k + s^2 + t^2$$

(46)

$$Q = -2\alpha^2 + k + 4s^2$$

(47)

$$R = \alpha^4\nu^2 - P(s^2 + t^2) + \alpha^2\nu Q$$

(48)

$$\bar{P} = -2\alpha^2 + k + s^2 + t^2$$

(49)

$$\bar{Q} = 2\alpha^2 - k + 4t^2$$

(50)

$$\bar{R} = \bar{P}(s^2 + t^2) + \alpha^2\nu\bar{Q}^2 - \alpha^4\nu^2$$

(51)

$$COEF_{CCR,(9,6)} = -\alpha^4\nu^2 + (s^2 + t^2)(2\alpha^2 - k - 7s^2 + t^2) - \alpha^2\nu Q$$

(52)

$$\begin{aligned} COEF_{CCR,(11,6)} = {} & 4\alpha^2\nu[-k(4s^2 + t^2) + 2\alpha^2(4s^2 + t^2) + 12s^2t^2] - 4\alpha^4\nu^2(4s^2 + t^2) + 4(s^2 \\ & + t^2)[k(4s^2 - t^2) + 4s^4 - s^2(8\alpha^2 + 3t^2) + t^4 + 2\alpha^2t^2] \end{aligned}$$

(53)

$$\begin{aligned} & COEF_{CCR,(11,16)} \\ & = k^2[4s^6 + s^4(9t^2 - 8\alpha^2\nu) + s^2(4\alpha^4\nu^2 + 6t^4 - 6\alpha^2\nu t^2) + (t^3 + \alpha^2\nu t)^2] + 4s^{10} \\ & + s^8(17t^2 - 16\alpha^2) + 4s^6\{-2\alpha^4[(\nu - 2)\nu - 2] + 7t^4 + \alpha^2(6\nu - 11)t^2\} \\ & + 2s^4\{8\alpha^6(\nu - 2)\nu + 11t^6 + 6\alpha^2(4\nu - 3)t^4 - \alpha^4[\nu(\nu + 22) - 18]t^2\} \\ & + 4s^2[\alpha^8(\nu - 2)^2\nu^2 + 2t^8 + \alpha^2(6\nu - 1)t^6 + \alpha^4(\nu(13\nu - 14) + 6)t^4 \\ & - 3\alpha^6(\nu - 2)\nu(2\nu - 1)t^2] + t^2(-\alpha^4(\nu - 2)\nu + t^4 + 2\alpha^2t^2)^2 \\ & + 2k\{4s^8 + s^6[11t^2 - 4\alpha^2(\nu + 2)]\} \\ & + 2k\{s^4[-4\alpha^4(\nu - 4)\nu + 9t^4 + \alpha^2(11\nu - 18)t^2] - [t^2 - \alpha^2(\nu - 2)](t^3 + \alpha^2\nu t)^2\} \\ & + 2k\{s^2[4\alpha^6(\nu - 2)\nu^2 + t^6 + 2\alpha^2(7\nu - 6)t^4 + 3\alpha^4(4 - 5\nu)\nu t^2]\} \end{aligned}$$

Table 5. Terms in characteristic stability equation when $\Delta > 0$ and $2\alpha^2 - k > \sqrt{\Delta}$ (Case 1: real roots)

[RT]	$[TRIG]_1$	$[TRIG]_2$	$[TRIG]_3$	$[TRIG]_4$
	$\cosh(2b\beta_1 - 2b\beta_2)$	$\cosh(2b\beta_1 + 2b\beta_2)$	$\sinh(2b\beta_1 - 2b\beta_2)$	$\sinh(2b\beta_1 + 2b\beta_2)$
$R_aT_aR_bT_b$	$2(\beta_1+\beta_2)^2$	$-2(\beta_1+\beta_2)^2$	0	0
$R_aT_aT_b$	0	0	$2(\beta_1-\beta_2)(\beta_1+\beta_2)^2$	$\sinh(2b\beta_1+2b\beta_2)$
$R_aT_aR_b$	0	0	$-2\beta_1(\beta_1-\beta_2)\beta_2(\beta_1+\beta_2)^2$	$-2\beta_1(\beta_1-\beta_2)^2\beta_2(\beta_1+\beta_2)$
$R_bT_bR_a$	0	0	$-2\beta_1(\beta_1-\beta_2)\beta_2(\beta_1+\beta_2)^2$	$-2\beta_1(\beta_1-\beta_2)^2\beta_2(\beta_1+\beta_2)$
$R_bT_bT_a$	0	0	$2(\beta_1-\beta_2)(\beta_1+\beta_2)^2$	$-2(\beta_1-\beta_2)^2(\beta_1+\beta_2)$
R_aT_a	$-2(\beta_1+\beta_2)^2G$	$2(\beta_1-\beta_2)^2H$	0	0
R_bT_b	$-2(\beta_1+\beta_2)^2G$	$2(\beta_1-\beta_2)^2H$	0	0
R_aR_b	$2\beta_1^2\beta_2^2(\beta_1^2-\beta_2^2)^2$	$-2\beta_1^2\beta_2^2(\beta_1^2-\beta_2^2)^2$	0	0
T_aT_b	$2(\beta_1^2-\beta_2^2)^2$	$-2(\beta_1^2-\beta_2^2)^2$	0	0
R_bT_a	$-2\beta_1\beta_2(\beta_1^2-\beta_2^2)^2$	$-2\beta_1\beta_2(\beta_1^2-\beta_2^2)^2$	0	0
R_aT_b	$-2\beta_1\beta_2(\beta_1^2-\beta_2^2)^2$	$-2\beta_1\beta_2(\beta_1^2-\beta_2^2)^2$	0	0
R_a	0	0	$2\beta_1(\beta_1-\beta_2)\beta_2(\beta_1+\beta_2)^2G$	$2\beta_1(\beta_1-\beta_2)^2\beta_2(\beta_1+\beta_2)H$
R_b	0	0	$2\beta_1(\beta_1-\beta_2)\beta_2(\beta_1+\beta_2)^2$	$2\beta_1(\beta_1-\beta_2)^2\beta_2(\beta_1+\beta_2)H$
T_a	0	0	$-2(\beta_1-\beta_2)(\beta_1+\beta_2)^2G$	$2(\beta_1-\beta_2)^2(\beta_1+\beta_2)H$
T_b	0	0	$-2(\beta_1-\beta_2)(\beta_1+\beta_2)^2$	$2(\beta_1-\beta_2)^2(\beta_1+\beta_2)H$
1	$2(\beta_1+\beta_2)^2G^2$	$-2(\beta_1-\beta_2)^2H^2$	0	0

Table 6. Terms in characteristic stability equation when $\Delta > 0$ and 2α2−k<Δ (Case 2: complex roots): Part 1

$[RT]$	$[TRIG]_1$	$[TRIG]_2$	$[TRIG]_3$	$[TRIG]_4$
	$\sin^2(b\beta_2)\sinh(2b\beta_1)$	$\cos^2(b\beta_2)\cosh(2b\beta_1)$	$\sin^2(b\beta_2)\cosh(2b\beta_1)$	$\cos^2(b\beta_2)\sinh(2b\beta_1)$
$R_aT_aR_bT_b$	0	$-4\beta_1\beta_2$	$4\beta_1\beta_2$	0
$R_aT_aT_b$	$2\beta_2(\beta_1^2+\beta_2^2)$	0	0	$-2\beta_2(\beta_1^2+\beta_2^2)$
$R_aT_aR_b$	$-2\beta_1^2\beta_2(\beta_1^2+\beta_2^2)$	0	0	$2\beta_1^2\beta_2(\beta_1^2+\beta_2^2)$
$R_bT_bR_a$	$-2\beta_1^2\beta_2(\beta_1^2+\beta_2^2)$	0	0	$2\beta_1^2\beta_2(\beta_1^2+\beta_2^2)$
$R_bT_bT_a$	$2\beta_2(\beta_1^2+\beta_2^2)$	0	0	$-2\beta_2(\beta_1^2+\beta_2^2)$
R_aT_a	0	$2\beta_1\beta_2L$	$-2\beta_1\beta_2L$	0
R_bT_b	0	$2\beta_1\beta_2L$	$-2\beta_1\beta_2L$	0
R_aR_b	0	0	0	0
T_aT_b	0	0	0	0
R_bT_a	0	$2\beta_1\beta_2(\beta_1^2+\beta_2^2)^2$	$-2\beta_1\beta_2(\beta_1^2+\beta_2^2)^2$	0

R_aT_b	0	$2\beta_1\beta_2(\beta_1^2+\beta_2^2)^2$	$-2\beta_1\beta_2(\beta_1^2+\beta_2^2)^2$	0
R_a	$2\beta_1^2\beta_2(\beta_1^2+\beta_2^2)J$	0	0	$-2\beta_1^2\beta_2(\beta_1^2+\beta_2^2)J$
R_b	$2\beta_1^2\beta_2(\beta_1^2+\beta_2^2)J$	0	0	$-2\beta_1^2\beta_2(\beta_1^2+\beta_2^2)J$
T_a	$-2\beta_2(\beta_1^2+\beta_2^2)K$	0	0	$2\beta_2(\beta_1^2+\beta_2^2)K$
T_b	$-2\beta_2(\beta_1^2+\beta_2^2)K$	0	0	$2\beta_2(\beta_1^2+\beta_2^2)K$
1	0	$-4\beta_1\beta_2JK$	$4\beta_1\beta_2JK$	0

Table 7. Terms in characteristic stability equation when $\Delta > 0$ and 2α2−k<Δ (Case 2: complex roots): Part 2

$[RT]$	$[TRIG]_5$	$[TRIG]_6$	$[TRIG]_7$	$[TRIG]_8$
	$\sin(b\beta_2)\cos(b\beta_2)\sinh(2b\beta_1)$	$\sin(b\beta_2)\cos(b\beta_2)\cosh(2b\beta_1)$	$\sin^2(b\beta_2)$	$\cos^2(b\beta_2)$
$R_aT_aR_bT_b$	$-4(\beta_2^2-\beta_1^2)$	0	$4\beta_1\beta_2$	$4\beta_1\beta_2$
$R_aT_aT_b$	0	$4\beta_1(\beta_1^2+\beta_2^2)$	0	0
$R_aT_aR_b$	0	$4\beta_1\beta_2^2(\beta_1^2+\beta_2^2)$	0	0
$R_bT_bR_a$	0	$4\beta_1\beta_2^2(\beta_1^2+\beta_2^2)$	0	0
$R_bT_bT_a$	0	$4\beta_1(\beta_1^2+\beta_2^2)$	0	0
R_aT_a	$4COEF_{CR,(5,6)}$	0	$\beta_1\beta_2COEF_{CR,(7,6)}$	$\beta_1\beta_2COEF_{CR,(7,6)}$
R_bT_b	$4COEF_{CR,(5,6)}$	0	$\beta_1\beta_2COEF_{CR,(7,6)}$	$\beta_1\beta_2COEF_{CR,(7,6)}$
R_aR_b	$-4\beta_1^2\beta_2^2(\beta_1^2+\beta_2^2)^2$	0	0	0
T_aT_b	$4(\beta_1^2+\beta_2^2)^2$	0	0	0
R_bT_a	0	0	0	0
R_aT_b	0	0	0	0
R_a	0	$-4\beta_1\beta_2^2(\beta_1^2+\beta_2^2)K$	0	0
R_b	0	$-4\beta_1\beta_2^2(\beta_1^2+\beta_2^2)K$	0	0
T_a	0	$-4\beta_1(\beta_1^2+\beta_2^2)J$	0	0
T_b	0	$-4\beta_1(\beta_1^2+\beta_2^2)J$	0	0
1	$4COEF_{CR,(5,16)}$	0	$4\beta_1\beta_2JK$	$4\beta_1\beta_2J$

Table 8. Terms in characteristic stability equation when Δ<0 (Case 3: complex conjugate roots): Part 1

$[RT]$	$[TRIG]_1$	$[TRIG]_2$	$[TRIG]_3$	$[TRIG]_4$	$[TRIG]_1$
	$\cosh(4bs)\cos^4(bt)$	$\sinh(4bs)\cos^4(bt)$	$\cosh(4bs)\sin^4(bt)$	$\sinh(4bs)\sin^4(bt)$	$\sin^3(bt)\cos(bt)$
$R_aT_aR_bT_b$	$-2t^2$	0	$-2t^2$	0	0
$R_aT_aT_b$	0	$-4st^2$	0	$-4st^2$	$-16s^2t$

$R_aT_aR_b$	0	$-4st^2(s^2+t^2)$	0	$-4st^2(s^2+t^2)$	$16s^2t(s^2+t^2)$
$R_bT_bR_a$	0	$-4st^2(s^2+t^2)$	0	$-4st^2(s^2+t^2)$	$16s^2t(s^2+t^2)$
$R_bT_bT_a$	0	$-4st^2$	0	$-4st^2$	$-16s^2t$
R_aT_a	$2t^2R$	0	$2t^2R$	0	0
R_bT_b	$2t^2R$	0	$2t^2R$	0	0
R_aR_b	$-8s^2t^2(s^2+t^2)^2$	0	$-8s^2t^2(s^2+t^2)^2$	0	0
T_aT_b	$-8s^2t^2$	0	$-8s^2t^2$	0	0
R_bT_a	$-8s^2t^2(s^2+t^2)$	0	$-8s^2t^2(s^2+t^2)$	0	0
R_aT_b	$-8s^2t^2(s^2+t^2)$	0	$-8s^2t^2(s^2+t^2)$	0	0
R_a	0	$4st^2R(s^2+t^2)$	0	$4st^2R(s^2+t^2)$	$16s^2t\bar{R}(s^2+t^2)$
R_b	0	$4st^2R(s^2+t^2)$	0	$4st^2R(s^2+t^2)$	$16s^2t\bar{R}(s^2+t^2)$
T_a	0	$4st^2R$	0	$4st^2R$	$-16s^2t\bar{R}$
T_b	0	$4st^2R$	0	$4st^2R$	$-16s^2t\bar{R}$
1	$-2t^2R^2$	0	$-2t^2R^2$	0	0

Table 9. Terms in characteristic stability equation when Δ<0 (Case 3: complex conjugate roots): Part 2

$[RT]$	$[TRIG]_6$	$[TRIG]_7$	$[TRIG]_8$	$[TRIG]_9$	$[TRIG]_{10}$
	$\sin(bt)\cos^3(bt)$	$\cosh(4bs)\sin^2(bt)\cos^2(bt)$	$\sinh(4bs)\sin^2(bt)\cos^2(bt)$	$\sin^4(bt)$	$\cos^4(bt)$
$R_aT_aR_bT_b$	0	$-4t^2$	0	$2t^2$	$2t^2$
$R_aT_aT_b$	$16s^2t$	0	$-8st^2$	0	0
$R_aT_aR_b$	$-16s^2t(s^2+t^2)$	0	$-8st^2(s^2+t^2)$	0	0
$R_bT_bR_a$	$-16s^2t(s^2+t^2)$	0	$-8st^2(s^2+t^2)$	0	0
$R_bT_bT_a$	$16s^2t$	0	$-8st^2$	0	0
R_aT_a	0	$4t^2R$	0	$2t^2COEF_{CCR,(9,6)}$	$2t^2COEF_{CCR,(9,6)}$
R_bT_b	0	$4t^2R$	0	$2t^2COEF_{CCR,(9,6)}$	$2t^2COEF_{CCR,(9,6)}$
R_aR_b	0	$-16s^2t^2(s^2+t^2)^2$	0	$8s^2t^2(s^2+t^2)^2$	$8s^2t^2(s^2+t^2)^2$
T_aT_b	0	$-16s^2t^2$	0	$8s^2t^2$	$8s^2t^2$
R_bT_a	0	$-16s^2t^2(s^2+t^2)$	0	$-8s^2t^2(s^2+t^2)$	$-8s^2t^2(s^2+t^2)$
R_aT_b	0	$-16s^2t^2(s^2+t^2)$	0	$-8s^2t^2(s^2+t^2)$	$-8s^2t^2(s^2+t^2)$
R_a	$-16s^2t\bar{R}(s^2+t^2)$	0	$8st^2R(s^2+t^2)$	0	0
R_b	$-16s^2t\bar{R}(s^2+t^2)$	0	$8st^2R(s^2+t^2)$	0	0
T_a	$16s^2t\bar{R}$	0	$8st^2R$	0	0
T_b	$16s^2t\bar{R}$	0	$8st^2R$	0	0

1	0	$-4t^2R^2$	0	$2t^2R^2$	$2t^2R^2$

Table 10. Terms in characteristic stability equation when $\Delta < 0$ (Case 3: complex conjugate roots): Part 3

$[RT]$	$[TRIG]_{11}$
	$\sin^2(bt)\cos^2(bt)$
$R_aT_aR_bT_b$	$4(4s^2+t^2)$
$R_aT_aT_b$	0
$R_aT_aR_b$	0
$R_bT_bR_a$	0
$R_bT_bT_a$	0
R_aT_a	$COEF_{CCR,(11,6)}$
R_bT_b	$COEF_{CCR,(11,6)}$
R_aR_b	$-48s^2t^2(s^2+t^2)^2$
T_aT_b	$-48s^2t^2$
R_bT_a	$48s^2t^2(s^2+t^2)$
R_aT_b	$48s^2t^2(s^2+t^2)$
R_a	0
R_b	0
T_a	0
T_b	0
1	$4COEF_{CCR,(11,16)}$

Table 11. Terms in characteristic stability equation not multiplying trigonometric functions when $\Delta > 0$ and $2\alpha^2 - k > \sqrt{\Delta}$ (Case 1: real roots)

$[RT]$	$[REM]$
$R_aT_aR_bT_b$	$-8\beta_1\beta_2$
$R_aT_aT_b$	0
$R_aT_aR_b$	0
$R_bT_bR_a$	0
$R_bT_bT_a$	0
R_aT_a	$-4\beta_1\beta_2[\beta_1^4+\beta_2^4-2\alpha^2F\nu+\beta_1^2(k-2\alpha^2)+\beta_2^2(k-2\alpha^2)]$
R_bT_b	$-4\beta_1\beta_2[\beta_1^4+\beta_2^4-2\alpha^2F\nu+\beta_1^2(k-2\alpha^2)+\beta_2^2(k-2\alpha^2)]$

$R_a R_b$	0
$T_a T_b$	0
$R_b T_a$	0
$R_a T_b$	0
R_a	0
R_b	0
T_a	0
T_b	0
1	$-8\beta_1\beta_2(\beta_1^2 + F)(\beta_2^2 + F)(\alpha^2\nu - \beta_1^2)(\alpha^2\nu - \beta_2^2)$

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Notation

The following symbols are used in this paper:

$[A]$	=	matrix storing coefficients of c_1, c_2, c_3, and c_4;
B_p, B_m	=	addition and subtraction between β_1 and β_2;
b	=	twice the ratio between slab width and length;
c_1, c_2, c_3, c_4	=	constants in general solution of differential equation;
$[C]$	=	vector storing c_1, c_2, c_3, c_4;
$[COEF]$	=	vector of trigonometric functions;
D	=	slab flexural stiffness;
E	=	slab elastic modulus;
h	=	slab thickness;
i	=	$\sqrt{-1}$, complex unity;
k	=	dimensionless buckling load coefficient;
L	=	slab length;
M	=	slab bending moment;
N	=	axial load per unit length along slab edges;
R_a, R_b	=	rotational stiffness indexes;
$[REM]$	=	terms not multiplying trigonometric functions;
$[RT]$	=	vector containing combinations of R_a, R_b, T_a, and T_b;
S_a , S_b	=	translational springs;
s, t	=	real and complex part in roots of characteristic equation for Case 3;
T_a, T_b	=	translational stiffness indexes;
$[TRIG]$	=	matrix storing trigonometric function coefficients;
V	=	slab shear force;
w	=	slab vertical deflection;
x, y	=	normalized coordinates along slab length and width, respectively;
x' , y'	=	coordinates along slab length and width, respectively;
α	=	nπnπ, with n an integer;
β	=	unknown in characteristic equation;
β_1, β_2	=	roots of characteristic equation;
θ	=	slab rotation;
κ_a, κ_b	=	rotational springs;
Λ	=	elastic foundation constant;
λ	=	dimensionless subgrade stiffness coefficient; and
ν	=	slab Poisson's ratio.

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