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An LMI Approach to Discrete-Time Observer Design with Stochastic Resilience

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Abstract
Much of the recent work on robust control or observer design has focused on preservation of stability of the controlled system or the convergence of the observer in the presence of parameter perturbations in the plant or the measurement model. The present work addresses the important problem of stochastic resilience or non-fragility of a discrete-time Luenberger observer which is the maintenance of convergence and/or performance when the observer is erroneously implemented possibly due to computational errors i.e. round off errors in digital implementation or sensor errors, etc. A common linear matrix inequality framework is presented to address the stochastic resilient design problem for various performance criteria in the implementation based on

the knowledge of an upper bound on the variance of the random error in the observer gain. Present results are compared to earlier designs for stochastic robustness. Illustrative examples are given to complement the theoretical results.

Keywords

Discrete-time Luenberger observer, LMI, Resilence

1. Introduction

A controller for which the closed-loop system is destabilized by a small perturbation in the control gains is referred to as a “fragile” or “non-resilient” controller. In fact, the fragility problem is not new. Extreme fragility of various controllers is studied in [5]. It is shown that even vanishingly small perturbations in the control coefficients may destabilize the closed-loop system. After the publication of [5], the subject of fragility has gained more attention. A quadratic optimal state feedback controller that is non-fragile against perturbations in control gain is proposed in Ref. [4]. In Ref. [2], an overview of the non-fragile design techniques are presented. The robustness of control systems in digital implementation of a continuous time controller design is investigated in [6]. The synthesis of a resilient regulator for linear systems is described in [3]. In Ref. [8], the design of robust non-fragile state feedback controllers with controller gains in a given polytope is addressed. Robust non-fragile Kalman filter design for a class of linear systems with norm-bounded uncertainties and multiplicative uncertainties in the filter gain is given in [9]. Resilient filtering for a class of linear continuous-time systems with norm-bounded uncertainties and multiplicative and additive perturbations is investigated in [7].

In practice, more and more controllers and observers are implemented digitally. Thus implementation is subject to finite word length round off errors in numerical computations. Moreover, in some implementations, it is necessary to make manual tuning to obtain the desired performance for the closed-loop system. Therefore, the design process needs to be modified to accommodate perturbations in the controller and observer coefficients. This means that any useful design procedure should generate a controller or observer, which also has sufficient room for readjustment of its coefficients.

In this paper, in contrast with the earlier contributions, a stochastic approach to resilience is taken. A novel design of stability- and performance-resilient observers is introduced in discrete time. Process and measurement disturbances are modeled as random additive noise sequences with finite energy and the observer gain perturbations are modeled as white multiplicative noise sequences. Various design formulations are expressed in a general linear matrix inequality (LMI) [1] framework. The results obtained in this paper on stochastic resilience are compared with earlier ones on the robust observer design for stochastic parameters in system and measurement equations [10], [12]. Some illustrative examples are also included.

The following notation is utilized in this work: $x \in \mathbb{R}^n$ denotes an *n*-dimensional vector with real elements and with the associated norm $\| x \| = (x^{\mathrm{T}}x)^{1/2}$ where $(\cdot)^{\mathrm{T}}$ represents the transpose. $A \in \mathbb{R}^{m\times n}$ denotes an $m \times n$ matrix with real elements. A^{-1} is the inverse of matrix *A*, $A > 0 (A < 0)$ means *A* is a positive (negative) definite matrix, and I_m is an identity matrix of dimension m. $\lambda_{\min}(A)(\lambda_{\max}(A))$ denotes the minimum (maximum) eigenvalue of the symmetric matrix A. $E\{x\}$ and $E\{x/y\}$ denote the expectation of *x* and the expectation of *x* conditional on *y*. ℓ_2 is the space of all random infinite sequences of vectors $\{x_0, x_1, \ldots\}$ with finite energy $\lim_{N\to\infty} \sum_{i=0}^{N} E\{\| x_i \|^2\} < \infty$.

2. Signal and error dynamics

Consider the following discrete-time system and measurement equations

$$x_{k+1} = Ax_k + Fw_k,$$

$$y_k = Cx_k + Gw_k, \quad (1)$$

where $x_k \in \mathbb{R}^n$ is the state to be estimated from the knowledge of the measurements $y_k \in \mathbb{R}^p$. wk is an ℓ_2 disturbance input. Consider the following equation in the Luenberger observer form:

$$\hat{x}_{k+1} = A\hat{x}_k + (K + \Delta_k)(y_k - C\hat{x}_k), \hat{x}_0 = E\{x_0\}, \quad (2)$$

where Δ_k represents the time-varying error made in computing the observer gain *K*. In this work, a general stochastic description of the error in the filter gain is given as follows:

$$\Delta_k = \sum_{i=1}^{N} \gamma_k^i K^i, \quad (3)$$

where γ_k^i are zero mean mutually uncorrelated scalar white noise sequences with known variance upper bounds σ^i and K^i are known perturbation matrices. γ_k^i are assumed to be uncorrelated with the additive noise w_k. The zero mean property chosen for the multiplicative noise represents the physical situation where the perturbations can be positive or negative in an equally likely manner. The general time varying property is attributed to the gain perturbations by assuming γ_k^i as random sequences rather than random constants, because this allows different amounts of perturbations that may occur during operation. If only an a priori computation error in the gain is to be considered, then γ^i can be modeled as random constants and not as random sequences. This would be a special case of the general description in (3), which has been used in robustness studies involving structured parameter perturbations [11].

Let $e_k = x_k - \; x_{\;k}$ denote the estimation error. Substituting from Eqs. (1) and (2), we find that the error dynamics obey

$$e_{k+1} = (A - (K + \Delta_k)C)e_k + (F - (K + \Delta_k)G)w_k. \quad (4)$$

3. Performance criteria

Let Z_k denote the performance output where

$$Z_k = C_z e_k + D_z w_k. \quad (5)$$

Consider the general performance objective

$$E\{V_{k+1} - V_k + \delta \parallel Z_k \parallel^2 + \varepsilon \parallel w_k \parallel^2 - \beta Z_k^{\mathrm{T}} w_k | e_k, e_{k-1}, \dots, e_0\} \leqslant 0 \quad (6)$$

for a $V_k = e_k^{\mathrm{T}} P e_k$, where $P > 0$.

Notice that upon summation, taking expectation and using the interlacing property of expectation,

$$E\{E\{x/y\}\} = E\{x\}$$

inequality (6) yields

$$E\{e_N^{\mathrm{T}} P e_N\} \leqslant E\{e_0^{\mathrm{T}} P e_0\} - E\left\{\sum_{k=0}^{N} \; (\delta \parallel Z_k \parallel^2 + \varepsilon \parallel w_k \parallel^2 - \beta Z_k^{\mathrm{T}} w_k)\right\} \quad (7)$$

or

$$\lambda_{\min}(P)E\{\| e_N \|^2\} \leqslant \lambda_{\max}(P)E\{\| e_0 \|^2\} - \sum_{k=0}^{N} E\{(\delta \| Z_k \|^2 + \varepsilon \| w_k \|^2 - \beta Z_k^{\mathrm{T}} w_k)\}$$
(8)

by using Rayleigh's inequality ($\lambda_{\min}(P) \| e \|^2 \leqslant e^{\mathrm{T}}Pe \leqslant \lambda_{\max}(P) \| e \|^2$) twice, that allows several optimization formulations possible in a unified eigenvalue problem [1] framework.

First of all, we take $F = 0, G = 0$, and $D_z = 0$ to eliminate the additive noise dependence. In this case, if we let $\delta = \varepsilon = 0$, and $\beta = 0$, (8) yields

$$E\{\| e_N \|^2\} \leqslant \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} E\{\| e_0 \|^2\}. \text{ (9)}$$

This means that by minimizing $\lambda_{\max}(P)$ and maximizing $\lambda_{\max}(P)$, we can lower the bound on the mean square (m.s.) of the estimation error, which will guarantee a faster response for the observer. For systems such as (4), it was shown in [10] that this also guarantees almost sure (with probability one) boundedness of the estimation error.

For the same choice of parameter matrices, taking $\delta > 0, \beta = 0$, and $\varepsilon = 0$, (8) will yield a bound on the energy of the performance output in terms of the initial m.s. estimation error e_0

$$\sum_{k=0}^{N} E\{\| Z_k \|^2\} \leqslant \frac{1}{\delta} \lambda_{\max}(P) E\{\| e_0 \|^2\}. \text{(10)}$$

Minimizing $\lambda_{\max}(P)$ and maximizing δ will give us a smaller bound on the energy of the performance output. This is a sub-optimal H_2 observer.

In the case of additive noise w_k, and for general choices of F, G, and D_z, by setting $\delta = 1, \beta = 0$, and $\varepsilon < 0$ for $e_0 = 0$, gives the result

$$\sum_{k=0}^{N} E\{\| Z_k \|^2\} \leqslant -\varepsilon \sum_{k=0}^{N} E\{\| w_k \|^2\} \text{ (11)}$$

which means a bound on the ℓ_2 to ℓ_2 gain of the estimator, or a suboptimal H∞ design.

Again when $e_0 = 0$, if we use this formulation, we can design several m.s. dissipative observers by using different values of δ, β, and ε. All of these cases will require the choice $D_z + D_z^{\mathrm{T}} > 0$ in the performance output in (5).

For example, taking $\delta = 0, \beta = 1$, and $\varepsilon = 0$ will give m.s. passivity

$$\sum_{k=0}^{N} E\{Z_k^{\mathrm{T}} w_k\} \geqslant 0. \text{ (12)}$$

If we take $\delta = 0, \beta = 1$, and $\varepsilon > 0$, it will yield the m.s. input strict passivity result:

$$\sum_{k=0}^{N} E\{Z_k^{\mathrm{T}} w_k\} \geqslant \varepsilon \sum_{k=0}^{N} E\{\| w_k \|^2\}. \text{ (13)}$$

If we set $\delta > 0, \beta = 1$, and ε=0, we will get m.s. output strict passivity:

$$\sum_{k=0}^{N} E\{Z_k^{\mathrm{T}} w_k\} \geqslant \delta \sum_{k=0}^{N} E\{\|Z_k\|^2\}. \text{ (14)}$$

M.s. very strict passivity, which is the m.s. passivity both in the terms of the input and the output, will be obtained if we set $\delta > 0, \beta = 1$, and $\varepsilon > 0$:

$$\sum_{k=0}^{N} E\{Z_k^{\mathrm{T}} w_k\} \geqslant \varepsilon \sum_{k=0}^{N} E\{\| w_k \|^2\} + \delta \sum_{k=0}^{N} E\{\| Z_k \|^2\}. \text{ (15)}$$

As described above, the LMI formulation enables us to design various observers according to different performance criteria in a common framework.

4. LMI formulation

The non-noisy and noisy cases will be treated separately in the following development. First, consider inequality (6) with $F=0, G=0, D_z=0$, and $\varepsilon=\beta=0$. Substituting for the terms in the inequality from (1)–(5), and after some manipulations involving taking expectations and rearrangement, we obtain

$$E\{V_{k+1}-V_k+\delta \parallel Z_k \parallel^2 |e_k, e_{k-1}, \dots, e_0\} \leqslant E\left\{e_k^{\mathrm{T}}[-P+\delta C_z^{\mathrm{T}} C_z + C^{\mathrm{T}} \textstyle\sum(P) C + (A-KC)^{\mathrm{T}} P (A-KC)] e_k\right\} \quad (16)$$

for $\sum(P)=\sum_{i=1}^{N} \sigma^i K^{i^{\mathrm{T}}} P K^i$. This is negative semidefinite if and only if

$$\begin{bmatrix} P-\delta C_z^{\mathrm{T}} C_z - C^{\mathrm{T}} \sum(P) C & A^{\mathrm{T}} P - C^{\mathrm{T}} Y^{\mathrm{T}} \\ * & P \end{bmatrix} \geqslant 0 \quad (17)$$

by using Schur's complement [1] for $Y=PK$. Therefore, we have:

Theorem 1

Let (17) hold for $P>0$ and Y. Then, for $\delta=0$, this implies inequality (9) *and for $\delta>0$, this implies* (10). *The necessary resilient observer gain is found by $K=P^{-1}Y$.*

In the noisy case, similar arguments will lead to

$$E\left\{[e_k^{\mathrm{T}} w_k^{\mathrm{T}}]\left(\begin{bmatrix} P-\delta C_z^{\mathrm{T}} C_z - C^{\mathrm{T}} \sum(P) C & -\delta C_z^{\mathrm{T}} D_z + C^{\mathrm{T}} \sum(P) G + 0.5\beta C_z^{\mathrm{T}} \\ * & -\delta D_z^{\mathrm{T}} D_z - \varepsilon I + 0.5\beta (D_z + D_z^{\mathrm{T}}) - G^{\mathrm{T}} \sum(P) G \end{bmatrix} - \begin{bmatrix} (A-KC)^{\mathrm{T}} \\ (F-KG)^{\mathrm{T}} \end{bmatrix} P[(A-KC), (F-KG)]\right)\begin{bmatrix} e_k \\ w_k \end{bmatrix}\right\} > 0. \quad (18)$$

Using the Schur's complement result, (18) is equivalent to

$$Q=\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ * & q_{22} & q_{23} \\ * & * & q_{33} \end{bmatrix} \geqslant 0 \quad (19)$$

for

$$q_{11} = P-\delta C_z^{\mathrm{T}} C_z - C^{\mathrm{T}} \textstyle\sum(P) C,$$

$$q_{12} = -\delta C_z^{\mathrm{T}} D_z + 0.5\beta C_z^{\mathrm{T}} - C^{\mathrm{T}} \textstyle\sum(P) G,$$

$$q_{13} = A^{\mathrm{T}} P - C^{\mathrm{T}} Y^{\mathrm{T}},$$

$$q_{22} = -\delta D_z^{\mathrm{T}} D_z - \varepsilon I + 0.5\beta (D_z + D_z^{\mathrm{T}}) - G^{\mathrm{T}} \textstyle\sum(P) G,$$

$$q_{23} = F^{\mathrm{T}} P - G^{\mathrm{T}} Y^{\mathrm{T}},$$

$$q_{33} = P,$$

where $\sum(P)$ is defined above.

So, in the noisy case, we have:

Theorem 2

Let the LMI (19) *hold for* $P > 0$ *and* Y. *Then for different choice of design parameters* δ, β, and ε, *the inequalities*(11)–(15) *hold and the necessary resilient observer gain K is found from* $K = P^{-1}Y$.

5. Comparison with earlier results

Consider the following stochastically perturbed signal and measurement models:

$$x_{k+1} = \left(A + \sum_{i=1}^{N} \gamma_k^i A^i\right)x_k + Fw_k, \quad (20)$$

$$y_k = \left(C + \sum_{i=1}^{N} \gamma_k^i C^i\right)x_k + Gw_k, \quad (21)$$

where the definitions of the variables are the same. Let us again use the Luenberger observer

$$\hat{x}_{k+1} = A\hat{x}_k + K(y_k - C\hat{x}_k), \hat{x}_0 = E\{x_0\}. \quad (22)$$

Then we have the following robust observer result available:

Theorem 3

Yaz and Yaz [12]

Let the following LMI hold for some $X > 0$ *and scalar* $\alpha_1 > 0$:

$$\begin{bmatrix} X - I - A^{\mathrm{T}}XA - \sum_{i=1}^{N} \sigma^i A^{i\mathrm{T}} X A^i & -A^{\mathrm{T}}XF \\ * & \alpha_1 I - F^{\mathrm{T}}XF \end{bmatrix} > 0. \quad (23)$$

If there exist matrices $P > 0, Y$ *and scalar* $\alpha_2 > 0$, *such that*

$$\begin{bmatrix} P & 0 & AP - C^{\mathrm{T}}Y^{\mathrm{T}} \\ * & \alpha_2 I & F^{\mathrm{T}}P - G^{\mathrm{T}}Y^{\mathrm{T}} \\ * & * & P \end{bmatrix} \geqslant 0 \quad (24)$$

then $K = P^{-1}Y$ *and*

$$E\{e_N^{\mathrm{T}}Pe_N\} < E\{e_0^{\mathrm{T}}Pe_0\} \quad (25)$$

for all $N \geqslant 0$.

If instead of (24), *the following is true*:

$$\begin{bmatrix} P - C_z^{\mathrm{T}}C_z & 0 & AP - C^{\mathrm{T}}Y^{\mathrm{T}} \\ * & \alpha_2 I & F^{\mathrm{T}}P - G^{\mathrm{T}}Y^{\mathrm{T}} \\ * & * & P \end{bmatrix} \geqslant 0$$

together with (23), *then the performance output defined as*

$$Z_k = C_z e_k$$

satisfies the energy bound

$$\sum_{k=0}^{N} E\{\|Z_k\|^2\} < E\{e_0^{\mathrm{T}} P e_0\}$$

with $K = P^{-1}Y$.

One can see that the solution to the stochastically robust observer design problem necessitates also the satisfaction of the LMI (23) in addition to the main LMI. This additional LMI condition is interpreted by the following lemma in the same work [12]:

Lemma 1

The unforced system (20) with $F = 0$ *is m.s. exponentially stable if and only if there exist* $X > 0$ *and* $\alpha_1 > 0$ *such that (23) holds.*

Therefore, the solution of the robust stochastic observer problem necessitates the m.s. exponential stability of the system model, which is a more stringent requirement, whereas the solution of the stochastic resilient observer design problem does not.

6. Solution surfaces

The following section contains an investigation into the regions in the P and Y coordinates in which the LMIs (17) and (19) have solutions for a one-dimensional system and various design parameters. In this paper, we chose to work in the Y and P coordinates rather than K and P because Y vs. P feasibility regions directly describe the solution set of the corresponding LMIs. In a design situation, the corresponding resilient observer gains can be found from $K = P^{-1}Y$. The design parameters are given in Table 1 for three different performance indices: H_2 sub-optimal, m.s. input and output strict passivity.

Table 1. Design parameter values

	A	C	C_z	D_z	F	G	σ	K^i	δ	β	ε
H_2-observer	0.5	1	1	0	0	0	0.1	1	0.001, 0.01, 0.1, 1,5, 10	0	0
Input strict passivity	0.5	1	1	1	1	0.1	0.1	1	0	1	0.001, 0.01, 0.1, 0.3, 0.4, 0.49
Output strict passivity	0.5	1	1	1	1	0.1	0.1	1	0.001, 0.01, 0.1, 0.3, 0.4, 0.5	1	0

The corresponding areas are shaded differently to indicate how the shape of the regions change as the design parameters change. Large areas should be interpreted as containing the small areas inside. Magnified form of the critical parts of the regions in Fig. 1(a) are presented in (b).

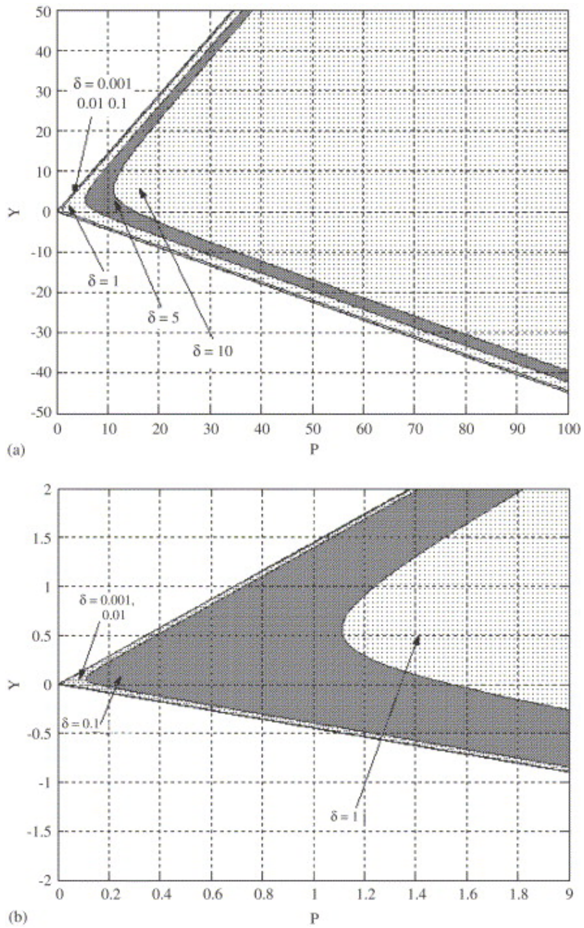

Fig. 1. (a) H2 sub-optimal observer feasibility regions. (b) H2 sub-optimal observer feasibility regions (vicinity of origin magnified).

Figs. 1(a) and (b) show how the feasibility region for the H2 sub-optimal resilient observer gets smaller as δ increases as expected. This is because it gets more difficult to satisfy the bound on the output energy which keeps getting smaller in (10).

Fig. 2. shows the feasibility region of the m.s. input strictly passive resilient observer. As ε increases from 0 to 0.49, the feasibility region gets smaller as expected. This is because the dissipation rate increases with ε and it becomes more difficult to satisfy (13) with increasing ε. When the maximum εvalue that is smaller than 0.49 is exceeded, LMI (19) ceases to have a positive definite solution.

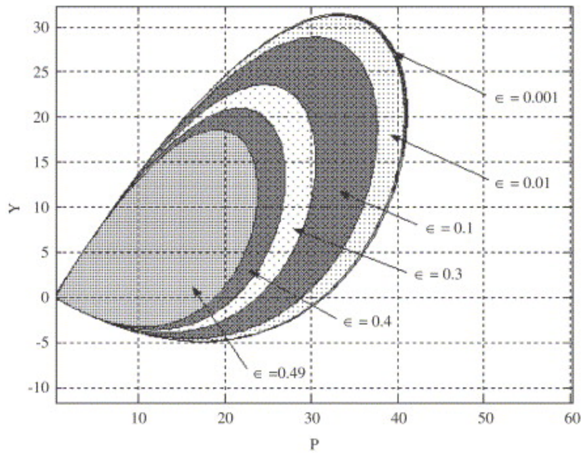

Fig. 2. M.s. input strictly passive observer feasibility regions.

The result given in Fig. 3. for m.s. output strict passivity is similar to the one given in Fig. 2. As δ increases, the feasibility region shrinks because it becomes more difficult to satisfy inequality (14) due to a higher required dissipation rate. For a δ value slightly larger than 0.5, the LMI ceases to be feasible.

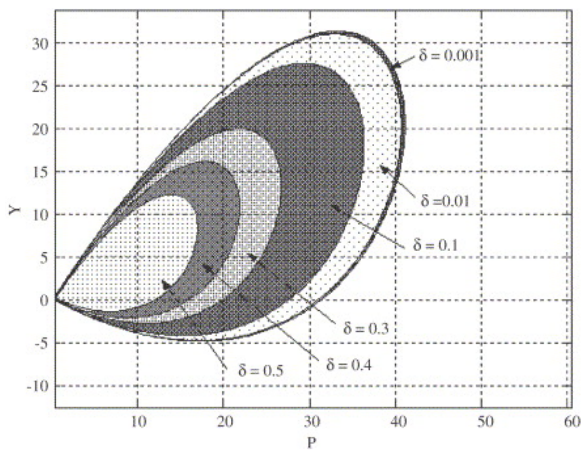

Fig. 3. M.s. output strictly passive observer feasibility regions.

7. Conclusions

This paper has presented a simple solution to the problem of non-fragile or resilient observer design for discrete-time systems with ℓ_2-type additive stochastic disturbances where the observer gain is randomly perturbed possibly due to computational errors. An LMI-based approach has been proposed to design observers with guaranteed performance and/or stability and the theoretical results introduced have been accompanied by illustrations of feasibility regions.

References

[1] S. Boyd, L.E. Ghaoui, E. Feron, V. Balakrishnan. **Linear matrix inequalities in system and control theory.** SIAM Studies in Applied Mathematics, SIAM, Philadelphia (1994)

[2] P. Dorato. **Non-fragile controller design: an overview.** Proceedings of ACC (1998), pp. 2829-2831

[3] D. Famularo, C.T. Abdallah, A. Jadbabaie, P. Dorato, W.H. Haddad. **Robust non-fragile LQ controllers: the static feedback case.** Proceedings of ACC (1998), pp. 1109-1113

[4] A. Jadbabaie, C.T. Abdallah, D. Famularo, P. Dorato. **Robust, non-fragile and optimal controller design via linear matrix inequalities.** Proceedings of ACC (1998), pp. 2832-2836

[5] L.H. Keel, S.P. Bhattacharyya. **Robust, fragile, or optimal?** IEEE Trans. Automat. Control, 42 (1997), pp. 1098-1105
[6] L.H. Keel, S.P. Bhattacharyya. **Stability margins and digital implementation of controllers.** Proceedings of ACC (1998), pp. 2852-2856
[7] M.S. Mahmoud. **Resilient linear filtering of uncertain systems.** Automatica, 40 (2004), pp. 1797-1802
[8] R.H.C. Takahashi, D.A. Dutra, R.M. Palhares, P.K.D. Peres. **On robust non-fragile static state-feedback controller synthesis.** Proceedings of CDC (2000), pp. 4909-4914
[9] G.-H. Yang, J.L. Wang. **Robust non-fragile Kalman filtering for uncertain linear systems.** IEEE Trans. Automat. Control, 46 (2001), pp. 343-348
[10] E. Yaz. **On the almost sure and mean square exponential convergence of some stochastic observers.** IEEE Trans. Automat. Control, 35 (1990), pp. 935-936
[11] E.E. Yaz. **Robust stability of discrete-time randomly perturbed systems.** Control and Dynamic Systems, vol. 73, Academic Press, New York (1995), pp. 89-120
[12] Y.I. Yaz, E.E. Yaz. **A new formulation of some discrete-time stochastic-parameter state estimation problems.** Appl. Math. Lett., 10 (1997), pp. 13-19