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Extremal H-colorings of graphs with fixed minimum degree

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Abstract:

For graphs G and H, a homomorphism from G to H, or H-coloring of G, is a map from the vertices of G to the vertices of H that preserves adjacency. When H is composed of an edge with one looped endvertex, an H-coloring of G corresponds to an independent set in G. Galvin showed that, for sufficiently large n, the complete bipartite graph $K_{\delta;n-\delta}$ is the n-vertex graph with minimum degree δ that has the largest number of independent sets.

In this paper, we begin the project of generalizing this result to arbitrary H. Writing hom(G,H) for the number of H-colorings of G, we show that for fixed H and δ = 1 or δ = 2,

$$\hom(G,H) \le \max\{\hom(K_{\delta+1},H)^{\frac{n}{\delta+1}}, \hom(K_{\delta,\delta},H)^{\frac{n}{2\delta}}, \hom(K_{\delta,n-\delta},H)\}$$

for any n-vertex G with minimum degree δ (for sufficiently large n). We also provide examples of H for which the maximum is achieved by $\hom(K_{\delta+1},H)^{n/\delta+1}$ and other H for which the maximum is achieved by $\hom(K_{\delta,\delta},H)^{n/2\delta}$. For $\delta \geq 3$ (and sufficiently large n), we provide a infinite family of H for which $\hom(G,H) \le \hom(K_{\delta,n-\delta},H)$ for any n-vertex G with minimum degree δ. The results generalize to weighted H-colorings.

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1 Introduction and statement of results

Let G = (V (G),E(G)) be a finite simple graph. A *homomorphism* from G to a finite graph H = (V (H),E(H)) (without multi-edges but perhaps with loops) is a map from V (G) to V (H) that preserves edge adjacency. We write

$$\mathrm{Hom}(G,H) = \{f : V(G) \to V(H) \mid v \sim_G w \implies f(v) \sim_H f(w)\}$$

for the set of all homomorphisms from G to H, and hom(G,H) for |Hom(G,H)|. All graphs mentioned in this paper will be finite without multiple edges. Those denoted by G will always be loopless, while those denoted by H may possibly have loops. We will also assume that H has no isolated vertices.

Graph homomorphisms generalize a number of important notions in graph theory. When $H = H_{ind}$, the graph consisting of a single edge and a loop on one endvertex, elements of Hom(G,H_{ind}) can be identified with the independent sets in G. When $H = K_q$, the complete graph on q vertices, elements of Hom(G,K_q) can be identified with the proper q-colorings of G. Motivated by this latter example, elements of Hom(G,H) are sometimes referred to as H-colorings of G, and the vertices of H are referred to as colors. We will utilize this terminology throughout the paper.

In statistical physics, H-colorings have a natural interpretation as configurations in *hard-constraint spin systems*. Here, the vertices of G are thought of as sites that are occupied by particles, with the edges of G representing pairs of bonded sites (for example by spatial proximity). The vertices of H represent the possible spins that a particle may have, and the occupation rule is that spins appearing on bonded sites must be adjacent in H. A valid configuration of spins on G is exactly an H-coloring of G. In the language of statistical physics, independent sets are configurations in the hard-core gas model, and proper q-colorings are configurations in the zero-temperature q-state antiferromagnetic Potts model. Another example comes from the Widom-Rowlinson graph $H = H_{WR}$, the fully-looped path on three vertices. If the endpoints of the path represent different particles and the middle vertex represents empty space, then the Widom-Rowlinson graph models the occupation of space by two mutually repelling particles.

Fix a graph H. A natural extremal question to ask is the following: for a given family of graphs $\mathcal{G}$, which graphs G in $\mathcal{G}$ maximize hom(G,H)? If we assume that all graphs in $\mathcal{G}$ have n vertices, then there are several cases where this question has a trivial answer. First, if H = K_q^{loop}, the fully looped complete graph on q vertices, then every map f : V (G) -> V (H) is an H-coloring (and so hom(G,K_q^{loop}) = q^n). Second, if the empty graph $\overline{K}_n$ is contained in $\mathcal{G}$, then again every map f : V ($\overline{K}_n$) -> V (H) is an H-coloring (and so hom($\overline{K}_n$,H) = $|V(H)|^n$). Motivated by this second trivial case, it is interesting to consider families $\mathcal{G}$ for which each G ∈ $\mathcal{G}$ has many edges.

For the family of n-vertex m-edge graphs, this question was first posed for H = K_q around 1986, independently, by Linial and Wilf. Lazebnik provided an answer for q = 2 [15], but for general q there is still not a complete answer. However, much progress has been made (see [16] and the references therein). Recently, Cutler and Radcliffe answered this question for H = H_{ind}, H = H_{WR}, and some other small H [2, 3]. A feature of the family of n-vertex, m-edge graphs emerging from the partial results mentioned is that there seems to be no uniform answer to the question,"which G in the family maximizes hom(G;H)?", with the answers depending very sensitively on the choice of H.

Another interesting family to consider is the family of n-vertex d-regular graphs. Here, Kahn [13] used entropy methods to show that every bipartite graph G in this family satisfies hom(G,H_{ind}) ≤ hom($K_{d,d}$;H_{ind})$^{n/2d}$, where $K_{d,d}$ is the complete bipartite graph with d vertices in each partition class. Notice that when 2d/n this bound is achieved by $\frac{n}{2d}K_{d,d}$, the disjoint union of n/2d copies of $K_{d,d}$. Galvin and Tetali [11] generalized this entropy argument, showing that for any H and any bipartite G in this family,

$$\hom(G,H) \le \hom(K_{d,d},H)^{n/2d}. \qquad (1)$$

Kahn conjectured that (1) should hold for H = H_{ind} for all (not necessarily bipartite) G, and Zhao [19] resolved this conjecture affirmatively, deducing the general result from the bipartite case.

Interestingly, (1) does not hold for general H when biparticity is dropped, as there are examples of n, d, and H for which $\frac{n}{d+1}K_{d+1}$, the disjoint union of n/(d + 1) copies of the complete graph K_{d+1}, maximizes the number of H-colorings of graphs in this family. (For example, take H to be the disjoint union of two looped vertices; here $\log_2(\hom(G,H))$ equals the number of components of G.) Galvin proposes the following conjecture in [8].

Conjecture 1.1. *Let G be an n-vertex d-regular graph. Then, for any H,*

$$\hom(G,H) \leq \max\{\hom(K_{d+1},H)^{\frac{n}{d+1}}, \hom(K_{d,d},H)^{\frac{n}{2d}}\}.$$

When 2d(d + 1)|n, this bound is achieved by either $\frac{n}{2d}K_{d,d}$ or $\frac{n}{d+1}K_{d+1}$. Evidence for this conjecture is given by Zhao [19, 20], who provided a large class of H for which $\hom(G,H) \leq \hom(K_{d,d},H)^{n/2d}$. Galvin [8, 9] provides further results for various H (including triples (n,d,H) for which $\hom(G,H) \leq \hom(K_{d+1},H)^{n/d+1}$) and asymptotic evidence for the conjecture.

It is clear that Conjecture 1.1 is true when d = 1, since the graph consisting of n/2 disjoint copies of an edge is the only 1-regular graph on n vertices. We prove the conjecture for d = 2 and also characterize the cases of equality.

Theorem 1.2. *Let G be an n-vertex 2-regular graph. Then, for any H,*

$$\hom(G,H) \leq \max\{\hom(C_3,H)^{\frac{n}{3}}, \hom(C_4,H)^{\frac{n}{4}}\}.$$

If $H \neq K_q^{loop}$, the only graphs achieving equality are $G = \frac{n}{3}C_3$ (when $\hom(C_3,H)^{n/3} > \hom(C_4,H)^{n/4}$), $G = \frac{n}{4}C_4$ ($\hom(C_3,H)^{n/3} < \hom(C_4,H)^{n/4}$), or the disjoint union of copies of C_3 and copies of C_4 (when $\hom(C_3,H)^{n/3} = \hom(C_4,H)^{n/4}$).

It is possible for each of the equality conditions in Theorem 1.2 to occur. The first two situations arise when H is a disjoint union of two looped vertices and $H = K_2$, respectively. For the third situation, we utilize that if G is connected and H is the disjoint union of H_1 and H_2, then $\hom(G,H) = \hom(G,H_1) + \hom(G,H_2)$. Letting H be the disjoint union of 8 copies of a single looped vertex and and 4 copies of K_2 gives $\hom(C_3,H)^{1/3} = \hom(C_4,H)^{1/4} = 2$.

Another natural and related family to study is **G** (n, δ), the set of all n-vertex graphs with minimum degree δ. Our question here becomes: for a given H, which G ∈ **G** (n, δ) maximizes hom(G,H)? Since removing edges increases the number of H-colorings, it is tempting to believe that the answer to this question will be a graph that is δ -regular (or close to δ -regular). This in fact is not the case, even for $H = H_{ind}$. The following result appears in [7].

Theorem 1.3. *For $\delta \geq 1$, $n \geq 8\delta^2$, and $G \in \mathcal{G}(n, \delta)$, we have*

$$\hom(G, H_{ind}) \leq \hom(K_{\delta,n-\delta}, H_{ind}),$$

with equality only for $G = K_{\delta,n-\delta}$.

Recently, Cutler and Radcliffe [4] have extended Theorem 1.3 to the range $n \geq 2\delta$. Further results related to maximizing the number of independent sets of a fixed size for $G \in \mathcal{G}(n, \delta)$ can be found in e.g. [1, 6].

With Conjecture 1.1 and Theorem 1.3 in mind, the following conjecture is natural.

Conjecture 1.4. Fix $\delta \geq 1$ and H. There exists a constant $c(\delta,H)$ (depending on δ and H) such that for $n \geq c(\delta,H)$ and $G \in \mathcal{G}(n, \delta)$,

$$\hom(G, H) \leq \max\{\hom(K_{\delta+1}, H)^{\frac{n}{\delta+1}}, \hom(K_{\delta,\delta}, H)^{\frac{n}{2\delta}}, \hom(K_{\delta,n-\delta}, H)\}.$$

This conjecture stands in marked contrast to the situation for the family of n-vertex m-edge graphs, where each choice of H seems to create a different set of extremal graphs. Here, we conjecture that for any H, one of exactly three situations can occur. For $2(\delta + 1)|n$ and n large, this represents the best possible conjecture, since for H consisting of a disjoint union of two looped vertices, $H = K_2$, and $H = H_{ind}$, the number of H-colorings of a graph $G \in \mathcal{G}(n, \delta)$ is maximized by $G = \frac{n}{\delta+1}K_{\delta+1}$, $G = \frac{n}{2\delta}K_{\delta,\delta}$, and $G = K_{\delta,n-\delta}$, respectively.

The purpose of this paper is to make progress toward Conjecture 1.4. We resolve the conjecture for $\delta = 1$ and $\delta = 2$, and characterize the graphs that achieve equality. We also find an infinite family of H for which hom(G,H) ≤ hom($K_{\delta,n-\delta}$,H) for all G ∈ $\mathcal{G}$ (n, δ) (for sufficiently large n), with equality only for G = $K_{\delta,n-\delta}$. Before we formally state these theorems, we highlight the degree conventions and notations that we will follow for the remainder of the paper.

Convention. For v ∈ V (H), let d(v) denote the degree of v, where loops count *once* toward the degree. While δ will always refer to the minimum degree of a graph G, Δ will always denote the maximum degree of a graph H (unless explicity stated otherwise).

Theorem 1.5. (δ = 1). Fix H, n ≥ 2 and G ∈ $\mathcal{G}$ (n, 1).

1. Suppose that H ≠ K_Δ^{loop} satisfies $\sum_{v\in V(H)} d(v) \geq \Delta^2$. Then

$$\hom(G,H) \leq \hom(K_2,H)^{n/2},$$

with equality only for G = $\frac{n}{2}K_2$.

2. Suppose that H satisfies $\sum_{v\in V(H)} d(v) \geq \Delta^2$, and let $n_0 = n_0(H)$ be the smallest integer in {3, 4,…} satisfying $\sum_{v\in V(H)} d(v) < \left(\sum_{v\in V(H)} d(v)^{n_0-1}\right)^{\frac{2}{n_0}}$

(a) If 2 ≤ n < n_0, then

$$\hom(G,H) \leq \hom(K_2,H)^{n/2},$$

with equality only for G = $\frac{n}{2}K_2$ [unless n = n_0 - 1 and $\sum_{v\in V(H)} d(v) = \left(\sum_{v\in V(H)} d(v)^{n_0-1}\right)^{\frac{2}{n_0-1}}$ in which case G = $K_{1,n-1}$ also achieves equality].

(b) If n ≥ n_0, then

$$\hom(G,H) \leq \hom(K_{1,n-1},H),$$

with equality only for G = $K_{1,n-1}$.

Remark. Notice that hom(K_2,H) = $\sum_{v\in V(H)} d(v)$, so the conditions on H in Theorem 1.5 may also be written as hom(K_2,H)$^{1/2}$ ≥ Δ and hom(K_2,H) > Δ.

Theorem 1.6. (δ = 2). Fix H.

1. Suppose that H ≠ K_Δ^{loop} satisfies max{hom(C_3,H)$^{1/3}$, hom(C_4,H)$^{1/4}$} ≥ Δ. Then for all n ≥ 3 and G ∈ $\mathcal{G}$ (n, 2)

$$\hom(G, H) = \max\{\hom(C_3,H)^{1/3}, \hom(C_4,H)^{1/4}\},$$

with equality only for G = $\frac{n}{3}C_3$ (when hom(C_3,H)$^{1/3}$ > hom(C_4,H)$^{1/4}$), G = $\frac{n}{4}C_4$ (when hom(C3;H) n

3 < hom(C4;H)

n

4), or the disjoint union of copies of C3

and copies of C4 (when hom(C3;H)

n

3 = hom(C4;H)

n

4).

2. Suppose that H satis_es maxfhom(C3;H)

1

3 ; hom(C4;H)

1

4 g < _. Then there

exists a constant cH such that for n > cH and G 2 G(n; 2),

hom(G;H) _ hom(K2;n☐2;H);

with equality only for G = K2;n☐2.

About the Authors

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