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CHAPTER

10 Concept of Sub-Independence

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ABSTRACT: Limit theorems as well as other well-known results in probability and statistics are often based on the distributions of the sums of independent random variables. The concept of sub-independence, which is weaker than that of independence, is shown to be sufficient to yield the conclusions of these theorems and results. It also provides a measure of dissociation between two random variables which is stronger than uncorrelatedness.

10.1 INTRODUCTION

Limit theorems as well as other well-known results in probability and statistics are often based on the distributions of the sums of independent (and often identically distributed) random variables rather than the joint distribution of the summands. Therefore, the full force of independence of the summands will not be required. In other words, it is the convolution of the marginal distributions which is needed, rather than the joint distribution of the summands which, in the case of independence, is the product of the marginal distributions. This is precisely the reason for the statement: "why assume independence when you can get by with sub-independence".

A comprehensive treatment of the concept of sub-independence from its beginning 1979 to 2011 will appear as an Expository Article elsewhere which will include the content of this article.

The concept of sub-independence can help to provide solution for some modeling problems where the variable of interest is the sum of a few components. Examples include household income, the total profit of major firms in an industry, and a regression model $Y = g(X) + \varepsilon$ where $g(X)$ and ε are uncorrelated, however, they may not be independent. For example, in Bazargan et al. (2007), the return value of significant wave height (Y) is modeled by the sum of a cyclic function of random delay D, $\hat{g}(D)$, and a residual term $\hat{\varepsilon}$. They found that the two components are at least uncorrelated but not independent and used sub-independence to compute the distribution of the return value. For the detailed application of the concept of sub-independence in this direction we refer the reader to Bazargan et al. (2007).

For the sake of completeness we restate some well-known definitions. Let X and Y be two $rv's$ (random variables) with joint and marginal $cdf's$ (cumulative distribution functions) $F_{X,Y}$, F_X and F_Y respectively. Then X and Y are said to be independent if and only if

$$F_{X,Y}(x,y) = F_X(x)F_Y(y), \text{ for all } (x,y) \in \mathbb{R}^2, \tag{10.1}$$

or equivalently, if and only if

$$\varphi_{X,Y}(s,t) = \varphi_X(s)\,\varphi_Y(t) \quad \text{for all} \quad (s,t) \in \mathbb{R}^2, \tag{10.2}$$

where $\varphi_{X,Y}(s,t)$, $\varphi_X(s)$ and $\varphi_Y(t)$, respectively, are the corresponding joint and marginal $cf's$ (characteristic functions). Note that (10.1) and (10.2) are also equivalent to

$$P(X \in A \text{ and } Y \in B) = P(X \in A)\, P(Y \in B), \text{ for all Borel sets } A, B. \tag{10.3}$$

The concept of sub-independence, as far as we have gathered, was formally introduced by Durairajan (1979), stated as follows: The $rv's$ X and Y with $cdf's$ F_X and F_Y are $s.i.$ (sub-independent) if the cdf of $X + Y$ is given by

$$F_{X+Y}(z) = (F_X * F_Y)(z) = \int_{\mathbb{R}} F_X(z-y)\, dF_Y(y), \quad z \in \mathbb{R}, \tag{10.4}$$

or equivalently if and only if

$$\varphi_{X+Y}(t) = \varphi_{X,Y}(t,t) = \varphi_X(t)\,\varphi_Y(t), \quad \text{for all } t \in \mathbb{R}. \tag{10.5}$$

The drawback of the concept of sub-independence in comparison with that of independence has been that the former does not have an equivalent definition in the sense of (10.3), which some believe, to be the natural definition of independence. We believe to have found such a definition now, which is stated below. We shall give two separate definitions, one for the discrete case (Definition 10.1) and the other for the continuous case (Definition 10.2).

Let $(X,Y):\Omega \to \mathbb{R}^2$ be a discrete random vector with range $\Re(X,Y)=\{(x_i,y_j): i,j=1,2,\ldots\}$ (finitely or infinitely countable). Consider the events

$$A_i=\{\omega\in\Omega : X(\omega)=x_i\}\ ,\ B_j=\{\omega\in\Omega : Y(\omega)=y_j\}$$

and

$$A^z=\{\omega\in\Omega : X(\omega)+Y(\omega)=z\}\ ,\ z\in\Re(X+Y).$$

DEFINITION 10.1 The discrete *rv's* X and Y are *s.i.* if for every $z\in\Re(X+Y)$

$$P(A^z)=\sum_{i,j,}\ \sum_{x_i+y_j=z} P(A_i)\,P(B_j). \tag{10.6}$$

To see that (10.6) is equivalent to (10.5), suppose X and Y are *s.i.* via (10.5), then

$$\sum_i\sum_j e^{it(x_i+y_j)} f(x_i,y_j)=\sum_i\sum_j e^{it(x_i+y_j)} f_X(x_i)f_Y(y_j),$$

where f, f_X and f_Y are probability functions of (X, Y), X and Y respectively. Let $z\in\Re(X+Y)$, then

$$e^{itz}\sum\ \sum_{i,j,x_i+y_j=z} f(x_i,y_j)=e^{itz}\sum\ \sum_{i,j,x_i+y_j=z} f_X(x_i)f_Y(y_j),$$

which implies (10.6).

For the continuous case, we observe that the half-plane $H=\{(x,y): x+y<0\}$ can be written as a countable disjoint union of rectangles:

$$H=\cup_{i=1}^{\infty} E_i\times F_i,$$

where E_i and F_i are intervals. Now, let $(X,Y):\Omega\to\mathbb{R}^2$ be a continuous random vector and for $c\in\mathbb{R}$, let

$$A_c=\{\omega\in\Omega : X(\omega)+Y(\omega)<c\}$$

and

$$A_i^{(c)}=\left\{\omega\in\Omega : X(\omega)-\frac{c}{2}\in E_i\right\}, B_i^{(c)}=\left\{\omega\in\Omega : Y(\omega)-\frac{c}{2}\in F_i\right\}.$$

DEFINITION 10.2. The continuous $rv's$ X and Y are $s.i.$ if for every $c \in \mathbb{R}$

$$P(A_c) = \sum_{i=1}^{\infty} P(A_i^{(c)})P(B_i^{(c)}). \tag{10.7}$$

To see that (10.7) is equivalent to (10.4), observe that (LHS of (10.7))

$$P(A_c) = P(X+Y<c) = P\big((X,Y)\in H_c\big), \tag{10.8}$$

where $H_c = \{(x,y): x+y<c\}$. Now, if X and Y are $s.i.$ then

$$P(A_c) = (P_X \times P_Y)\,(H_c)$$

where P_X, P_Y are probability measures on $\mathbb{R}$ defined by

$$P_X(B) = P(X\in B) \text{ and } P_Y\,(B) = P(Y \in B),$$

and $P_X \times P_Y$ is the product measure.

We also observe that (RHS of (10.7))

$$\begin{aligned}\sum_{i=1}^{\infty} P\left(A_i^{(c)}\right)P\left(B_i^{(c)}\right) &= \sum_{i=1}^{\infty} P\left(X-\frac{c}{2}\in E_i\right)P\left(Y-\frac{c}{2}\in F_i\right)\\ &= \sum_{i=1}^{\infty} P\left(X\in E_i+\frac{c}{2}\right)P\left(Y\in F_i+\frac{c}{2}\right)\\ &= \sum_{i=1}^{\infty} P_X \times P_Y\left(E_i+\frac{c}{2}\right)\times\left(F_i+\frac{c}{2}\right).\end{aligned} \tag{10.9}$$

Now, (10.8) and (10.9) will be equal if $H_c = \cup_{i=1}^{\infty}\left\{\left(E_i+\frac{c}{2}\right)\times\left(F_i+\frac{c}{2}\right)\right\}$, which is true since the points in H_c are obtained by shifting each point in H over to the right by $\frac{c}{2}$ units and then up by $\frac{c}{2}$ units.

REMARKS 10.1.

(*i*) Note that H can be written as a union of squares and triangles. The triangles are congruent to $0 \le y < x$, $0 \le x < 1$ which in turn can be written as a disjoint union of squares. For example, take [0, 1/2) × [0, 1/2) then [1/2, 3/4) × [0, 1/4) and so on.

(*ii*) The discrete $rv's$ X, Y and Z are $s.i.$ if (10.6) holds for any pair and

$$P(A^s) = \sum_{i,j,k,}\ \sum_{x_i+y_j+z_k=s}\ \sum P(A_i)P(B_j)P(C_k). \tag{10.10}$$

For p variate case we need $2^p - p - 1$ equations of the above form.

(*iii*) The representation (10.7) can be extended to the multivariate case as well.

(*iv*) For the sake of simplicity of the computations, (10.5) and its extension to the multivariate case will be taken as definition of sub-independence.

We may in some occasions have asked ourselves if there is a concept between "uncorrelatedness" and "independence" of two random variables. It seems that the concept of "sub-independence" is the one: it is much stronger than uncorrelatedness and much weaker than independence. The notion of sub-independence seems important in the sense that under usual assumptions, Khintchine's Law of Large Numbers and Lindeberg-Lévy's Central Limit Theorem as well as other important theorems in probability and statistics hold for a sequence of *s.i.* random variables. While sub-independence can be substituted for independence in many cases, it is difficult, in general, to find conditions under which the former implies the latter. Even in the case of two discrete identically distributed $rv's$ X and Y, the joint distribution can assume many forms consistent with sub-independence. In order for two random variables X and Y to be *s.i.*, the probabilities

$$p_i = P\left(X = x_i\right), \quad i = 1,2,\dots,n$$

and

$$q_{ij} = P\left(X = x_i, Y = x_j\right), \quad i,j = 1,2,\dots,n,$$

must satisfy the following conditions:

1. $\sum(q_{ij} - p_i p_j) = 0$, where the sum extends for all values of i and j for which $x_i + x_j = z$ and z takes all the values in the set $\left\{\min(x_i + x_j),\dots,\max(x_i + x_j)\right\}$;
2. $p_i = \sum_{j=1}^{n} q_{ij} = \sum_{j=1}^{n} q_{ji}$, $\quad i = 1,2,\dots,n$.

This linear system in n^2 variables q_{ij} is considerably underdetermined for all but the smallest value of n specially if a large number of points (x_i, x_j) lie on the line $x + y = z$. On the other hand, the only q_{ij} consistent with independence is $q_{ij} = p_i p_i$.

If X and Y are *s.i.*, then unlike independence, X and αY are not necessarily *s.i.* for any real $\alpha \neq 1$ as the following simple example shows.

EXAMPLE 10.1. Let X and Y have the joint cf given by

$$\varphi_{X,Y}(t_1,t_2) = \exp\{-(t_1^2 + t_2^2)/2\}[1 + \beta\, t_1 t_2 (t_1 - t_2)^2 \times \exp\{(t_1^2 + t_2^2)/4\}],\ (t_1,t_2) \in \mathbb{R}^2,$$

where β is an appropriate constant. Then X and Y are *s.i.* standard normal $rv's$, and hence $X + Y$ is normal with mean 0 and variance 2, but X and $-Y$ are not *s.i.* and consequently $X - Y$ does not have a normal distribution.

The concept of sub-independence defined earlier can be extended to n (> 2) $rv's$ as follows.

DEFINITION 10.3. The $rv's$ $X_1, X_2, ..., X_n$ are $s.i.$ if for each subset $\{X_{\alpha_1}, X_{\alpha_2}, ..., X_{\alpha_r}\}$ of $\{X_1, X_2, ..., X_n\}$

$$\varphi_{X_{\alpha_1},...,X_{\alpha_r}}(t,...,t) = \Pi_{i=1}^{r}\varphi_{X_{\alpha_i}}(t),\ forall\ t \in \mathbb{R}. \tag{10.11}$$

To see how weak the concept of sub-independence is in comparison with that of independence, even in the case involving normal distribution, Hamedani (1983) gave the following example.

EXAMPLE 10.2. Given $\{(a_k, b_k) : k = 1,2,...,N\}$ a finite set in $\mathbb{R}^2$. Consider the joint cf

$$\varphi_{X,Y}(t_1,t_2) = \exp\{-(t_1^2 + t_2^2)/2\} + t_1 t_2 (t_1^2 - t_2^2)$$
$$\times \exp\left\{-\frac{1}{2}[c_1 - c_2(t_1^2 + t_2^2)\right\} \Pi_{k=1}^{N} (b_k^2 t_1^2 - a_k^2 t_2^2),\ (t_1,t_2) \in \mathbb{R}^2,$$

where c_1 and c_2 are suitable constants. Then X and Y are standard normal $rv's$, X and Y, as well as, X and $-Y$ are $s.i.$ and more

$$\varphi_{X,Y}(a_k t, b_k t) = \varphi_X(a_k t)\, \varphi_Y(b_k t),\ \text{for all } t \in \mathbb{R},\ \ k = 1,2,...,N,$$

i.e., $a_k X$ and $b_k Y$, $k = 1,2,...,N$ are $s.i.$ and of course $a_k X + b_k Y$, $k = 1,2,...,N$ are all normally distributed, but X and Y are not independent.

REMARK 10.2. The set $\{(a_k, b_k) : k = 1,2,...,N\}$ in Example 10.2 cannot be taken to be infinitely countable. Hamedani and Tata (1975) showed that two normally distributed $rv's$ X and Y are independent only if they are uncorrelated and $a_k X$ and $b_k Y$, $k = 1, 2,...$ are $s.i.$; *i.e.*,

$$\varphi_{X,Y}(a_k t, b_k t) = \varphi_X(a_k t)\varphi_X(b_k t),\ \text{for all } t \in \mathbb{R},\ k = 1,2,...,$$

where, $\{(a_k, b_k) : k = 1,2,...\}$ is a distinct sequence in $\mathbb{R}^2$.

10.2 SOME APPLICATIONS OF THE CONCEPT OF SUB-INDEPENDENCE

We mention below a few results in which the assumption of independence is replaced by that of sub-independence, starting with the $s.i.$ version of Cramér's famous theorem (Theorem 1, 1936) which appeared in Hamedani and Walter $(1984b)$.

THEOREM 10.1. (Cramér). If the sum $X + Y$ of the $rv's$ X and Y is normally distributed and these $rv's$ are $s.i.$, then each of X and Y is normally distributed.

PROPOSITION 10.1. (Chung). Let X and Y be $s.i.i.d.$ (sub-independent and identically distributed) $rv's$ with mean 0 and variance 1 such that

(i) X and $-Y$ are $s.i.$,

(ii) $X + Y$ and $X - Y$ are $s.i.$.

Then, both X and Y have standard normal distributions.

PROPOSITION 10.2. (Chung). Let X and Y as well as X and $-Y$ be *s.i.* normally distributed $rv's$ with the same variance. Then $X+Y$ and $X-Y$ are *s.i.*.

THEOREM 10.2. Let X and Y be *s.i.i.d.* nondegenerate $rv's$. If X^2 and $\frac{1}{2}(X+Y)^2$ are *i.d.* chi-square with one degree of freedom, then the common distribution of X and Y is standard normal.

THEOREM 10.3. Let $X_1, X_2, \ldots, X_n$ be *s.i.i.d.* nondegenerate $rv's$. If $k\ \overline{X}_k^2$, $\overline{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i$, is distributed as chi-square with one degree of freedom for two positive integers m_1 and m_2, then $X_i's$ are normally distributed.

A rv X (or its *pdf* f_X) is called reciprocal if its *cf* is a multiple of a *pdf*. It is called self-reciprocal if there exist constants A and α such that $Af_X(\alpha t)$ is the *cf* of X. It is strictly self-reciprocal if $(2\pi)^{1/2}\ f_X(t)$ is the *cf* of X. Using the concepts of reciprocal, self-reciprocal, strictly self-reciprocal and sub-independence, Hamedani and Walter (1985) reported the following observations (Propositions 10.3, 10.4 and Theorem 10.7 below).

PROPOSITION 10.3 Let X be the standard normal rv, Y be any infinitely divisible rv *s.i.* of X. Then $X+Y$ is self-reciprocal if and only if Y is normally distributed.

THEOREM 10.4. Let X be the standard normal rv and Y be strictly self-reciprocal and *s.i.* of X. Then $X+Y$ is self-reciprocal if and only if it is normally distributed.

PROPOSITION 10.4. Let X be the standard normal rv, Y a symmetric (about 0) rv *s.i.* of X. Then Y is strictly self-reciprocal if and only if the *cf* φ of the rv $X+Y$ satisfies the functional equation

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\{(s+it)^2/2\}\, \varphi(s)\,ds, \text{forall } t \in \mathbb{R}.$$

THEOREM 10.5. Let X and Y be *s.i.i.d* $rv's$ whose sum $X+Y$, is symmetric. Then X and Y are symmetric $rv's$.

THEOREM 10.6. Let X and Y be *s.i.* and $X+Y$ symmetric. If X is symmetric with *cf* $\varphi_X(t) \neq 0$, for all t, then Y must be symmetric.

THEOREM 10.7. (Raikov). If X and Y are non-negative integer-valued $rv's$ such $X+Y$ has a Poisson distribution and X and Y are *s.i.*, then each of X and Y has a Poisson distribution.

As we mentioned before, the well-known Khintchine's Law of Large Numbers and Lindeberg-Lévy's Central Limit Theorem as well as other important results can be stated in terms of *s.i.* $rv's$. Hamedani and Walter (1984a) reported several version of the central limit theorems for *s.i.i.d.* $rv's$ to which we refer the reader for details.

10.3. A DIFFERENT BUT EQUIVALENT INTERPRETATION OF SUB-INDEPENDENCE AND RELATED RESULTS

Ebrahimi et al. (2010) look at the concept of sub-independence in different but equivalent definition which provides a better understanding of this concept. Here we copy a portion of their paper since it treats this notion in completely different direction than we have dealt with so far. They present models for the joint distribution of uncorrelated variables that are not independent, but the distribution of their sum is given by the product of their marginal distributions. These models are referred to as the summable uncorrelated marginals distributions. They are developed utilizing the assumption of sub-independence, which has been employed in the present work in various directions, for the derivation of the distribution of the sum of random variables.

We shall now revisit the definition of sub-independence of the *rv's* $X_1, X_2, \ldots, X_p$. We use p in place of n to be consistent with its use by Ebrahimi et al. (2010). Let $\mathbf{X} = (X_1, X_2, \ldots, X_p)'$ be a random vector with *cdf* F and *cf* $\Psi(\mathbf{t})$. Components of $\mathbf{X}$ are said to be *s.i.* if

$$\Psi(\mathbf{t}) = \Pi_{i=1}^{p} \psi_i(t), \quad \forall \mathbf{t} = (t, t, \ldots, t)' \in \mathbb{R}^p, \tag{10.12}$$

where $\psi_i(t)$ is *cf* of X_i. We first consider the bivariate case $p = 2$ and let F be the *cdf* of $\mathbf{X} = (X_1, X_2)$, and $\mathbf{X}^* = (X_1^*, X_2^*)$ denote the random vector with *cdf* $F^*(x_1, x_2) = F_1(x_1) F_2(x_2)$, where F_i, $i = 1, 2$ is the marginal *cdf* of X_i.

DEFINITION 10.4. F is said to be SUM (summable uncorrelated marginals) bivariate distribution if $X_1 + X_2 \overset{st}{=} X_1^* + X_2^*$, where $\overset{st}{=}$ denotes the stochastic equality. Random variables with a SUM joint distribution are referred to as SUM random variables.

It is clear that the SUM and sub-independence are equivalent, so the two terminologies can be used interchangeably. It is also clear that the class of SUM *rv's* are closed under scalar multiplication and addition under independence. That is, if $\mathbf{X} = (X_1, X_2)$ is a SUM random vector, so is $a\ \mathbf{X}$, and if $\mathbf{Y} = (Y_1, Y_2)$ is another SUM random vector independent of $\mathbf{X}$, then $\mathbf{X} + \mathbf{Y}$ is also SUM random vector. However, the SUM property is directional in that X_1 and X_2 being SUM *rv's* does not imply that X_1 and aX_2 are SUM. Definition 10.4 can be generalized to any specific direction by $a_1 X_1 + a_2 X_2 \overset{st}{=} a_1 X_1^* + a_2 X_2^*$.

We define a bivariate SUM copula to be a SUM distribution on the unit square $[0,1]^2$ with uniform marginals. We state the following lemma, due to Ebrahimi et al. (2010), which explains the construction of families of SUM models by linking the univariate *pdf's* $f_i(x_i)$, $i = 1, 2$.

LEMMA 10.1. Let $f_i(x_i)$, $i = 1, 2$ be *pdf's* and $g(x_1, x_2)$ a measurable function. Set

$$f_\beta(x_1,x_2) = f_1(x_1)f_2(x_2)+\beta\, g(x_1,x_2),\ (x_1,x_2)\in\mathbb{R}^2. \tag{10.13}$$

Then for some $\beta\in\mathbb{R}$, $f_\beta(x_1,x_2)$ is a SUM *pdf* with marginal *pdf's* $f_i(x_i)$, $i = 1, 2$, provided that:

(*a*) $f_\beta(x_1,x_2)\ge 0$

(*b*) $\int_{\mathbb{R}} g(x_1,x_2)dx_1 = \int_{\mathbb{R}} g(x_1,x_2)\,dx_2 = 0$ for all $(x_1,x_2)\in\mathbb{R}^2$

(*c*) $\int_{\mathbb{R}} g(c-t,t)\,dt = 0$ for all $c\in\mathbb{R}$.

The next example illustrates Lemma 10.1.

EXAMPLE 10.3. Let $f_i(x_i)$, $i = 1, 2$ be two *pdf's* on [0, 1] and set

$$f_\beta(x_1,x_2) = f_1(x_1)\, f_2(x_2)+\beta\sin[2\pi(x_2-x_1)],\ (x_1,x_2)\in[0,1]^2, \tag{10.14}$$

such that for some $\beta\in\mathbb{R}$, $f_\beta(x_1,x_2)$ is a *pdf* on $[0, 1]^2$.

We mention here examples in which (X_1,X_2) has a SUM distribution and X_1 and X_2 are *i.d.* with symmetric *pdf's* other than $N(0, 1)$.

(i) Standard Cauchy: $f(x)=\dfrac{1}{\pi(1+x^2)},\quad x\in\mathbb{R}$;

(ii) Laplace Double Exponential: $f(x)=\dfrac{1}{2\sigma}e^{-|x-\mu|/\sigma},\ x\in\mathbb{R}$;

(iii) Hyperbolic Secant: $f(x)=\dfrac{1}{2\gamma K_1(\alpha\gamma)}e^{-\alpha\sqrt{\gamma^2+(x-\mu)^2}}$, $x\in\mathbb{R}$, where K_1 is a modified Bessel function of the second kind;

(iv) Logistic or Sech-Square(d): $f(x)=\dfrac{e^{-(x-\mu)/s}}{s(1+e^{-(x-\mu)/s})^2}=\dfrac{1}{4s}\,Sec\,h\left(\dfrac{x-\mu}{2s}\right)$, $\mu = mean$ and s is proportion to standard deviation;

(v) Raised Cosine: $f(x)=\dfrac{1}{2s}\left[1+\cos\left(\dfrac{\pi(x-\mu)}{s}\right)\right]$, $\mu-s\le x\le\mu+s$;

(vi) Wigner Semicircle: $f(x)=\frac{2}{\pi r^2}\sqrt{r^2-x^2}\ ,\ -r<x<r\ ;$

(vii) $f(x)=\frac{1}{2\pi}\frac{\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2}\ ;\quad f(x)=\frac{2(2+x^2)\sin^2\left(\frac{x}{4}\right)+x\left(x+\sin\left(\frac{x}{2}\right)\right)}{2\pi x^2(1+x^2)}\ ;$

$$f(x)=\frac{4\sin^2\left(\frac{x}{4}\right)}{\pi x^2}\ ;\ f(x)=\frac{4\left(x-2\sin\left(\frac{x}{2}\right)\right)}{\pi x^3}\ ,\ x\in\mathbb{R}.$$

Note that the *cf's* corresponding to *pdf's* (*vii*) are respectively

$$\varphi(t)=\begin{cases}1-|t| & ,\ \text{if}\ |t|\le 1\\ 0 & ,\ \text{if}\ |t|>1\end{cases}\ ;\quad \varphi(t)=\begin{cases}1-|t| & \text{if}\ ,\ |t|\le\frac{1}{2}\\ \frac{1}{2}e^{-|t|+\frac{1}{2}} & \text{if}\ ,\ |t|>\frac{1}{2}\end{cases};$$

$$\varphi_1(t)=\begin{cases}1-2|t| & \text{if}\ ,\ |t|\le\frac{1}{2}\\ 0 & \text{if}\ ,\ |t|>\frac{1}{2}\end{cases};\quad \varphi_2(t)=|\varphi_1(t)|^2\ ,\ t\in\mathbb{R}\ .$$

The graphs of the first two *pdf's* in (*vii*) are bell-shaped and can be used to approximate normal *pdf*. Hamedani et al. (2011) have presented various examples of bivariate mixture SUM distributions based on the *pdf's* given in (vii).

We can consider multivariate SUM random variables. Let F be the *cdf* of $\mathbf{X}=(X_1,X_2,\ldots,X_p)'$ and $\mathbf{X}^*=(X_1^*,X_2^*,\ldots,X_p^*)'$ denote the random vector with *cdf* $F^*=\Pi_{i=1}^p F_i$, where F_i is the *cdf* of X_i.

DEFINITION 10.5. F is said to be a SUMp (SUM distribution of order p) if $\sum_{i=1}^p X_i \overset{st}{=} \sum_{i=1}^p X_i^*$.

The following example, due to Ebrahimi et al. (2010), shows a trivariate SUM distribution.

EXAMPLE 10.4. Let $\mathbf{X}=(X_1,X_2,X_3)'$ and consider the distribution with *pdf*

$$f_\beta(\mathbf{x})=\frac{1}{(2\pi)^{3/2}}e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}}\left(1+\beta(x_1-x_2)(x_1-x_3)(x_2-x_3)e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}}\right),\ \mathbf{x}\in\mathbb{R}^3,$$

where $\beta = B^{-1}$ and

$$\left|(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}}\right| \le B. \qquad (10.15)$$

The corresponding *cf* is

$$\Psi_\beta(\mathbf{t}) = e^{-\frac{1}{2}\mathbf{t}'\mathbf{t}} - \frac{1}{2^{9/2}}\beta i(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)e^{-\frac{1}{4}\mathbf{t}'\mathbf{t}}, \ \mathbf{t} \in \mathbb{R}^3,$$

where $\mathbf{t} = (t_1, t_2, t_3)'$. Clearly $f_\beta(\mathbf{x})$ is SUM3. It can be shown that $f_\beta(x_i, x_j)$, $i \ne j = 1,2,3$ are SUM2 for all β satisfying (10.15). So, $f_\beta(\mathbf{x})$ is a trivariate SUM distribution. The univariate marginals are $N(0, 1)$, so the distributions of $\mathbf{a}'\mathbf{X}$ where $\sum_{k=1}^{3} a_k = n \le 3$ are $N(0, n)$, $n = 2,3$, given by the independent trivariate normal model.

The following example is quite interesting in the sense that any subset of size $r < n$ is *s.i.* but not independent.

EXAMPLE 10.5. Let $(X_1, X_2, ..., X_n)$ have *pdf* given by

$$f_\beta(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}} \left\{ \begin{array}{l} 1 + \dfrac{\beta}{(2c)^{\frac{n}{2}+6}}(x_2^2 - x_1^2)\left[12c^2 - 2c\left(x_2^2 + x_1^2\right) + x_1^2 x_2^2\right] \times \\ \left[1 + \displaystyle\sum_{k=3}^{n}\left(\frac{1}{4c^2}\right)^{k-2} \Pi_{i=3}^{k}(2c - x_i^2)\right] e^{-\left(\frac{1}{4c} - \frac{1}{2}\right)\mathbf{x}'\mathbf{x}} \end{array} \right\}, \ \mathbf{x} \in \mathbb{R}^n,$$

where $0 < c < \dfrac{1}{2}$, $\beta = B^{-1}$ and

$$\left| \begin{array}{l} \dfrac{1}{(2c)^{\frac{n}{2}+6}}\left(x_2^2 - x_1^2\right)\left[12c^2 - 2c\left(x_2^2 + x_1^2\right) + x_1^2 x_2^2\right] \times \\ \left[1 + \displaystyle\sum_{k=3}^{n}\left(\frac{1}{4c^2}\right)^{k-2} \Pi_{i=3}^{k}\left(2c - x_i^2\right)\right] e^{-\left(\frac{1}{4c} - \frac{1}{2}\right)\mathbf{x}'\mathbf{x}} \end{array} \right| \le B.$$

The *cf* for f_β is

$$\Psi_\beta(t_1, t_2, ..., t_n) = e^{-\frac{1}{2}\sum_{j=1}^{n} t_j^2} + \beta e^{-c\sum_{j=1}^{n} t_j^2} \times \left(\sum_{k=2}^{n} \Pi_{i=1}^{k} t_i^2\right) (t_1^2 - t_2^2), (t_1, t_2, ..., t_n) \in \mathbb{R}^n.$$

Then X_j' *s* are *s.i.i.d.* $N(0, 1)$. The same is also true for random vector $(X_1, X_2, ..., X_j$ $j = 2,3,...,n-1$. So, $X_1, X_2, ..., X_n$ indeed form a sequence of *s.i.i.d rv's*.

We conclude this section with a characterization of the multivariate SUM distribution.

THEOREM 10.8 Let φ_j, $j = 1, 2, \ldots, n$ be *cf's* and let

$$\Psi_\beta(t_1, t_2, \ldots, t_n) = \Pi_{j=1}^{n} \varphi_j(t_j) + \beta \;\; q(t_1, t_2, \ldots, t_n),$$

where $q(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$ is non-negative definite, continuous at the origin and $q(t, t, \ldots, t) = 0$ for $t \in \mathbb{R}$. Then for some constant β, Ψ_β is *cf* of a SUM distribution if $\left|\Psi_\beta(\mathbf{t})\right| \le 1$ for all $\mathbf{t} \in \mathbb{R}^n$.

PROOF. Ψ_β is non-negative definite, continuous at the origin and $\Psi_\beta(\mathbf{0}) = 1$. Then by Bochner's Theorem Ψ_β is a *cf.*

10.4 EQUIVALENCE OF SUB-INDEPENDENCE AND INDEPENDENCE IN A SPECIAL CASE

The interesting question is that under what conditions sub-independence implies independence. It is possible to have an answer for this question if the underlying joint distribution has a specific form. The following result (Lemma 10.2), due to Ebrahimi et al. (2010), relates the SUM distributions to the well-known notions of POD (Positive Orthant Dependence) and NOD (Negative Orthant Dependence) defined below.

DEFINITION 10.6. A multivariate distribution F is said to be POD (NOD) if

$$\overline{F}(x_1, x_2, \ldots, x_p) \ge (\le) \, \Pi_{i=1}^{p} \, \overline{F}_i(x_i),$$

where $\overline{F}(x_1, x_2, \ldots, x_p) = P(X_1 > x_1, X_2 > x_2, \ldots, X_p > x_p)$ and $\overline{F}_i(x_i) = P(X_i > x_i)$.

It is known that under POD (NOD), if $\rho(X_i, X_j) = 0$ (correlation coefficient), then X_i and X_j are pairwise independent, without implying any higher order dependency among X_i's. For details about POD (NOD) and other notions of dependence see Barlow and Proschan (1981).

LEMMA 10.2. Let $\mathbf{X} = (X_1, X_2, \ldots, X_p)'$ be a nonnegative random vector with a POD (NOD) distribution F. Then F is a SUM distribution if and only if $F(\mathbf{x}) = \Pi_{i=1}^{p} F_i(x_i)$, where F_i is *cdf* of X_i.

10.5 DISSOCIATION AND SUB-INDEPENDENCE

De Paula (2008), presented a bivariate distribution for which

$$E(Y^n \mid X) = E(Y^n) \text{ and } E(X^n \mid Y) = E(X^n), \; n = 1, 2, \ldots \tag{10.16}$$

i.e., X^m and Y^n are uncorrelated for all positive integers m and n, but X and Y are not independent. De Paula's goal was to show a measure of dissociation between two dependent *rv's* X and Y beyond the concept of uncorrelatedness of X and Y. Hamedani and Volkmer (2009*a*, 2009b) showed that the *rv's* considered in De Paula (2008) are not *s.i.*. Then, they presented a bivariate distribution for which (10.16) holds, X and Y are *s.i.*, but not independent. This provides a stronger measure of dissociation between X and Y (see Hamedani and Volkmer (2009*b*).

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