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# On the uniqueness of solution of magnetostatic vector-potential problems by three-dimensional finite-element methods

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## ABSTRACT

In this paper, particular attention is paid to the impact of finite-element approximation on uniqueness and to approximations implicit in finite element formulations from the uniqueness requirements standpoint. It is also shown that the flux density is unique without qualifications. The theoretical and numerical uniqueness of the magnetic vector potential in three-dimensional problems is also given. This analysis is restricted to linear, isotropic media with Dirichlet Boundary conditions. As an interesting consequence of this analysis it is shown that, under usual conditions adopted in obtaining three-dimensional finite-element solutions, it is not necessary to specify  $\text{div } \bar{A}$  in order that  $\bar{A}$  be uniquely defined.

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## INTRODUCTION

Although for a relatively long time the magnetic vector-potential has been used for obtaining numerical solutions of magnetostatic problems, it seems that the question of uniqueness of the vector-potential itself has not been treated in detail, particularly in conjunction with three-dimensional finite-element solutions. (This, of course, is due to the fact that in the end  $\text{curl } \bar{A}$ , and not  $\bar{A}$  itself, is needed.) Interest in questions of validity and uniqueness of numerical 3-D finite element (FE) solutions to magnetostatic problems has been intensified as a result of the pre-sentation and publication of two papers, References [1] and [2], and their accompanying discussions.

This paper addresses the questions of validity and uniqueness of such 3-D FE solutions. For simplicity, we shall restrict consideration to linear, isotropic media with Dirichlet boundary conditions. Both the curl and divergence nature of the magnetic vector potential (mvp) shall be considered.

## STATEMENT OF THE PROBLEM

In this work, one is interested in obtaining a solution to the magnetostatic form of Maxwell's equations given by

$$\nabla \times \bar{H} = \bar{J} \quad (1a)$$

$$\nabla \cdot \bar{B} = 0 \quad (1b)$$

$$v\bar{B} = \bar{H} \quad (1c)$$

where  $v$  is the reluctivity of the medium.

It is well known that  $\bar{B}$  may be expressed as [3]

$$\bar{B} = \nabla \times \bar{A} \quad (2)$$

where  $\bar{A}$  is the magnetic vector potential (mvp), The mvp is a solution to

$$v\nabla \times (\nabla \times \bar{A}) = \bar{J} \quad (3)$$

It is assumed here that  $\bar{A}$  is the exact solution which satisfies the given boundary conditions. The object of the following section is to consider the uniqueness of  $\bar{A}$  obtained in the solution of (3).

## THE UNIQUENESS OF THE CURL ( $\nabla \times \bar{A} = \bar{B}$ )

If  $\bar{A}_1$  and  $\bar{A}_2$  are both solutions to (3) with  $\sigma\bar{A} = \bar{A}_1 - \bar{A}_2$ , then it follows that

$$\nabla \times \nu (\nabla \times \delta\bar{A}) = 0. \quad (4)$$

To define the uniqueness statement, we first consider the integral of  $\nu |\nabla \times \delta\bar{A}|^2$ . One obtains we first consider following through integration by parts:

$$\int_V \nu |\nabla \times \delta\bar{A}|^2 dv = \int_V [\delta\bar{A} \cdot \nabla \times \nu (\nabla \times \delta\bar{A})] dv + \oint_S [\delta\bar{A} \times \nu (\nabla \times \bar{A})] \cdot d\bar{s} \quad (5)$$

where S is the given Dirichlet boundary at which  $\hat{n} \times \bar{A} = 0$ . Based on this boundary condition, the surface integral in (5) vanishes. This integral would also vanish for the Neumann condition. Hence, upon substituting (4) into (5) one obtains

$$\int_V \nu |\nabla \times \delta\bar{A}|^2 dv = 0 \quad (6)$$

which, for positive  $\nu$ , requires that

$$\nabla \times \delta\bar{A} = 0 \quad (7)$$

and by the Helmholtz theorem [4]

$$\delta\bar{A} \approx \nabla\phi \quad (8)$$

Thus, we see that specifying tangential  $\bar{A}$  on the boundary surface requires  $\bar{A}$  to be unique to within the gradient of a potential.

The same results may be obtained for the finite element solution to (3). We begin with the following energy functional which was previously used in references [1] and [2]:

$$\rho(\bar{U}) = \int_V [(\bar{U} - \bar{A}) \cdot \nabla \times (\nu \times (\bar{U} - \bar{A}))] dv \quad (9)$$

where  $\bar{U}$  is the approximate solution of (3).

Expanding  $\bar{U}$  as

$$\bar{U} = \sum_{i=1}^N \varepsilon_i \bar{U}_i$$

with N approximation functions  $\bar{U}_i$  we may write the first variation of  $\rho$  as follows:

$$\frac{\partial \rho}{\partial \varepsilon_i} = - \int_V [\bar{U}_i \cdot \nabla \times (\nu \nabla \times (\bar{U} - \bar{A})) + (\bar{U} - \bar{A}) \cdot \nabla \times (\nu \nabla \times \bar{U}_i)] dv = 0 \quad (11)$$

Integrating (11) by parts yields

$$\int_V [2\bar{U}_i \cdot (\nabla \times (\nu \nabla \times \bar{U}) - \bar{J})] dv - \oint_S [(\bar{U} - \bar{A}) \times (\nu \nabla - \bar{U}_i) - \bar{U}_i \times (\nu \nabla \times (\bar{U} - \bar{A}))] \cdot d\bar{s} = 0 \quad (12)$$

which gives the Euler equation

$$\nabla \times (\nu \nabla \times \bar{U}) = \bar{J} \quad (13)$$

For computational purposes, it is convenient to rewrite (12) with Dirichlet conditions as

$$2 \int_V [\nu \nabla \times \bar{U}_i \cdot \nabla \times \bar{U} - \bar{U}_i \cdot \bar{J}] dv = 0 \quad (14)$$

for  $i = 1, 2, \dots, N$ . For linear approximation functions, the curls in (14) are piecewise constants requiring only simple integrals. If  $\bar{A}_1$  and  $\bar{A}_2$  are two solutions to  $\bar{U}$  in the numerical problem, then based on (10) and (14) it is easily shown that

$$\int_V [\nu |\nabla \times (\bar{A}_1 - \bar{A}_2)|^2] dv = 0 \quad (15)$$

and thus  $\bar{A}$  must also be unique to within  $\nabla\phi$  for the numerical problem. However, since the curl of  $\nabla\phi$  is zero, the solution for  $\nabla \times \bar{A}$ , that is the flux density  $\bar{B}$ , is unique with no qualifications.

## THE DIVERGENCE OF THE VECTOR POTENTIAL

In obtaining  $\bar{B}$  as the curl of  $\bar{A}$  as in the previous section, the divergence of  $\bar{A}$  is not required and the solution for  $\bar{B}$  has been shown above to be unique. However, in the numerical aspect of solving (14) this non-uniqueness of  $\bar{A}$  by  $\nabla\phi$  will create a singular numerical process for obtaining  $\bar{A}$  which must be solved by constraining the problem in some way. One method would be to add  $(\nabla \cdot \bar{U} - \nabla \cdot \bar{A})^2$  to the functional integrand and specify the gauge  $\nabla \cdot \bar{A}$  as suggested by Van Bladel [5] to satisfy the sufficiency requirements of the Helmholtz equation. A second method would be to solve the resultant matrix equation for  $\bar{A}$  by pseudoinverse methods [6]. A third method which has been used successfully in references [1] and [2] is to specify  $\hat{n} \cdot \bar{A}$  (or  $(\hat{n} \cdot \nabla)(\hat{n} \cdot \bar{A})$ ) at the boundary surface. It has been found that this last technique gives good results for piecewise linear, continuous approximation functions. It is the purpose of this section to show that this uniqueness may be predicted a priori and thus a specific gauge need not be imposed.

A simplistic view of this uniqueness may be obtained in a numerical form by counting the number of unknowns and equations (independence assumed). For the finite-element formulation we obtain three equations and three unknowns at each interior node. To complete the problem, we require boundary conditions on all three components of the mvp at boundary nodes. This is consistent with the additional constraint suggested above.

Let us consider  $\bar{A}_1$  and  $\bar{A}_2$  to be solutions of (12) with Dirichlet boundary conditions on  $\bar{A}$  (not just  $\hat{n} \times \bar{A}$ ). We wish to show that the difference,  $\nabla\phi$ , between  $\bar{A}_1$  and  $\bar{A}_2$  must vanish in  $V$  for piecewise linear, continuous bases (using tetrahedral elements).

Let us consider an element at the boundary surface. Since  $\nabla\phi$  is linear and must vanish at the boundary, one can write

$$\nabla\phi = C_1 (\hat{n}_1 \cdot (r - r_s)) \quad (16)$$

Where  $\hat{n}_1$  is the boundary surface normal and  $r_s$  is on the surface. Taking the curls of  $\nabla\phi$  we obtain

$$\nabla \times (\nabla \phi) = \hat{n}_1 \times \bar{C}_1 \quad (17)$$

which must be identically zero. Thus  $\bar{C}_1$  must be parallel to  $\hat{n}_1$  and we may write (16) as

$$\nabla \phi = c_1 [\hat{n}_1 \cdot (\bar{r} - \bar{r}_s)] \hat{n}_2 \quad (19)$$

If these elements fall along an edge such that

$$\hat{n}_1 \neq \hat{n}_2 \quad (20)$$

then  $c$  and  $c$  must be zero since  $\nabla \phi$  must be continuous at the 1 adjoining 2 faces. Hence, these two tetrahedrons may be deleted from  $\nabla$  to determine  $\nabla \phi$ . This result may be generalized to elements with adjacent edges, but non-adjacent faces. Since for a closed, finite volume there must always be two such tetrahedrons, all tetrahedrons may be deleted to obtain  $\nabla \phi = 0$ . Thus the  $\nabla \cdot \bar{A}$  in the solution is unique (though we have not determined the value). This result may be extended to the problem with Neumann boundary conditions.

## NUMERICAL COMPARISONS

The air-cored coil described in reference (2) has been solved using the discretization grids given in that reference. The solution is obtained here with the boundary conditions  $\bar{A} = 0$  on the outermost surface (same as in reference (2)), and also with the condition on the normal component of  $A$  replaced by setting the normal derivative to zero (Neumann condition). Table (1) shows these results for arbitrarily chosen tetrahedral elements in the given volume. The results for the curl and the divergence of the vector potential are also shown for both of the above cases in that table. As can be easily noticed, the values (single precision on an IBM 370) of the curl of the vector potential  $\bar{A}$  are the same in both cases, hence the curl is unique. However, there is no fixed pattern for the divergence of  $\bar{A}$ , and the value of the divergence of  $\bar{A}$  is inconsequential to the flux densities as expected.

**Table (1)** Values of Curl (lines /sq. inch) and Divergence of the Vector Potential at Arbitrary Points

Pt.	Dirichlet Boundaries				Neumann Boundaries			
	$B_x$	$B_y$	$B_z$	$\nabla \cdot \bar{A}$	$B_x$	$B_y$	$B_z$	$\nabla \cdot \bar{A}$
1	64.4	0.0	2055.	2.7	64.3	0.0	2058.	6.5
2	40.4	57.8	2159.	-7.7	40.5	57.5	2161.	-0.7
3	22.0	-2.1	-67.	-2.8	22.5	-2.6	-64.	-15.1
4	69.3	-6.9	-190.	1.3	68.9	-7.5	-190.	-2.2
5	2.3	17.8	-50	-13.4	2.5	17.7	-50.	1.0
6	220.0	22.5	-274.	-22.7	220.2	22.4	-271.	-2.3
7	48.4	19.7	-45.	-0.9	48.9	18.3	-44.	-7.0
8	87.7	548.5	-166.	-4.4	87.6	547.9	-165.	-22.5
9	-102.3	26.1	310.	85.7	-102.4	25.8	310.	90.8
10	89.7	95.3	-113.	2.7	89.6	95.1	-114.	9.1

## CONCLUSION

A theoretical proof of uniqueness for a three dimensional finite element magnetostatic field solution has been given for a class of 3-D problems. It has been shown that the flux density solution is unique with no qualifications. The theoretical and numerical uniqueness of the magnetic vector potential for linear finite-

elements have also been derived, where the derivation has been restricted to linear, isotropic media with Dirichlet boundary conditions. The consequence of this analysis is that under usual conditions assumed in obtaining a three-dimensional finite-element solution, it is not necessary to explicitly, specify the gauge  $\nabla \cdot \bar{A}$  in order to uniquely define the vector potential  $\bar{A}$ .

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