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## Free Split Bands

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# Free Split Bands

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**Abstract:** We solve the word problem for the free objects in the variety consisting of bands with a semilattice transversal. It follows that every free band can be embedded into a band with a semilattice transversal.

**Keywords:** Free band, Split band, Semilattice transversal

## 1 Introduction

We refer to<sup>3</sup> and<sup>6</sup> for a general background and as references to terminology used in this paper.

Recall that a *band* is a semigroup where every element is an idempotent. The *Green relation*  $\mathcal{D}$  is the least semilattice congruence on a band, and so every band is a semilattice of its  $\mathcal{D}$ -classes; the  $\mathcal{D}$ -classes themselves form rectangular bands.<sup>5</sup> We shall be interested in bands  $S$  for which the least semilattice congruence  $\mathcal{D}$  *splits*, that is, there exists a subsemilattice  $S^\circ$  of  $S$  which intersects each  $\mathcal{D}$ -class in exactly one element. Such a subsemilattice  $S^\circ$  of  $S$  will then be called a *semilattice transversal* of  $S$ .

If the band  $S$  has a semilattice transversal  $S^\circ$ , then we can associate to every  $a \in S$  the unique  $a^\circ \in S^\circ$  which is  $\mathcal{D}$ -related to  $a$ . The unary operation  $S \mapsto S, a \mapsto a^\circ$  is an idempotent  $\mathcal{D}$ -class preserving endomorphism of  $S$  which induces the  $\mathcal{D}$ -relation on  $S$ , and in particular,  $S^\circ$  is a retract of  $S$ . The unary band  $((S, \cdot, \circ))$  thus obtained obviously satisfies, apart from the associative law, the identities

$$\begin{aligned} x &\approx x^2, & xx^\circ x &\approx x, & x^\circ xx^\circ &\approx x^\circ, \\ (xy)^\circ &\approx x^\circ y^\circ \approx y^\circ x^\circ, & (x^\circ)^\circ &\approx x^\circ. \end{aligned}$$

(1)

One readily verifies that the last identity  $(x^\circ)^\circ \approx x^\circ$  follows in fact from the previous ones. It is not difficult to see that if a unary band  $((S, \cdot, \circ))$  satisfies the identities (1), then  $S^\circ = \{a^\circ \mid a \in S\}$  is a semilattice transversal of the band  $(S, \cdot)$ . For this reason we call the unary bands which satisfy the identities (1) *split bands*.

The variety of bands will be denoted by  $\mathbf{B}$  and the variety of all split bands will be denoted  $\mathbf{B}^\circ$ . For a nonempty set  $X$ ,  $F_{\mathbf{B}}(X)$  and  $F_{\mathbf{B}^\circ}(X)$  denote the free objects on  $X$  in  $\mathbf{B}$  and  $\mathbf{B}^\circ$ , respectively. As the abstract indicates, the purpose of this paper is to solve the word problem for  $F_{\mathbf{B}^\circ}(X)$  and to show that  $F_{\mathbf{B}}(X)$  can be isomorphically embedded into the multiplicative reduct of  $F_{\mathbf{B}^\circ}(X)$ . The solution of the word problem for  $F_{\mathbf{B}}(X)$  is well understood and the reader will find all the necessary details in Sect 4.5 of [3] where appropriate references to [1] and [2] are given. Our solution to the word problem for  $F_{\mathbf{B}^\circ}(X)$  is similar though slightly more complicated. While  $\mathbf{B}^\circ$  is, like  $\mathbf{B}$ , locally finite, for any finite nonempty set  $X$  the (finite) free object  $F_{\mathbf{B}^\circ}(X)$  is much larger than  $F_{\mathbf{B}}(X)$ .

There is something more enigmatic about all this. It turns out that if a band  $S$  has a semilattice transversal  $S^\circ$  then the union of all the  $\mathcal{L}$ -classes [ $\mathcal{R}$ -classes] in  $S$  of the elements of  $S^\circ$  is a left [right] regular band which is a transversal of the  $\mathcal{R}$ -classes [ $\mathcal{L}$ -classes] of  $S$ . This follows from a very special application of Proposition 2.3 and Corollary 2.4 of [8] and below we intend to give a short independent proof of this fact in the special circumstances we consider here.

## Result 1.1

Let  $(S, \cdot, \circ)$  be a split band with semilattice transversal  $S^\circ = \{a^\circ \mid a \in S\}$ . Then  $I_S = \{aa^\circ \mid a \in S\}$  is a left [right] regular subband of  $(S, \cdot)$  with semilattice transversal  $S^\circ$ , and  $\Lambda_S$  is a transversal of the  $\mathcal{R}$ -classes [ $\mathcal{L}$ -classes] of  $(S, \cdot)$ .

### *Proof*

Clearly, for any  $a \in S$ , the intersection of  $I_S$  and the  $\mathcal{D}$ -class  $D_a$  of  $a$  is the  $\mathcal{L}$ -class  $L_{a^\circ}$  of  $a^\circ$  and  $I_S$  intersects the  $\mathcal{R}$ -class  $R_a$  of  $a$  in the unique element  $aa^\circ$ . It suffices to prove that  $I_S$  is a subsemigroup of  $S$ , or in particular, that for any  $a, b \in S$  we have that  $(aa^\circ)(bb^\circ)\mathcal{L}(ab)^\circ$ . In any case  $a^\circ b^\circ = b^\circ a^\circ = (ab)^\circ$  is  $\mathcal{D}$ -related to  $(aa^\circ)(bb^\circ)$  in  $S$  and since this  $\mathcal{D}$ -class is a rectangular band it follows that  $(b^\circ a^\circ)(aa^\circ bb^\circ)\mathcal{L}(aa^\circ)(bb^\circ)$ . Applying the identities (1) we find that

$$\begin{aligned} (b^\circ a^\circ)(aa^\circ bb^\circ) &= b^\circ(a^\circ aa^\circ)bb^\circ \\ &= b^\circ a^\circ bb^\circ \\ &= a^\circ b^\circ bb^\circ \\ &= a^\circ b^\circ = (ab)^\circ. \end{aligned}$$

Thus  $(aa^\circ)(bb^\circ)\mathcal{L}(ab)^\circ$  as required.  $\square$

The  $I_S$  and  $\Lambda_S$  mentioned in Result 1.1 may well serve as a means to coordinatize  $S$  and one would expect that such a coordinatization would set the stage for a structure theorem of split bands in terms of the left and right regular bands  $I_S$  and  $\Lambda_S$  akin to, but simpler than the construction in II.1 of [6]. For a free split band  $F = F_{\mathbf{B}^\circ}(X)$ , it is easy to characterize the elements of the left [right] regular split band  $\Lambda_F$  (see Corollary 2.2 and Theorem 2.5). By left-right duality,  $I_F$  is anti-isomorphic to  $\Lambda_F$ .

The variety  $\mathbf{RRB}^\circ \mathbf{LRB}^\circ$  of right [left] regular split bands is the subvariety of  $\mathbf{B}^\circ$  determined by the additional identity  $xyx \approx yx \ xyx \approx xy$ . Thus, with the notation of Result 1.1,  $I_S$  and  $\Lambda_S$  belong to  $\mathbf{LRB}^\circ$  and  $\mathbf{RRB}^\circ$ , respectively. As we shall see, if  $F = F_{\mathbf{B}^\circ}(X)$  is a free split band, then  $I_S$  and  $\Lambda_S$  should not be assumed to be free on  $X$  in  $\mathbf{LRB}^\circ$  and  $\mathbf{RRB}^\circ$ , respectively.

It is time to put our paper in the context of current research. The adequate terminology *split band* is not of our invention but already occurs in [4] where the authors give

a structure theorem for orthodox semigroups for which the least inverse semigroup congruence splits. Theorem 2 of<sup>9</sup> gives a structure theorem for the members of  $\mathbf{RRB}^\circ$  in the manner of Theorem II.1.6 of.<sup>6</sup> Combining this result of Yoshida with its dual and with Theorem 2 of,<sup>7</sup> one obtains a structure theorem for the members of  $\mathbf{B}^\circ$ . We would, however, like to draw the reader's attention to the all encompassing paper<sup>8</sup> which has already been mentioned above, and which in its Example 2.15 introduces a variety of unary semigroups (whose members are all regular semigroups) which contains  $\mathbf{B}^\circ$  as a subvariety.

## 2 Free split bands

In this section we give a solution of the word problem for the free object  $F_{\mathbf{B}^\circ}(X)$  in the variety  $\mathbf{B}^\circ$  on a nonempty set  $X$  of *variables*.

We let  $X^\circ$  be a set disjoint of  $X$  and  $X \mapsto X^\circ, x \mapsto x^\circ$ , a bijection. The elements of  $X \cup X^\circ$  will be called *letters*. The identity of the free monoid  $(X \cup X^\circ)^*$  is the empty word  $1$ , thus  $(X \cup X^\circ)^* = (X \cup X^\circ)^+ \cup \{1\}$ . For any  $w \in (X \cup X^\circ)^*$  we define the *content*  $c(w)$  of  $w$  inductively by

$$\begin{aligned} c(1) &= \emptyset, \\ c(x) &= c(x^\circ) = \{x\}, \quad x \in X, \\ c(y_1 \cdots y_{n-1}y_n) &= c(y_1 \cdots y_{n-1}) \cup c(y_n), \\ &n > 1, y_1, \dots, y_n \in X \cup X^\circ. \end{aligned}$$

We let  $\beta$  be the congruence relation of the free semigroup  $(X \cup X^\circ)^+$  generated by the pairs

$$\begin{aligned} (w, w^2), \quad w &\in (X \cup X^\circ)^+, \\ (x, xx^\circ x), (x^\circ, x^\circ xx^\circ), \quad x &\in X, \\ (x^\circ y^\circ, y^\circ x^\circ), \quad x, y &\in X. \end{aligned}$$

(2)

One readily verifies that  $(X \cup X^\circ)/\beta$  is a band generated by the elements of the form  $x\beta$  or  $x^\circ\beta$ , and Green's  $\mathcal{D}$ -relation on this band is given by

$$v\beta \mathcal{D} w\beta \Leftrightarrow c(v) = c(w).$$

(3)

Also, the element of the form  $(x_1^\circ \dots x_n^\circ)\beta$ ,  $n \geq 1$ ,  $x_1, \dots, x_n \in X$ , constitute a subsemilattice which intersects every  $\mathcal{D}$ -class exactly once. Given  $w \in (X \cup X^\circ)^+$  with content  $c(w) = \{x_1, \dots, x_n\}$  we let  $(w\beta)^\circ$  be the unique element  $(x_1^\circ \dots x_n^\circ)\beta$  of this semilattice which is  $\mathcal{D}$ -related to  $w\beta$  in  $(X \cup X^\circ)^+/\beta$ : the mapping  $w\beta \rightarrow (w\beta)^\circ$  yields an idempotent endomorphism which induces the  $\mathcal{D}$ -relation. The unary band thus obtained will be denoted by  $F$ .

For the sake of simplicity we drop the notation  $\beta$  and we use “=” to denote equality in  $F$ . Thus for  $v, w \in (X \cup X^\circ)^+$  we write  $v = w$  (in  $F$ ) instead of  $v\beta w$ . We shall reserve “ $\equiv$ ” for the equality in  $(X \cup X^\circ)^*$ . We shall denote the semilattice transversal of  $F$  consisting of the elements  $x_1^\circ \dots x_n^\circ$ ,  $n \geq 1$ ,  $x_1, \dots, x_n \in X$ , by  $F^\circ$ . Clearly  $F$  is a model of  $F_{\mathbf{B}^\circ}(X)$ : if  $\iota: X \rightarrow F$ , then for every  $S \in \mathbf{B}^\circ$  and every mapping  $\varphi: X \rightarrow S$ , there exists a (unique) homomorphism of unary bands  $\bar{\varphi}: F \rightarrow S$  such that  $\varphi = \iota\bar{\varphi}$ . If  $X$  is finite then so is  $X \cup X^\circ$ , and therefore  $F$  is also finite, since finitely generated bands are finite. In other words, the variety  $\mathbf{B}^\circ$  is *locally finite*. From this it follows that there exists an algorithm which decides whether for given  $v, w \in (X \cup X^\circ)^+$  we have that  $v = w$  in  $F$ . We shall give an algorithm which is transparent enough to be useful. The algorithm which we set out to describe is similar to the algorithm given in<sup>2</sup> for free bands.

We shall need some invariants. In our context, a property of a word which belongs to  $(X \cup X^\circ)^+$  is called an *invariant* if whenever  $v = w$  in  $F$  and  $v$  satisfies this property, then so does  $w$ . To “have the same content” is such an invariant: recall that for  $v, w \in (X \cup X^\circ)^+$ ,  $c(v) = c(w)$  if and only if  $v\mathcal{D}w$  in  $F$ , or if and only if  $v\mathcal{D}w$  in  $F$ .

For any  $w \in (X \cup X^\circ)^*$  we define  $w_L \in X^*$  to be the word obtained from  $w$  by deleting first every occurrence of an  $x \in X$  in  $w$  when preceded somewhere in  $w$  by  $x$  or  $x^\circ$ , and then deleting every occurrence of any  $y^\circ \in X^\circ$ . Thus for instance, for  $x, y, z \in X$ ,

$$\begin{aligned} (x^\circ y^\circ x x^\circ z^\circ y z)_L &\equiv 1, \\ (x^\circ y x x^\circ y^\circ z y z^\circ)_L &\equiv y z. \end{aligned}$$

The letters of such words  $w_L$  (if any) belong to  $X$  and are necessarily distinct. If  $w_L \neq 1$ , then the last letter of  $w_L$  will be denoted by  $\bar{l}(w)$ . We can then uniquely write  $w \equiv l(w)\bar{l}(w)u$  for some prefix  $l(w)$  of  $w$  where  $\bar{l}(w) \notin c(l(w))$  and some suffix  $u$ . Here  $l(w)$  or  $u$  may well be empty. Thus, for instance,

$$\begin{aligned}\bar{l}(x^\circ y x x^\circ y^\circ z y z^\circ) &= z \\ l(x^\circ y x x^\circ y^\circ z y z^\circ) &= x^\circ y x x^\circ y^\circ.\end{aligned}$$

If  $w_L = 1$ , it will be convenient to put  $l(w) \equiv \bar{l}(w) \equiv 1$ .

In a left-right dual way we define, for  $w \in (X \cup X^\circ)^*$ , the word  $w_R \in X^*$  and, whenever  $w_R \neq 1$ , the variable  $\bar{r}(w)$  and the suffix  $r(w)$  of  $w$ . Then if  $w_R \neq 1$ ,  $w \equiv v\bar{r}(w)r(w)$  in  $(X \cup X^\circ)^*$  for some prefix  $v$ . If  $w_R \equiv 1$  we put  $r(w) \equiv \bar{r}(w) \equiv 1$ .

### Lemma 2.1

For  $w \in (X \cup X^\circ)^+$  with  $c(w) = \{x_1, \dots, x_n\}$ ,

$$\begin{aligned}w \quad \mathcal{R}w x_1^\circ \cdots x_n^\circ &= x_1^\circ \cdots x_n^\circ \text{ in } F \text{ if } w_L \equiv 1, \text{ and} \\ w \quad \mathcal{R}w x_1^\circ \cdots x_n^\circ &= l(w)\bar{l}(w)x_1^\circ \cdots x_n^\circ \text{ in } F \text{ otherwise.}\end{aligned}$$

### *Proof*

If  $w \in (X \cup X^\circ)^+$  with  $c(w) = \{x_1, \dots, x_n\}$ , then  $w \in X^+$  or  $w \equiv w_0 y_1^\circ w_1 \dots y_k^\circ w_k$  where  $w_0, \dots, w_k \in X^*$  and  $y_1, \dots, y_k \in c(w)$  for some  $k \geq 1$ .

First we assume that  $w_L \equiv 1$ . Then  $w$  can be written as  $w = y_1^\circ w_1 \cdots y_k^\circ w_k$ ,  $k \geq 1$ , where  $c(w) = \{y_1, \dots, y_k\}$  such that, for every  $1 \leq i \leq k$ , we have that  $c(w_i) \subseteq \{y_1, \dots, y_i\}$ . We prove by induction that for all  $1 \leq i \leq k$  we have that  $w = y_1^\circ \cdots y_i^\circ w$  in  $F$ . This is obviously true for  $i = 1$ . Suppose that for  $i < k$  we have that

$$w = y_1^\circ \cdots y_i^\circ w = y_1^\circ \cdots y_i^\circ y_1^\circ w_1 y_2^\circ w_2 \cdots y_i^\circ w_i y_{i+1}^\circ \cdots y_k^\circ w_k.$$

Since

$$c(y_1^\circ \cdots y_i^\circ y_1^\circ w_1 \cdots y_i^\circ w_i) = y_1, \dots, y_i = c(y_1^\circ \cdots y_i^\circ),$$

it follows that

$$y_1^\circ \cdots y_i^\circ \mathcal{R}y_1^\circ \cdots y_i^\circ y_1^\circ w_1 \cdots y_i^\circ w_i$$

in  $F$ . Therefore

$$\begin{aligned} y_1^\circ \cdots y_i^\circ y_1^\circ w_1 \cdots y_i^\circ w_i y_{i+1}^\circ &= y_{i+1}^\circ y_1^\circ \cdots y_i^\circ y_1^\circ w_1 \cdots y_i^\circ w_i y_{i+1}^\circ \\ &= y_1^\circ \cdots y_i^\circ y_{i+1}^\circ y_1^\circ w_1 \cdots y_i^\circ w_i y_{i+1}^\circ, \end{aligned}$$

since the elements of  $F$  which are  $\mathcal{R}$ -related to an element of the semilattice transversal  $F^\circ$  of  $F$  form a right regular band by Result 1.1. It follows that  $w = y_1^\circ \cdots y_i^\circ y_{i+1}^\circ w$ . Using induction and the fact that  $x_1^\circ \cdots x_n^\circ = y_1^\circ \cdots y_k^\circ$  we find that  $w = x_1^\circ \cdots x_n^\circ w$ . Since  $w \mathcal{D} x_1^\circ \cdots x_n^\circ$  in  $F$ , we thus have that  $w \mathcal{R} w x_1^\circ \cdots x_n^\circ = x_1^\circ \cdots x_n^\circ w x_1^\circ \cdots x_n^\circ = x_1^\circ \cdots x_n^\circ$ .

We next consider the case where  $w_L \neq 1$ . Then there exists  $\bar{l}(w) \in X$  such that  $w \equiv l(w)\bar{l}(w)u$  for some  $l(w), u \in (X \cup X^\circ)^*$ . If  $c(w) = \{x_1, \dots, x_n\}$ , we may as well assume that  $c(l(w)\bar{l}(w)) = \{x_1, \dots, x_j\}$  for some  $j \leq n$ . Then

$$\begin{aligned} w &= l(w)\bar{l}(w)(l(w)\bar{l}(w))^\circ l(w)\bar{l}(w)u \text{ (since } l(w)\bar{l}(w) \mathcal{D} (l(w)\bar{l}(w))^\circ \text{ in } F) \\ &= l(w)\bar{l}(w)x_1^\circ \cdots x_j^\circ l(w)\bar{l}(w)u \\ &= l(w)\bar{l}(w)v \end{aligned}$$

where  $v \equiv x_1^\circ \cdots x_j^\circ l(w)\bar{l}(w)u$  is such that  $v_L = 1$  and  $c(v) = \{x_1, \dots, x_n\} = c(w)$ . By the first part of the proof,  $v = x_1^\circ \cdots x_n^\circ v$  and thus  $w = l(w)\bar{l}(w)x_1^\circ \cdots x_n^\circ v$ . Since  $w$  and  $l(w)\bar{l}(w)x_1^\circ \cdots x_n^\circ$  have the same content, they must then be  $\mathcal{R}$ -related in  $F$ . Further,  $x_1^\circ \cdots x_n^\circ v x_1^\circ \cdots x_n^\circ = x_1^\circ \cdots x_n^\circ$  and so  $w x_1^\circ \cdots x_n^\circ = l(w)\bar{l}(w)x_1^\circ \cdots x_n^\circ$ .  $\square$

From Lemma 2.1 and its dual we have the following.

### Corollary 2.2

Let  $w \in (X \cup X^\circ)^+$ ,  $c(w) = \{x_1, \dots, x_n\}$ .

If  $w_L \neq 1 \neq w_R$ , then

$$w \mathcal{R} l(w)\bar{l}(w)x_1^\circ \cdots x_n^\circ \mathcal{L} x_1^\circ \cdots x_n^\circ \mathcal{R} x_1^\circ \cdots x_n^\circ \bar{r}(w)r(w) \mathcal{L} w$$

and  $w = l(w)\bar{l}(w)x_1^\circ \cdots x_n^\circ \bar{r}(w)r(w)$ .

If  $w_L \neq 1 \equiv w_R$ , then

$$w = l(w)\bar{l}(w)x_1^\circ \cdots x_n^\circ \mathcal{L} x_1^\circ \cdots x_n^\circ.$$

If  $w_L \equiv 1 \not\equiv w_R$ , then

$$w = x_1^\circ \cdots x_n^\circ \bar{r}(w) r(w) \mathcal{R} x_1^\circ \cdots x_n^\circ.$$

If  $w_L \equiv 1 \equiv w_R$ , then  $w = x_1^\circ \cdots x_n^\circ$ .

That the  $l, t, r$  and  $\bar{r}$  yield invariants follows from the following sequence of results.

### Lemma 2.3

Let  $v, w \in (X \cup X^\circ)^+$  be such that  $v = w$  in  $F$ . Then  $v_L \not\equiv 1$  if and only if  $w_L \not\equiv 1$ . If this is the case, then  $\bar{l}(v) \equiv \bar{l}(w)$  and either  $l(v) \equiv 1 \equiv l(w)$  or  $l(v) = l(w)$  in  $F$ .

#### *Proof*

If  $v = w$  in  $F$  then  $v$  can be transformed into  $w$  by using a finite sequence of transformations of the form  $psq \rightarrow ptq$  where  $p, q \in (X \cup X^\circ)^*$  and where  $(s, t)$  or  $(t, s)$  is one of the pairs listed in (2). It therefore suffices to prove the statement of the theorem for the special case where  $v \equiv psq$  and  $w \equiv ptq$  where  $(s, t)$  is any of the pairs listed in (2). The proof is not much harder than the corresponding verification for free bands as in the proof of Lemma 4.5.1 of 3  $\square$

Using an inductive argument we thus have the following.

### Corollary 2.4

If  $v, w \in (X \cup X^\circ)^+$  are such that  $v = w$  in  $F$ , then  $v_L \equiv w_L$ .

It will be convenient to denote by  $F^1$  the unary band  $F$  with an identity 1 adjoined. The unary operation of  $F$  can be extended to  $F^1$  by putting  $1^\circ = 1$ . From Corollaries 2.2 and 2.4, Lemma 2.3 and their duals, we conclude the following.

### Theorem 2.5

For  $v, w \in (X \cup X^\circ)^+$  we have: if  $v = w$  in  $F$ , then

$$c(v) = c(w), \quad v_L \equiv w_L, \quad v_R \equiv w_R,$$

and

$$\begin{aligned} l(v) &= l(w) \text{ in } F^1 \text{ if } v_L \equiv w_L \not\equiv 1, \\ r(v) &= r(w) \text{ in } F^1 \text{ if } v_R \equiv w_R \not\equiv 1. \end{aligned}$$

Conversely, if  $c(v) = c(w)$ , then  $v = w$  in  $F$  if either one of the following occur:

1. (i)  $v_L \equiv w_L \not\equiv 1, v_R \equiv w_R \not\equiv 1, l(v) = l(w)$  in  $F^1$  and  $r(v) = r(w)$  in  $F^1$ ,
- (ii)  $v_L \equiv w_L \equiv 1, v_R \equiv w_R \not\equiv 1$ , and  $r(v) = r(w)$  in  $F^1$ ,
- (iii)  $v_L \equiv w_L \not\equiv 1, v_R \equiv w_R \equiv 1$ , and  $l(v) = l(w)$  in  $F^1$ ,
- (iv)  $v_L \equiv w_L \equiv 1 \equiv v_R \equiv w_R$ .

□

We remark here that the task for testing whether  $v = w$  in  $F$  is, by Theorem 2.5, reduced to a similar task for words of smaller content. Proceeding inductively, Theorem 2.5 thus allows us to verify whether  $v = w$  in  $F$  in a finite number of steps. We shall be more explicit in the following.

Let  $\{l, r\}^*$  be the free monoid on the set containing the symbols  $l$  and  $r$ , and again let  $1$  stand for the identity element of  $\{l, r\}^*$ . For  $k \in \{l, r\}^*$  and  $w \in (X \cup X^\circ)^*$ , we define  $k(w)$  inductively by:

$$1(w) = w$$

and for  $k = k_1 \cdots k_n$  with  $k_1, \dots, k_n \in \{l, r\}, n \geq 1$ ,

$$k(w) = k_1(k_2 \dots k_n(w)).$$

The following follows from Theorem 2.5 using induction. For  $k \in \{l, r\}^*$ , we let  $|k|$  be the length of  $k$ , and, for a set  $A$ , we let  $|A|$  be the cardinality of  $A$ .

## Theorem 2.6

For  $v, w \in (X \cup X^\circ)^+$  we have that  $v = w$  in  $F$  if and only if  $c(v) = c(w)$  and for any  $k \in \{l, r\}^*$  with  $|k| \leq |c(v)| = |c(w)|$ ,

$$\begin{aligned} c(k(v)) &= c(k(w)), \\ k(v)_L &\equiv k(w)_L \text{ and } k(v)_R \equiv k(w)_R. \end{aligned}$$

□

From Theorem 2.6 and the solution of the word problem for the free band  $F_{\mathbf{B}}(X)$  on the set  $X$  (see [2] or Section 4.5 in [3]) it follows that for  $v, w \in X^+$ ,  $v$  and  $w$  represent the same element of the free band  $F_{\mathbf{B}}(X)$  if and only if  $v = w$  in  $F$ . We can state this as in the following corollary.

### Corollary 2.7

Let  $X$  be a nonempty set. Then the mapping  $X \mapsto X \cup X^\circ, x \mapsto x$ , can be extended to an embedding of the free band  $F_{\mathbf{B}}(X)$  as a subband of  $F = F_{\mathbf{B}^\circ}(X)$ .

In other words, any free band can be isomorphically embedded into a band which has a semilattice transversal. Therefore, the free objects in the quasivariety consisting of the bands which can be embedded into some band which has a semilattice transversal, are the familiar free bands.

### Example 1

The  $\mathcal{D}$ -classes of  $F = F_{\mathbf{B}^\circ}(X)$  are much larger than the corresponding  $\mathcal{D}$ -classes of the free band  $F_{\mathbf{B}}(X)$ . Thus, if  $x$  and  $y$  are distinct elements of  $X$ , then the  $\mathcal{L}$ -class of  $x^\circ y^\circ$  in  $F$  consists of the 11 elements  $x^\circ y^\circ, xx^\circ y^\circ, y^\circ xx^\circ y^\circ, y^\circ yxx^\circ y^\circ, yx^\circ y^\circ, x^\circ yx^\circ y^\circ, x^\circ xyx^\circ y^\circ, xyx^\circ y^\circ, xx^\circ yx^\circ y^\circ, yxx^\circ y^\circ$ , and  $yy^\circ xx^\circ y^\circ$ . Therefore the  $\mathcal{D}$ -class of  $xy$  in  $F$  contains exactly 121 elements. The  $\mathcal{D}$ -class  $xy$  in the free band  $F_{\mathbf{B}}(X)$  contains only the 4 elements  $xy, xyx, yx$ , and  $xyx$ .

If  $X = \{x, y\}$  then the 11 elements listed above, together with  $x^\circ, xx^\circ, y^\circ$ , and  $yy^\circ$  form the left regular band which is a transversal of the  $\mathcal{R}$ -classes of  $F_{\mathbf{B}^\circ}(X)$ . The 6-element free band  $F_{\mathbf{B}}(X)$  is a normal band, whereas the 129-element free split band  $F_{\mathbf{B}^\circ}(X)$  is not:  $x^\circ F_{\mathbf{B}^\circ}(X)x^\circ$  is not a semilattice, but a 9-element (square) rectangular band with an identity adjoined. If  $\rho$  is the smallest fully invariant congruence on  $F_{\mathbf{B}^\circ}(X)$  which identifies these 9 elements, then  $\rho$  separates the elements of  $F_{\mathbf{B}}(X)$ : the free normal band  $F_{\mathbf{B}}(X)$  is embeddable into the split band  $F_{\mathbf{B}^\circ}(X)/\rho$ , which is a normal band.

### 3 Normal forms

We continue to use the notation introduced in the first few paragraphs of the previous section. Thus  $\beta$  is the congruence relation on  $(X \cup X^\circ)^+$  generated by the pairs in (2), and as we have seen  $(X \cup X^\circ)^+ / \beta$  yields the free split band  $F_{\mathbf{B}^\circ}(X)$  on the set  $X$ . We would like to find a subset of  $(X \cup X^\circ)^+$  which is a cross section of the  $\beta$ -classes. The elements of this subset are then called *of normal form*, and every  $w \in (X \cup X^\circ)^+$  is  $\beta$  related to a unique element of normal form, called the *normal form of  $w$* . We would like to do all this in such a way that, given any  $w \in (X \cup X^\circ)^+$ , there exists an algorithm which produces the normal form of  $w$ . After this is accomplished, we have a concrete model of  $F_{\mathbf{B}^\circ}(X)$  at hand.

In order to be able to find a normal form for the elements of  $F_{\mathbf{B}^\circ}(X)$ , we shall assume that there exists a total order on the set  $X$  of generators. For any finite nonempty set  $C$  of  $X$ , say  $C = \{x_1, \dots, x_n\}$  with  $x_1 < x_2 \dots < x_n$  for the total order, we let  $C^\circ$  be the uniquely defined word  $x_1^\circ \cdots x_n^\circ$  of  $(X^\circ)^+$ . In particular, if  $w \in (X \cup X^\circ)^+$ , then  $c(w)^\circ \in (X^\circ)^+$  is uniquely defined and  $c(c(w)^\circ) = c(w)$ .

From the results of the previous section, the following holds.

#### Lemma 3.1

For any  $w \in (X \cup X^\circ)^+$ ,  $w = l(w)\bar{l}(w)c(w)^\circ\bar{r}(w)r(w)$ .

It follows that we can, in a unique way, parse every  $w \in (X \cup X^\circ)^+$  following the rule given by Lemma 3.1. We may visualize our parsing of  $w$  by considering a tree with a root labeled  $w$ , and where every node labeled  $v \in (X \cup X^\circ)^+$  is expanded as in Fig. 1.

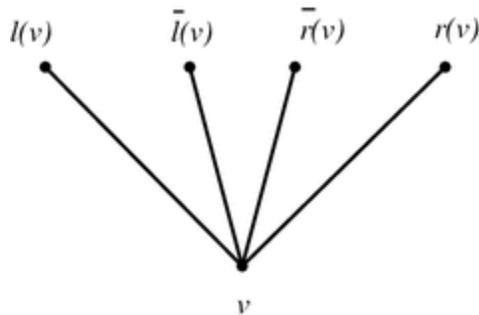


Fig. 1 Expansion of node  $v$

We shall, however, not consider such a further expansion if  $v \in X$ ,  $v \equiv 1$  or  $v \in (X^\circ)^+$ . In the latter case we rewrite  $v$  as  $c(v)^\circ$ . When parsing any  $w \in (X \cup X^\circ)^+$  we thus obtain a labeled tree whose leaves are labeled with either 1, or a letter of  $X$ , or a word of  $(X^\circ)^+$ .

By way of example, we give the parsing tree for the word  $w \equiv yx^\circ yq^\circ xs^\circ yzyt^\circ z^\circ s$  with  $q < s < t < x < y < z$  in  $X$ : see Fig. 2. The corresponding normal form for  $w$  may then be calculated, using Lemma 3.1, as

$$\begin{aligned} w &= [yx^\circ yq^\circ xs^\circ y] \cdot z \cdot (q^\circ s^\circ t^\circ x^\circ y^\circ z^\circ) \cdot x \cdot [s^\circ yzyt^\circ z^\circ s] \\ &= 1 \cdot y \cdot (q^\circ s^\circ x^\circ y^\circ) \cdot x \cdot [s^\circ y] \cdot z \cdot (q^\circ s^\circ t^\circ x^\circ y^\circ z^\circ) \cdot x \cdot [s^\circ y] \cdot \\ &\quad z \cdot (s^\circ t^\circ y^\circ z^\circ) \cdot y \cdot [t^\circ z^\circ s] \\ &= 1 \cdot y \cdot (q^\circ s^\circ x^\circ y^\circ) \cdot x \cdot s^\circ \cdot y \cdot (s^\circ y^\circ) \cdot y \cdot 1 \cdot z \cdot (q^\circ s^\circ t^\circ x^\circ y^\circ z^\circ) \cdot x \cdot \\ &\quad s^\circ \cdot y \cdot (s^\circ y^\circ) \cdot y \cdot 1 \cdot z \cdot (s^\circ t^\circ y^\circ z^\circ) \cdot y \cdot (t^\circ z^\circ) \cdot s \cdot (s^\circ t^\circ z^\circ) \cdot s \cdot 1. \end{aligned}$$

Here the square brackets indicate that further expansion is required, and dots suggest that some branching in the parsing tree is involved.

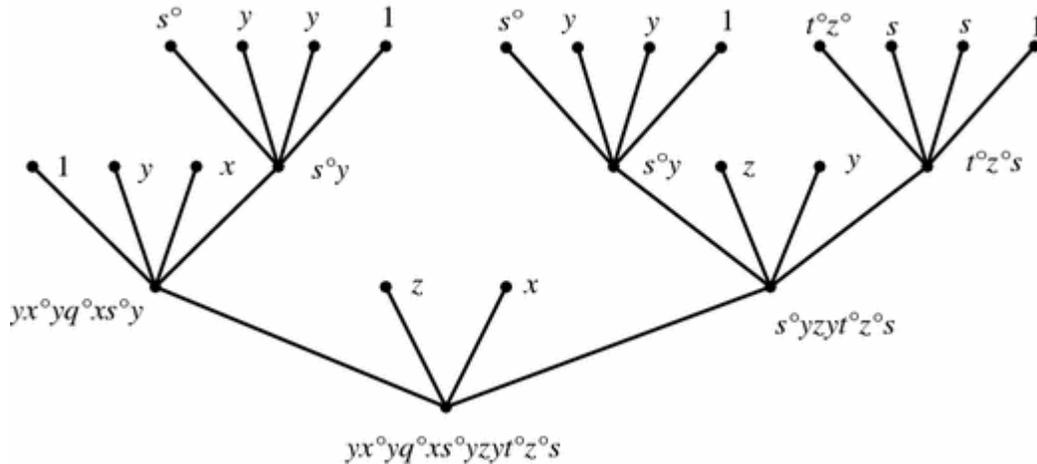


Fig. 2 Parsing of the word  $w$

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