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Abstract. In [5], Mann proves some bounds on the size of gaps between character degrees of maximal class p-groups. In this note we contruct a family of examples that shows that one of these bounds is sharp.

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There are many theorems relating the structure of a finite group to the set of irreducible character degrees for that group. Recently, a number of papers have focused on p-groups of maximal class and normally monomial groups (e.g. [3], [4], [6]). A character of a group is said to be normally monomial if it is induced from a linear character of a normal subgroup, and we say that a group is normally monomial if all of its irreducible characters are. Any finite p-group is monomial, but not all are normally monomial.

The paper [6] shows that there are some restrictions on possible sets of degrees of irreducible characters for normally monomial 5-groups of maximal class. That paper conjectures that similar restrictions exist for any 5-group of maximal class (not necessarily normally monomial) and further for any p-group of maximal class with $p \geq 5$. In [5], A. Mann proves:

Proposition. *Let G be a p-group of maximal class. If G has irreducible characters of degrees higher than p, then it has at least one such character of degree at most $p^{\frac{p+1}{2}}$.*

We will construct a family of groups (see Theorem 3 below) that shows that this bound is sharp. Since several of Mann's other results refer to normally monomial groups, it is of interest to note that these examples are normally monomial.

The following known result will be useful in our construction.

Lemma 1. *Let A be a finite abelian p-group with cyclic decomposition*

$$A = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_n \rangle,$$

and let z be an element of order p in $\mho_1(A)$. Then, for any integers $e_1, \ldots, e_n$ there is a unique automorphism θ of A with $\theta(x_i) = x_i z^{e_i}$ for $i = 1, \ldots, n$. Furthermore, θ^p is the identity map.

Proof. We can write any element $a \in A$ uniquely in the form

$$a = x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n},$$

where $0 \leq c_i < \operatorname{order}(x_i)$. We define

$$\theta(a) = x_1^{c_1} z^{e_1 c_1} x_2^{c_2} z^{e_2 c_2} \cdots x_n^{c_n} z^{e_n c_n}.$$

Factoring the powers of z to the right, we see that for any $a \in A$ there is an integer t_a such that $\theta(a) = a z^{t_a}$. Since z has order p, it is clear that θ preserves multiplication.

Furthermore, since $z \in \mho_1(A)$, $z = b^p$ for some $b \in A$ and

$$\theta(z) = \theta(b^p) = (b z^{t_b})^p = b^p z^{p t_b} = b^p = z.$$

Consequently, the image of θ includes z and so x_i for all i. Thus θ is an automorphism of A. Since θ fixes z, we see that

$$\theta^p(a) = a z^{p t_a} = a$$

as claimed. $\square$

We will also need the following surprising construction of R. Baer.

Notation. If G is a group of odd order and $x \in G$, there is a unique element of G whose square is x. We will denote this element by $\sqrt{x}$.

Lemma 2 (Baer trick). *Let G be a finite group of odd order and nilpotence class 2. We define a new binary operation ("addition") on the set G by*

$$x + y = xy\sqrt{[y, x]}.$$

Then $(G, +)$ forms an abelian group with the following properties.

- *If $xy = yx$ for $x, y \in G$, then $x + y = xy$.*
- *For any $x \in G$ and integer n, $x^n = nx$ (where nx denotes repeated addition).*
- *Every automorphism of G is also an automorphism of $(G, +)$.*

Proof. See [1], Appendix B. A modern treatment can be found in [2], Lemma 4.37. $\square$

Theorem 3. *For $p \geq 5$ there is a normally monomial p-group of maximal class with character degrees $\{1, p, p^{\frac{p+1}{2}}\}$ of multiplicity $\{p^2, p^{p-1} - 1, p - 1\}$ respectively.*

Proof. Let $k = \frac{p-1}{2}$, and let A be an abelian p-group of type $(1, \ldots, 1, 2, 2)$ with k cyclic factors. That is,

$$A = \langle a_0 \rangle \times \langle a_1 \rangle \times \cdots \times \langle a_{k-1} \rangle$$

where $a_0, \ldots, a_{k-3}$ have order p and a_{k-2} and a_{k-1} have order p^2.

Now let $z = a_{k-2}^{-p}$ and for $j = 0, 1, \ldots, k-1$ let θ_j be an automorphism of A as in Lemma 1 such that $\theta_j(a_i) = a_i z^{(-1)^i \binom{i}{j}}$ (where $\binom{i}{j}$ is zero if $j > i$). Then by the lemma, each of the automorphisms θ_j has order p and, since they each act only by multiplying by powers of z, they all commute.

Let $B = \langle b_0, b_1, \ldots, b_{k-1} \rangle$ be an elementary abelian p-group of rank k. Define an action of B on A by specifying that b_j acts like θ_j, and let $E = A \rtimes B$ be the semidirect product.

The group E will be a maximal subgroup of the group we are constructing, so we now want to define an automorphism of E. To do this it will be useful to have a power-commutator presentation of E. For the pc-generating sequence, we will take

$$a_{k-1}, a_{k-2}, \ldots, a_0, b_0, b_1, \ldots, b_{k-1}, y, z$$

where $y = a_{k-1}^{-p}$ and as above $z = a_{k-2}^{-p}$. These elements generate E, and every element of E can be written uniquely as a product of powers of these generators in the order given with exponents at most $p-1$. The relations in the presentation fall into two categories: power relations and commutator relations. The power relations are $a_{k-1}^p = y^{-1}$, $a_{k-2}^p = z^{-1}$, and the other p-th powers are trivial. The only non-trivial commutator relations are $[b_j, a_i] = z^{(-1)^{i+1}\binom{i}{j}}$, which are given by the action of B on A above. Since the b_j appears first in the commutator, this power of z is the negative of the one specified earlier.

We will define a map from E to itself by giving images of the pc-generators and verify that this gives a well-defined endomorphism by checking that the images satisfy the original relations for the group. We define $\sigma : E \to E$ by mapping each pc-generator to itself times the next generator in the sequence (and z to itself). That is

$\sigma(a_{k-1}) = a_{k-1}a_{k-2}, \sigma(a_{k-2}) = a_{k-2}a_{k-3}, \ldots, \sigma(a_0) = a_0 b_0,$
$\sigma(b_0) = b_0 b_1, \sigma(b_1) = b_1 b_2, \ldots, \sigma(b_{k-1}) = b_{k-1} y,$
$\sigma(y) = yz, \sigma(z) = z.$

To check the power relations, we compute

$$\sigma(a_{k-1})^p = (a_{k-1}a_{k-2})^p = a_{k-1}^p a_{k-2}^p = y^{-1}z^{-1} = \sigma(y)^{-1}.$$

For $p \geq 7$: The second power relation follows from

$$\sigma(a_{k-2})^p = (a_{k-2}a_{k-3})^p = a_{k-2}^p a_{k-3}^p = z^{-1} = \sigma(z)^{-1}.$$

For $p = 5$: $k = 2$ and the second check becomes

$$\sigma(a_0)^p = (a_0 b_0)^p = a_0^p b_0^p z^{-\binom{p}{2}} = z^{-1} = \sigma(z)^{-1}.$$

In checking the commutator relations, it is useful to note that commutators in E are central, and so $[x_1x_2, y_1y_2] = [x_1, y_1][x_1, y_2][x_2, y_1][x_2, y_2]$ for all $x_j, y_i \in E$. There are several cases of commutators to consider.

Case $[a_j, a_i]$ for $0 < j < i$: Here the generators and their images under σ lie entirely in the abelian group A, so they are all trivial.

Case $[a_0, a_i]$ for $0 < i$: $[a_0, a_i] = 1$ and

$$[\sigma(a_0), \sigma(a_i)] = [a_0 b_0, a_i a_{i-1}] = [b_0, a_i][b_0, a_{i-1}] = z^{(-1)^{i+1}\binom{i}{0}} z^{(-1)^i\binom{i-1}{0}} = 1.$$

Case $[b_j, b_i]$ for $i < j < k-1$: Here the generators and their images under σ lie entirely in the abelian group B, so they are all trivial.

Case $[b_{k-1}, b_i]$ for $i < k-1$: $[b_{k-1}, b_i] = 1$ and

$$[\sigma(b_{k-1}), \sigma(b_i)] = [b_{k-1}y, b_i b_{i+1}] = 1.$$

Case $[b_j, a_i]$ for $j < k-1$ and $0 < i$:

$$\begin{aligned}[\sigma(b_j), \sigma(a_i)] &= [b_j b_{j+1}, a_i a_{i-1}] = [b_j, a_i][b_j, a_{i-1}][b_{j+1}, a_i][b_{j+1}, a_{i-1}] \\ &= [b_j, a_i] z^{(-1)^i\binom{i-1}{j}} z^{(-1)^{i+1}\binom{i}{j+1}} z^{(-1)^i\binom{i-1}{j+1}}.\end{aligned}$$

The exponent of z becomes $(-1)^i[\binom{i-1}{j} - \binom{i}{j+1} + \binom{i-1}{j+1}] = 0$, and so

$$[\sigma(b_j), \sigma(a_i)] = [b_j, a_i] = \sigma([b_j, a_i]).$$

Case $[b_j, a_0]$ for $j < k-1$:

$$[\sigma(b_j), \sigma(a_0)] = [b_j b_{j+1}, a_0 b_0] = [b_j, a_0][b_{j+1}, a_0] = [b_j, a_0] = \sigma([b_j, a_0])$$

using the fact that $[b_{j+1}, a_0]$ is trivial since $j+1 > 0$ and so $\binom{0}{j+1}$ is zero.

Case $[b_{k-1}, a_i]$ for $0 < i$:

$$\begin{aligned}[\sigma(b_{k-1}), \sigma(a_i)] &= [b_{k-1}y, a_i a_{i-1}] = [b_{k-1}, a_i][b_{k-1}, a_{i-1}] \\ &= [b_{k-1}, a_i] = \sigma([b_{k-1}, a_i])\end{aligned}$$

using the fact that $[b_{k-1}, a_{i-1}]$ is trivial since $k-1 > i-1$ and so $\binom{i-1}{k-1}$ is zero.

Case $[b_{k-1}, a_0]$:

$$[\sigma(b_{k-1}), \sigma(a_0)] = [b_{k-1}y, a_0 b_0] = [b_{k-1}, a_0] = \sigma([b_{k-1}, a_0]).$$

Finally, any commutator relation involving y or z will not change since both these generators and their images under σ are central in E.

Since the relations are all preserved by σ, it defines an endomorphism of E. Furthermore, it is clear from the images of the pc-generators that σ is onto and so is an automorphism of E.

Next, we wish to show that σ has order p. It is apparently quite hard to compute powers of σ directly because of the fact that a_0 does not commute with $\sigma(a_0)$. Consequently, we are going to use the Baer trick to allow us to compute with σ acting on an abelian group.

The group E has nilpotence class 2 and odd order, and so by Lemma 2 we can define an abelian operation (denoted $x + y$) on the set E by

$$x + y = xy\sqrt{[y, x]}.$$

Now σ is a permutation of the elements of E and we would like to write that permutation in terms of the new operation. For most of the pc-generators, σ simply maps the generator to itself *plus* the next generator in the sequence (and z to itself), however the situation is different for a_0. We compute

$$a_0 + b_0 = a_0 b_0 \sqrt{z^{-1}} = a_0 b_0 + \sqrt{z^{-1}}$$

since $\sqrt{z^{-1}}$ is central in E. Consequently,

$$\sigma(a_0) = a_0 b_0 = a_0 + b_0 - \sqrt{z^{-1}} = a_0 + b_0 + \sqrt{z},$$

with the last equality using the fact that square root commutes with inversion and inverses are the same in E and $(E, +)$. We also will need

$$\begin{aligned}\sigma(b_0 + \sqrt{z}) &= \sigma(b_0\sqrt{z}) = b_0 b_1 \sqrt{z} \\ &= b_0 + b_1 + \sqrt{z} = (b_0 + \sqrt{z}) + b_1,\end{aligned}$$

where $\sqrt{z}$ is fixed by σ since it is a power of z.

At this point it is convenient to relabel the pc-generators of E as $u_1, u_2, \ldots, u_{p+1}$ in the same order as given above with the exception of u_{k+1}. That is, u_1 is a_{k-1}, u_k is a_0, u_{k+2} is b_1, u_{p+1} is z, and so on, but $u_{k+1} = b_0 + \sqrt{z}$. With this labeling, $\sigma(u_i) = u_i + u_{i+1}$ for $1 \leq i < p+1$ and σ fixes u_{p+1}. It follows that

$$\sigma^n(u_i) = u_i + \binom{n}{1} u_{i+1} + \cdots + \binom{n}{j} u_{i+j} + \cdots + \binom{n}{n} u_{i+n},$$

where we define u_j to be the identity element of E for $j > p+1$.

For $i > 1$

$$\sigma^p(u_i) = u_i + \binom{p}{1} u_{i+1} + \cdots + \binom{p}{p+1-i} u_{p+1} = u_i$$

since the coefficients of $u_{i+1}, \ldots$ are all divisible by p and all of the last $p-1$ pc-generators have order p.

For $i = 1$ we see

$$\sigma^p(u_1) = u_1 + \binom{p}{1} u_2 + \cdots + \binom{p}{p} u_{p+1} = u_1 + p u_2 + u_{p+1},$$

and since $pu_2 = a_{k-2}^p = z^{-1}$, $u_{p+1} = z$, and inverses are the same in E and $(E, +)$, we find that σ has order p as claimed.

Define $G = E \rtimes \langle s \rangle$ where s has order p and acts on E like σ. So G has order p^{p+2} and from the definition of σ one can see that G has maximal class. In particular, $[u_i, s] = u_{i+1}$ for $i = 1, \ldots, p$ except when $i = k$. There, because of our tweaked definition of u_{k+1}, we have $[u_k, s] = u_{k+1}\sqrt{z^{-1}}$.

We now consider the irreducible character degrees of G. Let $Y = \langle y, z \rangle$. We begin with the degrees of E. From the defined action of B, it is easy to see that B fixes Y elementwise and has only orbits of length p on $A \backslash Y$. Hence there are $p^2 + (p^{k+1} - p)$ orbits in total. Brauer's permutation lemma says that B has the same number of orbits on $\mathrm{Irr}(A)$. As the irreducible characters of A having z in their kernel are B-fixed, it follows that B has only regular orbits on the remaining characters. Consequently, those characters of A which do not extend to E, induce to irreducibles of degree p^k. Hence the multiset of character degrees of E is $[(1)^{p^{2k+1}}, (p^k)^{p(p-1)}]$.

It is clear that some of the linear characters of E extend to G and the rest induce irreducibly, so G has characters of degree 1 and p. Since G has maximal class, we know it has p^2 linears and the remaining linears of E induce to $p^{2k} - 1$ characters of degree p.

Next we consider the nonlinear characters of E. Since Y is central in E, the restriction ϕ_Y has a unique irreducible constituent for any $\phi \in \mathrm{Irr}(E)$. Consequently, the action of s on the nonlinear characters of E is the same as the action on the linear characters of Y which lie below some nonlinear ones of E. Those are the characters of Y without z in their kernel. Now s on Y fixes the elements of $\langle z \rangle$ and moves the $p^2 - p$ other elements in orbits of size p. Consequently, s moves the elements of $\mathrm{Irr}(Y)$ which do not have z in their kernel in orbits of size p and so, the characters of degree p^k all induce irreducibly to G giving $p-1$ characters of degree p^{k+1} as the only other degree of G. That is, the irreducible character degrees of G are $\{1, p, p^{k+1}\}$ with the claimed multiplicities. Recall that $k = \frac{p-1}{2}$, hence $k+1 = \frac{p+1}{2}$.

Finally, to see that G is normally monomial, note that any character of degree p is induced from a maximal, hence normal, subgroup. Further, if $\lambda \in \mathrm{Irr}(Y)$ does not have z in its kernel, then λ extends to $B \times Y$. Since λ^G has a single constituent, we see the characters of degree $p^{k+1} = p^{\frac{p+1}{2}}$ are induced from the normal abelian subgroup $B \times Y$. $\square$

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