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On Characterizations and Infinite Divisibility of Recently Introduced Distributions

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On Characterizations and Infinite Divisibility of Recently Introduced Distributions

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Abstract

We present here characterizations of the most recently introduced continuous univariate distributions based on: (*i*) a simple relationship between two truncated moments; (*ii*) truncated moments of certain functions of the 1th order statistic; (*iii*) truncated moments of certain functions of the *n*th order statistic; (*iv*) truncated moment of certain function of the random variable. We like to mention that the characterization (*i*) which is expressed in terms of the ratio of truncated moments is stable in the sense of weak convergence. We will also point out that some of these distributions are infinitely divisible via Bondesson's 1979 classifications.

Key words and phrases
Characterizations, univariate distributions, truncated moments, order statistics, infinitely divisible distributions.

1. Introduction

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions. The present work deals with the characterizations of *sixty five* new univariate continuous distributions. These characterizations are based on: (*i*) a simple relationship between two truncated moments; (*ii*) truncated moments of certain functions of the 1st order statistic; (*iii*) truncated moments of certain functions of the *n*th order statistic; (*iv*) truncated moment of certain function of the random variable.

The presentation of the content of this work is as follows: In Section 2, we present our characterization results in four subsections representing directions (*i*)–(*iv*) mentioned above. Infinite divisibility is an important property which is shared by certain distributions. Section 3 is devoted to infinite divisibility of some of these distributions via Bondesson's 1979 classifications. Section 4 deals with a very short concluding remarks. For further information regarding the domain of applicability of the distributions mentioned in the following section, we refer the interested readers to their corresponding papers cited in the references.

2. Characterization Results

In this section we present characterizations of sixty five recently introduced univariate continuous distributions in four different directions (*i*)–(*iv*) mentioned in the Introduction.

2.1. Characterizations based on two truncated moments

In this subsection we present characterizations of these distributions in terms of a simple relationship between two truncated moments. We like to mention here the works of Galambos and Kotz [29], Kotz and Shanbhag [43], Glänzel ([31, 32]), Glänzel et al. [33], Glänzel and Hamedani [34] and Hamedani ([36]–[38]) in this direction. Our characterization results presented here will employ an interesting result due to Glänzel [31] (Theorem 2.1.1 below).

THEOREM 2.1.1. *Let* $(\Omega, F, \mathbf{P})$ *be a given probability space and let* $H = [a, b]$ *be an interval for some* $a < b$ ($a = -\infty, b = \infty$ *might as well be allowed*). *Let* $X : \Omega \to H$ *be a continuous random variable with the distribution function* F *and let* g *and* h *be two real functions defined on* H *such that*

$$E[g(X) \mid X \geqq x] = E[h(X) \mid X \geqq x]\eta(x), \qquad x \in H,$$

is defined with some real function η. *Assume that* g, h *are continuous functions,* η *has continuous derivative and* F *is twice continuously differentiable and strictly monotone function on the set* H.

Finally, assume that the equation $h\eta = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g, h and η, particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)h(u) - g(u)} \right| \exp(-s(u))\, du,$$

where the function s is a solution of the differential equation $s' = \frac{\eta' h}{\eta h - g}$ and C is a constant, chosen to make $\int_H dF = 1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence, in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions g_n, h_n and η_n $(n \in \mathbb{N})$ satisfy the conditions of Theorem 2.1.1 and let $g_n \to g, h_n \to h$ for some continuously differentiable real functions g and h. Let, finally, X be a random variable with distribution F. Under the condition that $g_n(X)$ and $h_n(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if η_n converges to η, where

$$\eta(x) = \frac{E[g(X) \mid X \geqq x]}{E[h(X) \mid X \geqq x]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions g, h and η, respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lévy–Smirnov distribution if $\alpha \to \infty$, as was pointed out in Glänzel and Hamedani [34].

A further consequence of the stability property of Theorem 2.1.1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions g, h and, specially, η should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose η as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.

REMARK 2.1.2. In Theorem 2.1.1, the interval H need not be closed since the condition is only on the interior of H.

We will take up the distributions discussed in this section in the alphabetical order rather than their order of importance. In expressing these distributions we will use the same symbols for the parameters as employed by the original authors. We shall use $f(\cdot)$ for pdf (probability density function) and $F(\cdot)$ for its corresponding cdf (cumulative distribution function). The advantage of this kind of characterization is that the cdf, $F(\cdot)$ does not have to have a closed form as long as its corresponding pdf has a closed form.

1) Beta Pareto (BPa) Distribution (Akinsete et al., [1]): (2.1.1)

$$f(x) = f(x;\ \alpha,\beta,\theta,k) = \frac{k}{\theta B(\alpha,\beta)}[1 - (x/\theta)^{-k}]^{\beta-1}\,(x/\theta)^{-k\alpha-1}, x > \theta,$$

and

(2.1.2) $F(x) = F(x;\ \alpha,\beta,\theta,k) = \int_{\left(\frac{x}{\theta}\right)^{-k}}^{1} \frac{1}{B(\alpha,\beta)} u^{\alpha-1}(1-u)^{\beta-1}\,du,\ x > \theta,$

where α, β, θ, k are all positive parameters and

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1 - x)^{\beta-1}\,dx.$$

PROPOSITION 2.1.3. *Let* $X : \Omega \rightarrow (\theta,\infty)$ *be a continuous random variable and let* $h(x) = \left(\frac{x}{\theta}\right)^{k(\alpha-1)}\left[1 - \left(\frac{x}{\theta}\right)^{-k}\right]^{1-\beta}$ *and*

$$g(x) = \left(\frac{x}{\theta}\right)^{k(\alpha-1)}\left[1 - \left(\frac{x}{\theta}\right)^{-k}\right]^{2-\beta}$$

for $x \in (\theta,\infty)$. *The* pdf *of* X *is* (2.1.1) *if and only if the function* η *defined in Theorem* 2.1.1 *has the form*

$$\eta(x) = 1 - \frac{1}{2}\left(\frac{x}{\theta}\right)^{-k}, \qquad x > \theta.$$

PROOF. Let X have density (2.1.1), then

$$\big(1 - F(x)\big)\mathbf{E}[h(X) \mid X \geqq x] = \frac{1}{B(\alpha,\beta)}\left(\frac{x}{\theta}\right)^{-k}, \qquad x > \theta,$$

and

$$\big(1 - F(x)\big)\mathbf{E}[g(X) \mid X \geqq x] = \frac{1}{2B(\alpha,\beta)}\left(\frac{x}{\theta}\right)^{k(\alpha-1)}\left[2 - \left(\frac{x}{\theta}\right)^{-k}\right], \qquad x > \theta,$$

and finally

$$\eta(x)h(x) - g(x) = \frac{1}{2}\left(\frac{x}{\theta}\right)^{k(\alpha-2)}\left[1 - \left(\frac{x}{\theta}\right)^{-k}\right]^{1-\beta} > 0 \text{ for } x > \theta.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)h(x)}{\eta(x)h(x) - g(x)} = \frac{k}{\theta}\left(\frac{x}{\theta}\right)^{-1}, \qquad x > \theta,$$

and hence

$$s(x) = k\ln\left(\frac{x}{\theta}\right), x > \theta.$$

Now, in view of Theorem 2.1.1, X has density (2.1.1).

COROLLARY 2.1.4. *Let* $X : \Omega \to (\theta, \infty)$ *be a continuous random variable and let* $h(x)$ *be as in Proposition* 2.1.3. *The* pdf *of* X *is* (2.1.1) *if and only if there exist functions* g *and* η *defined in Theorem* 2.1.1 *satisfying the differential equation*

$$\frac{\eta'(x)h(x)}{\eta(x)h(x) - g(x)} = \frac{k}{\theta}\left(\frac{x}{\theta}\right)^{-1}, \qquad x > \theta.$$

REMARKS 2.1.5. (*a*) The general solution of the differential equation in Corollary 2.1.4 is

$$\eta(x) = \left(\frac{x}{\theta}\right)^{k} - \left[\int \frac{k}{\theta}\left(\frac{x}{\theta}\right)^{-(k\alpha+1)}\left[1-\left(\frac{x}{\theta}\right)^{-k}\right]^{1-\beta} g(x)dx \, + \, D\right],$$

for $x > \theta$, where D is a constant. One set of appropriate functions is given in Proposition 2.1.3 with $D = \frac{1}{2}$.

(*b*) Clearly there are other triplets of functions (h, g, η) satisfying the conditions of Theorem 2.1.1. We presented one such triplet in Proposition 2.1.3.

A Proposition, a Corollary and Remarks similar to Proposition 2.1.3, Corollary 2.1.4 and Remarks 2.1.5 can be stated for each of the following remaining distributions. For each of these distributions, however, we give below, the pdf f, cdf F and functions h, g, η corresponding to Theorem 2.1.1.

2) Beta Burr XII (BBXII) Distribution (Paranaíba et al., [63]):

$$f(x) = f(x;\, a,b,c,s,k) = \frac{ck}{s^{c}B(a,b)} x^{c-1}\left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-(kb+1)} \times \left\{1 - \left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-k}\right\}^{a-1},$$

$x > 0$ and

$$F(x) = F(x;\, a,b,c,s,k) = I_{1-\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}}(a,b) = \frac{1}{B(a,b)} \int_{0}^{1-\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}} \omega^{a-1}(1 - \omega)^{b-1}d\omega,$$

$x \geqq 0$, where a, b, c, s, k are all positive parameters. Taking, e.g., $h(x) = \left\{1 - \left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-k}\right\}^{1-a}\left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-kb}$ and $g(x) = \left\{1 - \left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-k}\right\}^{1-a}$, we have $\eta(x) = 2\left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-kb}$ for $x > 0$.

3) Beta-Cauchy (BC) Distribution (Alshawarbeh et al., [4]):

$$\begin{aligned} f(x;\, \alpha,\beta,\lambda,\theta) &= \frac{\lambda}{\pi B(\alpha,\beta)}\left[\frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x-\theta}{\lambda}\right)\right]^{\alpha-1} \\ &\times \left[\frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x-\theta}{\lambda}\right)\right]^{\beta-1} \frac{1}{\lambda^{2} + (x-\theta)^{2}}, \end{aligned}$$

$x \in \mathbb{R}$, where α, β, λ all positive and $\theta \in \mathbb{R}$ are parameters. Taking, e.g., $h(x) = \left[\frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x-\theta}{\lambda}\right)\right]^{\beta-1}$ and $g(x) = h(x)\left[\frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x-\theta}{\lambda}\right)\right]^{\alpha}$, we will have $\eta(x) = \frac{1}{2}\left\{1 + [\left[\frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x-\theta}{\lambda}\right)\right]^{\alpha}\right\}$, for $\theta \in \mathbb{R}$.

4) Beta-Dagum (BDa) Distribution (Domma and Condino, [25]):

$$f(x) = f(x;\, \beta, \lambda, \delta, a, b) = \frac{\beta\lambda\delta}{B(a,b)} x^{-(\delta+1)}\left(1 + \lambda x^{-\delta}\right)^{-a\beta-1}\left[1 - \left(1 + \lambda x^{-\delta}\right)^{-\beta}\right]^{b-1},$$

$x > 0$ and $\beta, \lambda, \delta, a, b$ are all positive parameters. Taking, e.g., $h(x) = \left[1 - \left(1 + \lambda x^{-\delta}\right)^{-\beta}\right]^{1-b}$ and $g(x) = h(x)\left(1 + \lambda x^{-\delta}\right)^{-a\beta}$, we will have $\eta(x) = \frac{1}{2}\left\{1 + \left(1 + \lambda x^{-\delta}\right)^{-a\beta}\right\}$, for $x > 0$.

5) Beta Exponentiated Pareto (BEP) Distribution (Zea et al., [75]):

This distribution is the same as (BGP) distribution of (8) below.

6) Beta Generalized Half-Normal (BGHN) Distribution (Pescim et al., [65]):

$$f(x) = f(x;\, a, b, \alpha, \theta)$$
$$= \frac{\alpha 2^b}{\sqrt{2\pi}\theta^{\alpha} B(a,b)} x^{\alpha-1} \exp\left\{-\frac{1}{2}\left(\frac{x}{\theta}\right)^{2\alpha}\right\} \times \left\{2\Phi\left[\left(\frac{x}{\theta}\right)^{\alpha}\right] - 1\right\}^{a-1}\left\{1 - \Phi\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]\right\}^{b-1},$$

$x > 0$ and

$$F(x) = F(x;\, a, b, \alpha, \theta) = \frac{1}{B(a,b)} \int_0^{2\Phi\left[\left(\frac{x}{\theta}\right)^{\alpha}\right] - 1} \omega^{a-1}(1 - \omega)^{b-1} d\omega, \qquad x \geqq 0$$

where a, b, α, θ are all positive parameters and Φ is the cdf of the standard normal distribution. Taking, e.g., $h(x) = \left\{2\Phi\left[\left(\frac{x}{\theta}\right)^{\alpha}\right] - 1\right\}^{1-a}$ and $g(x) = h(x)\Phi\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]$, we have $\eta(x) = \frac{1}{b+1}\left\{1 + b\Phi\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]\right\}$ for $x > 0$.

7) Beta Generalized Pareto (BGP) Distribution (Mahmoudi, [50]):

$$f(x) = f(x;\, \alpha, \beta, \sigma, \mu, \xi) = \frac{1}{\sigma B(\alpha,\beta)}\left[1 - \frac{\xi(x-\mu)}{\sigma}\right]^{\left(\frac{\beta}{\xi}\right)-1}\left\{1 - \left[1 - \frac{\xi(x-\mu)}{\sigma}\right]^{\frac{1}{\xi}}\right\}^{\alpha-1},$$

$x > \mu$ and

$$F(x) = F(x;\, \alpha, \beta, \sigma, \mu, \xi) = I_{\left[1 - \frac{\xi(x-\mu)}{\sigma}\right]^{\frac{1}{\xi}}}(\alpha, \beta), \qquad x \geqq \mu$$

where α, β, σ all positive, $\mu \in \mathbb{R}$ and $\xi \in \mathbb{R}\{0\}$ are parameters. Taking, e.g., $h(x) = \left\{1 - \left[1 - \frac{\xi(x-\mu)}{\sigma}\right]^{\frac{1}{\xi}}\right\}^{1-\alpha}$ and $g(x) = h(x)\left[1 - \frac{\xi(x-\mu)}{\sigma}\right]$, we have $\eta(x) = \frac{\beta}{\beta+\xi}\left[1 - \frac{\xi(x-\mu)}{\sigma}\right]$ for $x > \mu$.

8) Beta Generalized Pareto (BGP) Distribution (Nassar and Nada, [57]):

$$f(x) = f(x;\ \alpha,\beta,\gamma,a,b) = \frac{\gamma}{B(a,b)}\beta\alpha^{\beta}x^{-(\beta+1)}\left[1-\left(\frac{x}{\alpha}\right)^{-\beta}\right]^{\alpha\gamma-1}\left\{1-\left[1-\left(\frac{x}{\alpha}\right)^{-\beta}\right]^{\gamma}\right\}^{b-1},$$

$x > \alpha$ and α,β,γ,a,b are all positive parameters. Taking, e.g., $h(x) = \left\{1-\left[1-\left(\frac{x}{\alpha}\right)^{-\beta}\right]^{\gamma}\right\}^{b-1}$ and $g(x) = h(x)\left[1-\left(\frac{x}{\alpha}\right)^{-\beta}\right]^{\alpha\gamma}$, we have $\eta(x) = \frac{1}{2}\left\{1+\left[1-\left(\frac{x}{\alpha}\right)^{-\beta}\right]^{\alpha\gamma}\right\}$ for $x > \alpha$.

9) Beta Log-Logistic (BLLog) Distribution (Lemonte, [44]):

$$f(x) = f(x;\ \alpha,\beta,a,b) = \frac{(\beta/\alpha)}{B(a,b)}\frac{(x/\alpha)^{\alpha\beta-1}}{[1+(x/\alpha)^{\beta}]^{a+b}}, \qquad x>0,$$

and

$$F(x) = F(x;\ \alpha,\beta,a,b) = I_{\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}}(a,b), x \geqq 0$$

where α,β,a,b are all positive parameters. Taking, e.g., $h(x) = x^{-\alpha\beta+\beta}\left[1+(x/\alpha)^{\beta}\right]^{-2}, g(x) = x^{-\alpha\beta+\beta}\left[1+(x/\alpha)^{\beta}\right]^{-1}$, we have

$$\eta(x) = \frac{(a+b+1)}{(a+b)}\left[1+(x/\alpha)^{\beta}\right]$$

for $x > 0$.

10) Beta Power (BP) Distribution (Cordeiro and Brito, [16]):

$$f(x) = f(x;\ \alpha,\beta,a,b) = \frac{1}{B(a,b)}\{\alpha\beta(\beta x)^{\alpha a-1}[1-(\beta x)^{\alpha}]^{b-1}\}, 0 < x < \frac{1}{\beta},$$

and

$$F(x) = F(x;\ \alpha,\beta,a,b) = \frac{1}{B(a,b)}\int_0^{(\beta x)^{\alpha}} t^{a-1}(1-t)^{b-1}\,dt, 0 \leqq x \leqq \frac{1}{\beta}$$

where α,β,a,b are all positive parameters. Taking, e.g.,

$$h(x) = [1-(\beta x)^{\alpha}]^{2-b}, \qquad g(x) = [1-(\beta x)^{\alpha}]^{3-b},$$

we have $\eta(x) = \frac{2}{3}[1-(\beta x)^{\alpha}]$ for $0 < x < \frac{1}{\beta}$.

11) Compound Exponentiated Logarithmic (CEL) Distribution (Hakamipour et al., [35]; Personal Communication):

$$f(x) = f(x;\ \alpha,p) = -\frac{\alpha(1-p)g_1(x)[G_1(x)]^{\alpha-1}}{(\log p)\log\{p+(1-p)[G_1(x)]^{\alpha}\}}, x>0,$$

and

$$F(x) = F(x;\ \alpha, p) = 1 - \frac{1}{\log p} \log\{p + (1 - p)[G_1(x)]^\alpha\}, x \geqq 0$$

where $\alpha > 0, 0 < p < 1$ are parameters and $G_1(x)$ is a cdf on $[0, \infty)$ with the corresponding pdf $g_1(x)$. Taking, e.g., $h(x) = \{p + (1 - p)[G_1(x)]^\alpha\}$ and $g(x) = [G_1(x)]^\alpha\{p + (1 - p)[G_1(x)]^\alpha\}$, we have $\eta(x) = \frac{1}{2}\{1 + [G_1(x)]^\alpha\}$ for $x > 0$.

12) Compound Rayleigh (CR) Distribution (Khan, [42]):

$$f(x) = f(x;\ \gamma, \xi) = \frac{\gamma \xi^\gamma x}{(\xi^2 + x^2)^{\frac{\gamma}{2}+1}}, \qquad x > 0,$$

and

$$F(x) = F(x;\ \gamma, \xi) = 1 - \frac{\xi^\gamma}{(\xi^2 + x^2)^{\frac{\gamma}{2}}}, \qquad x \geqq 0$$

where $\gamma > 0$ and $\xi > 0$ are parameters. Taking, e.g., $h(x) = (\xi^2 + x^2)^{\frac{\gamma}{2}-2}$ and $g(x) = (\xi^2 + x^2)^{\frac{\gamma}{2}-1}$, we have $\eta(x) = 2(\xi^2 + x^2)$ for $x > 0$.

We can also characterize this distribution based on the hazard function as follows: The random variable X has a CR distribution if and only if its hazard function $\lambda(x)$ satisfies the following differential equation $\lambda'(x) + \frac{2x}{(\xi^2+x^2)}\ \lambda(x) = \frac{\gamma}{(\xi^2+x^2)}$.

13) Exponentiated Exponential-Geometric (EEG) Distribution (Louzada et al., [48]):

$$f(x) = f(x;\ \alpha, \lambda, \theta) = \frac{\alpha\lambda\theta e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\alpha-1}}{[1 - (1 - \theta)(1 - (1 - e^{-\lambda x})^\alpha)]^2}, \qquad x > 0,$$

and

$$F(x) = F(x;\ \alpha, \lambda, \theta) = \frac{1 - \left[1 - \left(1 - e^{-\lambda x}\right)^\alpha\right]}{[1 - (1 - \theta)(1 - (1 - e^{-\lambda x})^\alpha)]}, \qquad x \geqq 0$$

where $\alpha > 0, \lambda > 0, 0 < \theta < 1$ are parameters. Taking, e.g., $h(x) = \left[1 - (1 - \theta)\left(1 - \left(1 - e^{-\lambda x}\right)\alpha\right)\right]^2$ and $g(x) = h(x)\left(1 - e^{-\lambda x}\right)^\alpha$, we have $\eta(x) = \frac{1}{2}\left[1 + \left(1 - e^{-\lambda x}\right)^\alpha\right]$ for $x > 0$.

14) Exponentiated Generalized Inverse Gaussian (EGIG) Distribution (Lemonte and Cordeiro, [46]):

$$f(x) = f(x;\ \lambda, \omega, \xi, \beta) = C\beta x^{\lambda-1} \exp[-(\xi x + \omega x^{-1})] \times \{1 - C\Gamma(\lambda, \xi x;\ \xi\omega)\}^{\beta-1},$$

$x > 0$ and

$$F(x) = F(x;\ \lambda, \omega, \xi, \beta) = \{1 - C\Gamma(\lambda, \xi x;\ \xi\omega)\}^{\beta-1}, \qquad x \geqq 0$$

where $\lambda, \omega, \xi, \beta$ are all positive parameters, C is a normalizing constant and $\Gamma(\lambda, \xi x;\ \xi\omega) = \int_{\xi x}^{\infty} t^{\lambda-1} e^{-(t+\xi\omega t^{-1})}\, dt$. Taking, e.g., $h(x) = 1$ and $g(x) = \{1 - C\Gamma(\lambda, \xi x;\ \xi\omega)\}^{\beta}$, we have $\eta(x) = \frac{1}{2}\left[1 + \{1 - C\Gamma(\lambda, \xi x;\ \xi\omega)\}^{\beta-1}\right]$ for $x > 0$.

15) Exponentiated Generalized (EG) Distribution (Cordeiro et al., [17]):

$$f(x) = f(x;\ \alpha, \beta) = \alpha\beta g_1(x)[1 - G_1(x)]^{\alpha-1}\,[1 - \{1 - G_1(x)\}^{\alpha}]^{\beta-1}, \qquad x \in \mathbb{R},$$

and

$$F(x) = F(x;\ \alpha, \beta) = [1 - \{1 - G_1(x)\}^{\alpha}]^{\beta}, \qquad x \in \mathbb{R}$$

where α, β are positive parameters and $G_1(x)$ is a cdf with support in $\mathbb{R}$ and with the corresponding pdf $g_1(x)$. Taking, e.g., $h(x) = [1 - \{1 - G_1(x)\}^{\alpha}]^{1-\beta}$ and $g(x) = [1 - \{1 - G_1(x)\}^{\alpha}]^{2-\beta}$, we have $\eta(x) = \frac{1}{2}\,[1 - \{1 - G_1(x)\}^{\alpha}]$ for $x \in \mathbb{R}$.

Some Special Cases of Exponentiated Generalized Distribution:

(*i*) Exponentiated Generalized Fréchet Distribution:

$$G_1(x;\ \lambda, \sigma) = exp\{-(\sigma/x)^{\lambda}\},$$

$x \geqq 0$ and λ, σ are positive parameters.

(*ii*) Exponentiated Generalized Normal Distribution:

$$G_1(x;\ \mu, \sigma) = \Phi\left(\frac{x - \mu}{\sigma}\right),$$

$x \in \mathbb{R}, \mu \in \mathbb{R}, \sigma > 0$ and $\Phi(\cdot)$ is cdf of standard normal random variable.

(*iii*) Exponentiated Generalized Gamma Distribution:

$$G_1(x;\ a, b) = \frac{1}{\Gamma(a)}\gamma[a, bx],$$

$x \geqq 0$ and a, b are positive parameters.

(*iv*) Exponentiated Generalized Gumbel Distribution:

$$G_1(x;\ \mu, \sigma) = \exp\left\{-\exp\left(-\frac{x - \mu}{\sigma}\right)\right\},$$

$x \in \mathbb{R}$ and $\mu \in \mathbb{R}, \sigma > 0$ are parameters.

16) Exponentiated Generalized Gamma (EGG) Distribution (Cordeiro et al., [21]):

$$f(x) = f(x;\ \alpha, \beta, k, \lambda) = \frac{\lambda\beta}{\alpha\Gamma(k)}\left(\frac{x}{\alpha}\right)^{\beta k-1} \times \exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\left\{\frac{1}{\Gamma(k)}\gamma\left[k, \left(\frac{x}{\alpha}\right)^{\beta}\right]\right\}^{\lambda-1}, \qquad x > 0,$$

and

$$F(x) = F(x;\ \alpha,\beta,k,\lambda) = \left\{\frac{1}{\Gamma(k)}\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\beta}\right]\right\}^{\lambda}, \qquad x \geqq 0$$

where α,β,k,λ are all positive parameters and $\gamma[k;\ t] = \int_0^t u^{k-1}e^{-u}\,du$.

Taking, e.g., $h(x) = \left(\frac{x}{\alpha}\right)^{\beta(1-k)}\left\{\frac{1}{\Gamma(k)}\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\beta}\right]\right\}^{1-\lambda}$ and

$$g(x) = \left(\frac{x}{\alpha}\right)^{\beta(2-k)}\left\{\frac{1}{\Gamma(k)}\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\beta}\right]\right\}^{1-\lambda},$$

we have $\eta(x) = 1 + \left(\frac{x}{\alpha}\right)^{\beta}$ for $x > 0$.

17) Exponentiated Lomax Poisson (ELP) Distribution (Ramos et al., [68]):

$$\begin{aligned} f(x) = f(x;\ \alpha,\beta,\gamma,\lambda) & \\ &= \frac{e^{\lambda}}{e^{\lambda}-1}\{\alpha\beta\gamma\lambda(1+\beta x)^{-\gamma-1}[1-(1+\beta x)^{-\gamma}]^{\alpha-1}\} \\ &\times \exp\{-\lambda[1-(1+\beta x)^{-\gamma}]^{\alpha}\}, \end{aligned}$$

$x > 0$, and

$$F(x) = F(x;\ \alpha,\beta,\gamma,\lambda) = \frac{e^{\lambda}}{e^{\lambda}-1}[1-\exp\{-\lambda[1-(1+\beta x)^{-\gamma}]^{\alpha}\}], \qquad x \geqq 0,$$

where $\alpha,\beta,\gamma,\lambda$ are all positive parameters. Taking, e.g., $h(x) = \exp\{\lambda[1-(1+\beta x)^{-\gamma}]^{\alpha}\}$ and $g(x) = h(x)[1-(1+\beta x)^{-\gamma}]^{\alpha}$, we have $\eta(x) = \frac{1}{2}\{1+[1-(1+\beta x)^{-\gamma}]^{\alpha}\}$ for $x > 0$.

18) Extended Lomax (EL) Distribution (Lemonte and Cordeiro, [45]):

$$\begin{aligned} f(x) &= f(x;\ \alpha,\beta,\eta,a,c) \\ &= \frac{c\alpha\beta^{\alpha}}{B(ac^{-1},\eta+1)(\beta+x)^{\alpha+1}}\left\{1-\left(\frac{\beta}{\beta+x}\right)^{\alpha}\right\}^{a-1} \times \left[1-\left\{1-\left(\frac{\beta}{\beta+x}\right)^{\alpha}\right\}^{c}\right]^{\eta}, \end{aligned}$$

$x > 0$, and

$$F(x) = F(x;\ \alpha,\beta,\eta,a,c) = I_{\{1-\beta^{\alpha}(\beta+x)^{-\alpha}\}^{c}}(ac^{-1},\eta+1), \qquad x \geqq 0$$

where α,β,η,a,c are all positive parameters. Taking, e.g., $h(x) = \left[1-\left\{1-\left(\frac{\beta}{\beta+x}\right)^{\alpha}\right\}^{c}\right]^{-\eta}$ and $g(x) = h(x)\left\{1-\left(\frac{\beta}{\beta+x}\right)^{\alpha}\right\}^{a}$, we have $\eta(x) = \frac{1}{2}\left[1+\left\{1-\left(\frac{\beta}{\beta+x}\right)^{\alpha}\right\}^{a}\right]$ for $x > 0$.

19) Extension of the Exponential (EE) Distribution (Nadarajah and Haghighi, [53]):

$$f(x) = f(x;\ \alpha,\lambda) = \alpha\lambda(1+\lambda x)^{\alpha-1}\exp\{1-(1+\lambda x)^{\alpha}\}, \qquad x > 0,$$

and

$$F(x) = F(x;\ \alpha,\lambda) = 1-\exp\{1-(1+\lambda x)^{\alpha}\}, \qquad x \geqq 0$$

where α and λ are positive parameters. Taking, e.g., $h(x) = 1$ and $g(x) = (1 + \lambda x)^{\alpha}$, we have $\eta(x) = 1 + (1 + \lambda x)^{\alpha}$ for $x > 0$.

20) Four-Parameter Generalized Gamma (FPGG) Distribution (Ali et al., [3]):

$$f(x) = f(x;\ \alpha,\beta,\gamma,\theta) = \frac{\beta\alpha^{\gamma}}{\Gamma(\alpha)}(x - \theta)^{\alpha-1}e^{-\beta(x-\theta)} \times \left[\int_{\theta}^{x} \frac{\beta^{\alpha}}{\Gamma(\alpha)}(t - \theta)^{\alpha-1}e^{-\beta(t-\theta)}\, dt\right]^{\gamma-1},$$
$$x > \theta,$$

and

$$F(x) = F(x;\ \alpha,\beta,\gamma,\theta) = \left[\int_{\theta}^{x} \frac{\beta^{\alpha}}{\Gamma(\alpha)}(t - \theta)^{a-1}e^{-\beta(t-\theta)}\, dt\right]^{\gamma-1}, \qquad x \geqq \theta$$

where α,β,γ all positive and θ are parameters. Without loss of generality we take $\theta = 0$. Then taking, e.g., $h(x) = \left[\int_0^x \frac{\beta^{\alpha}}{\Gamma(\alpha)}\ t^{a-1}e^{-\beta t}\ dt\right]^{2-\gamma}$ and $g(x) = \left[\int_0^x \frac{\beta^{\alpha}}{\Gamma(\alpha)}\ t^{a-1}e^{-\beta t}\ dt\right]^{4-\gamma}$, we have $\eta(x) = \frac{1}{2}\left\{1 + \left[\int_0^x \frac{\beta^{\alpha}}{\Gamma(\alpha)}\ t^{a-1}e^{-\beta t}\ dt\right]^2\right\}$ for $x > 0$.

21) Gamma Pareto (GP) Distribution (Alzaatreh et al., [6]):

$$f(x) = f(x;\ \alpha,\theta,c) = \frac{1}{c^{\alpha}\Gamma(\alpha)}x^{-1}\left(\frac{\theta}{x}\right)^{1/c} \left[\log\left(\frac{x}{\theta}\right)\right]^{\alpha-1}, \qquad x > \theta,$$

and

$$F(x) = F(x;\ \alpha,\theta,c) = \frac{1}{\Gamma(\alpha)}\gamma\left\{\alpha, c^{-1}\log\left(\frac{x}{\theta}\right)\right\}, \qquad x \geqq \theta$$

where α,θ,c are all positive parameters. Then taking, e.g., $h(x) = x^{-1}\left[\log\left(\frac{x}{\theta}\right)\right]^{1-\alpha}$ and $g(x) = x^{-1}\left[\log\left(\frac{x}{\theta}\right)\right]^{1-\alpha}$, we have $\eta(x) = (c+1)x$ for $x > \theta$.

22) Gamma Extended Fr´echet (GEF) Distribution (da Silva et al., [23]):

$$\begin{aligned} f(x) = f(x;\ \alpha,\lambda,\sigma) \\ &= \frac{\alpha\lambda\sigma^{\lambda}}{\Gamma(\alpha)}x^{-(\lambda+1)}\exp\left[-\left(\frac{\sigma}{x}\right)^{\lambda}\right]\left\{1 - exp - \left[-\left(\frac{\sigma}{x}\right)^{\lambda}\right]\right\}^{\alpha-1} \\ &\times\left\{-\log\left[1 - \exp\left(-\left(\frac{\sigma}{x}\right)^{\lambda}\right)\right]^{\alpha}\right\}^{\alpha-1}, \end{aligned}$$

for $x > 0$ and

$$F(x) = F(x;\ \alpha,\lambda,\sigma) = \frac{1}{\Gamma(\alpha)}\gamma\left(\alpha, -\alpha\log\left[1 - \exp\left(-\left(\frac{\sigma}{x}\right)^{\lambda}\right)\right]\right), x \geqq 0$$

where α, λ, σ are all positive parameters. Taking, e.g., $h(x) = \left\{-\log\left[1 - \exp\left(-\left(\frac{\sigma}{x}\right)^{\lambda}\right)\right]^{\alpha}\right\}^{1-\alpha}$ and $g(x) = h(x)\left\{1 - \exp\left[-\left(\frac{\sigma}{x}\right)^{\lambda}\right]\right\}$, we have $\eta(x) = \frac{\alpha}{\alpha+1}\left\{1 - \exp\left[-\left(\frac{\sigma}{x}\right)^{\lambda}\right]\right\}$ for $x > 0$.

23) Gamma Extended Weibull (GEW) Distribution (Nascimento et al., [56]):

$$f(x) = f(x;\ \delta, \alpha, \xi) = \frac{\alpha^{\delta}}{\Gamma(\delta)} h_1(x;\ \xi)[H_1(x;\ \xi)]^{\delta-1} \exp[-\alpha H_1(x;\ \xi)],$$

for $a < x < b$ and

$$F(x) = F(x;\ \delta, \alpha, \xi) = \gamma[\delta, \alpha H_1(x;\ \xi)], a \leqq x \leqq b$$

where δ, α are positive and ξ is a vector of parameters. $H_1(x;\ \xi)$ corresponds to a special distribution whose cdf is given by $G\ (x;\ \delta, \alpha, \xi) = 1 - \exp[-\alpha H_1(x;\ \xi)]$ and $h_1(x;\ \xi) = \frac{d}{dx} H_1(x;\ \xi)$. We like to mention here that $H_1(x;\ \xi)$ is a non-negative function of x satisfying the following conditions $\lim_{x\to a} H_1(x;\ \xi) = 0$ and $\lim_{x\to b} H_1(x;\ \xi) = \infty$ and $\gamma\delta, [\ \alpha H_1(x;\ \xi)] = [\Gamma(\delta)]^{-1} \int_{a}^{\alpha H_1(x;\xi)} t^{\delta-1} e^{-t}\, dt$. Taking, e.g., $h(x) = [H_1(x;\ \xi)]^{1-\delta}$ and $g(x) = [H_1(x;\ \xi)]^{2-\delta}$, we have $\eta(x) = H_1(x;\ \xi) + \frac{1}{\alpha}$ for $a < x < b$.

24) Gamma Weibull (GW) Distribution (Razaq, [69]):

$$f(x) = f(x;\ \beta, \gamma, \mu, b, k) = kx^{\gamma+\beta-2} e^{-(\mu x + b x^{\beta})}, \qquad x > 0,$$

and

$$F(x) = F(x;\ \beta, \gamma, \mu, b, k) = k \int_{0}^{x} u^{\gamma+\beta-2} e^{-(\mu u + b u^{\beta})}\, du\,, \qquad x \geqq 0$$

where β, γ, μ, b, k are all positive parameters. Taking, e.g., $h(x) \equiv x^{2-\gamma-\beta} e^{bx^{\beta}}$ and $g(x) = x^{3-\gamma-\beta} e^{bx^{\beta}}$, we have $\eta(x) = x + \mu^{-1}$ for $x > 0$.

25) Generalized Beta Generated (GBG) Distribution (Alexander et al., [2]):

$$f(x) = f(x;\ \tau, a, b, c) = c[B(a,b)]^{-1} g(x;\ \tau)[G(x;\ \tau)]^{ac-1} \times \{1 - [G(x;\ \tau)]^{c}\}^{b-1},$$

where a, b, c are all positive parameters, $G(x;\ \tau), x \in \ell$ is parent cdf with parameter vector τ and pdf $g(x;\ \tau)$. Taking, e.g., $h(x) \equiv \{1 - [G(x;\ \tau)]^{c}\}^{1-b}$, and $g(x) = h(x)[G(x;\ \tau)]^{ac}$, we have $\eta(x) = \frac{1}{2}\{1 + [G(x;\ \tau)]^{ac}\}$ for $x \in \ell$.

26) Generalized Class (GC) of Distributions:

$$f(x) = f(x;\ \alpha, \beta) = \frac{1}{B(\alpha, \beta)} [Q(x)]^{\alpha-1}[1 - Q(x)]^{\beta-1}\, Q'(x), \qquad x \in \mathbb{R},$$

and

$$F(x) = F(x;\ \alpha,\beta) = \frac{1}{B(\alpha,\beta)} \int_0^{Q(x)} t^{\alpha-1}(1 - t)^{\beta-1}\, dt, \qquad x \in \mathbb{R}$$

where α and β are positive parameters and $Q(x)$ is an absolutely continuous cdf. Taking, e.g., $h(x) = [Q(x)]^{1-\alpha}$ and $g(x) = [Q(x)]^{1-\alpha}\,[1 - Q(x)]$, we have $\eta(x) = \frac{\beta}{\beta+1}\,[1 - Q(x)]$ for $x \in \mathbb{R}$.

An interesting special case of the above distribution appears to be when $Q(x) = \Phi\left(\frac{x-\mu}{\sigma}\right), \mu \in \mathbb{R}, \sigma > 0$ and $\Phi(\cdot)$ is the cdf of standard normal random variable. The properties and other important features of this special case have been discussed in (Eugene et al., [28]).

27) Generalized Exponential Model (GEM) Distribution (Mabrouk, [49]):

$$f(x) = f(x;\ \xi,\delta,\nu,\tau,\rho) = cx^{\xi+\delta}e^{-(\nu x^{\delta}+\tau x^{-\rho})}, \qquad x > 0,$$

where all five parameters are positive and c is the normalizing constant. Taking, e.g., $h(x) = x^{-\xi-1}e^{\tau x^{-\rho}}$ and $g(x) = x^{-\xi+\delta-1}e^{\tau x^{-\rho}}$, we have $\eta(x) = x^{\delta} + \frac{1}{\nu}$ for $x > 0$.

Some special cases of (GEM) Distribution given in (Mabrouk, [49]) are: Gamma; Weibull; Maxwell; Half-Normal; Chi-square; Rayleigh; Inverse Gaussian; Generalized Inverse Gaussian.

28) Generalized Log-Logistic (GLL) Distribution (Hanagal and Pandey, [39]):

$$f(x) = f(x;\ \alpha,\gamma,\lambda) = \alpha\gamma\lambda^{\alpha\gamma}x^{\alpha\gamma-1}(1 + (\lambda x)^{\gamma})^{-\alpha-1}, \qquad x > 0,$$

and

$$F(x) = F(x;\ \alpha,\gamma,\lambda) = (\lambda x)^{\alpha\gamma}(1 + (\lambda x)^{\gamma})^{-\alpha}, \qquad x \geqq 0$$

where $\alpha,\lambda > 0, \gamma \geqq 1$ are parameters. Taking, e.g., $h(x) \equiv x^{\gamma(1-\alpha)}$ and $g(x) = h(x)x^{\gamma}$, we have for $\alpha > 1, \eta(x) = \frac{1}{(\alpha-1)\lambda^{\gamma}}(1 + (\lambda x)^{\gamma})$ for $x > 0$.

29) Generalized Logistic Type I (GLTI) Distribution (Johnson et al., [41]):

$$f(x) = f(x;\ \alpha) = \alpha e^{-x}(1 + e^{-x})^{-\alpha-1}, \qquad x \in \mathbb{R},$$

and

$$F(x) = F(x;\ \alpha) = (1 + e^{-x})^{-\alpha}, \qquad x \in \mathbb{R}$$

where $\alpha > 0$ is a parameter. Taking, e.g., $h(x) \equiv 1$ and $g(x) = (1 + e^{-x})^{-\alpha}$, we have $\eta(x) = \frac{1}{2}\{1 + (1 + e^{-x})^{-\alpha}\}$ for $x \in \mathbb{R}$.

30) Generalized Modified Weibull (GMW) Distribution (Carrasco et al., [13]):

$$f(x) = f(x;\ \alpha,\beta,\gamma,\lambda) = \frac{\alpha\beta x^{\gamma-1}(\gamma + \lambda x)\exp\{\lambda x - \alpha x^{\gamma}\exp(\lambda x)\}}{[1 - exp - \alpha x^{\gamma}\exp\,(\lambda x)\_\}]^{1-\beta}}, \qquad x > 0,$$

and

$$F(x) = F(x;\ \alpha,\beta,\gamma,\lambda) = 1 - [1 - \exp(-\alpha x^{\gamma}\exp(\lambda x))\}]^{\beta}, x \geqq 0$$

where $\alpha > 0, \beta > 0, \gamma \geqq 0, \lambda \geqq 0$ are parameters. Taking, e.g., $h(x) = [1 - \exp(-\alpha x^{\gamma} \exp(\lambda x))\}]^{1-\beta}$ and $g(x) = h(x)\exp(\lambda x)$, we have $\eta(x) = \frac{1}{\alpha} + x^{\gamma}\exp(\lambda x)$ for $x > 0$.

31) Generalized Weibull Linear (GWL) Distribution (Prudente and Cordeiro, [67]):

$$f(x) = f(x;\ \phi, \lambda) = C\phi\lambda^{-\phi}x^{\phi-1}e^{-C\left(\frac{x}{\lambda}\right)^{\phi}}, \qquad x > 0,$$

and

$$F(x) = F(x;\ \phi, \lambda) = 1 - e^{-C\left(\frac{x}{\lambda}\right)^{\phi}}, \qquad x \geqq 0$$

where ϕ, λ are positive parameters and $C = e^{1-\gamma}, \gamma = 0.577215\ldots$ is the Euler's constant. Taking, e.g., $h(x) \equiv 1$ and $g(x) = x^{\phi}$, we have $\eta(x) = x^{\phi} + C^{-1}\lambda^{\phi}$ for $x > 0$.

32) Inverted Gamma-Inverted Weibull (IGIW) Distribution (Razaq, [69]):

$$f(x) = f(x;\ \beta, \alpha, \mu, b, k) = kx^{-\alpha-\beta-2}e^{-(\mu x^{-1}+bx^{-\beta})}, \qquad x > 0,$$

and

$$F(x) = F(x;\ \beta, \alpha, \mu, b, k) = k\int_0^x u^{-\alpha-\beta-2}e^{-(\mu u^{-1}+bu^{-\beta})}\,du, \qquad x \geqq 0$$

where β, α, μ, b, k are all positive parameters. Taking, e.g., $h(x) = x^{\alpha+\beta}e^{bx^{-\beta}}$ and $g(x) = x^{\alpha+\beta}e^{-\mu x^{-1}+bx^{-\beta}}$, we have $\eta(x) = \frac{1}{2}1 + e^{-\mu x^{-1}}$ for $x > 0$.

33) Inverted Weibull (IW) Distribution (Razaq, [69]):

$$f(x) = f(x;\ \alpha, \beta) = \alpha\beta x^{-\alpha-1}e^{-\alpha x^{-\beta}}, \qquad x > 0,$$

and

$$F(x) = F(x;\ \alpha, \beta) = \int_0^x \alpha\beta u^{-\alpha-1}e^{-\alpha u^{-\beta}}du, \qquad x \geqq 0$$

where $\alpha > 0, \beta > 0$ are parameters. Taking, e.g., $h(x) = x^{\alpha-\beta}$ and $g(x) = x^{\alpha-\beta}e^{-\alpha x^{-\beta}}$, we have $\eta(x) = \frac{1}{2}\left(1 + e^{-\alpha x^{-\beta}}\right)$ for $x > 0$.

34) Inverted Weibull-Gamma (IWG) Distribution (Razaq, [69]):

$$f(x) = f(x;\ \beta, \alpha, \mu, b, k) = kx^{\alpha-\beta-2}e^{-(\mu x+bx^{-\beta})}, \qquad x > 0,$$

and

$$F(x) = F(x;\ \beta, \alpha, \mu, b, k) = k\int_0^x u^{\alpha-\beta-2}e^{-(\mu u+bu^{-\beta})}\,du, \qquad x \geqq 0$$

where β, α, μ, b, k are all positive parameters. Taking, e.g., $h(x) = x^{-\alpha+\beta+2} e^{bx^{-\beta}}$ and $g(x) = x^{-\alpha+\beta+3} e^{bx^{-\beta}}$, we have $\eta(x) = x + \mu^{-1}$ for $x > 0$.

Special cases of IWG as appeared in Razaq, [69]:

i) Two-parameter IW distribution with pdf,

$$f(x) = f(x;\ \beta, \theta) = \beta\theta x^{-\beta-1} e^{-\theta x^{-\beta}}, \qquad x > 0.$$

ii) The inverse Maxwell distribution with pdf,

$$f(x) = f(x;\ \phi) = \frac{\sqrt{2}}{\sqrt{\pi\phi^3}}\ x^{-4} e^{-\frac{x^{-2}}{2\phi^2}}, \qquad x > 0.$$

iii) The inverse half-normal distribution with pdf,

$$f(x) = f(x;\ \phi) = \frac{2\phi}{\pi} x^{-2} e^{-\frac{x^{-2}\phi^2}{\pi}}, \qquad x > 0.$$

iv) The inverse Rayleigh distribution with pdf,

$$f(x) = f(x;\ \theta) = 2\theta x^{-3} e^{-\theta x^{-2}}, \qquad x > 0.$$

35) Kumaraswamy Generalized Gamma (KGG) Distribution (Pascoa et al., [64]):

$$f(x) = f(x;\ \alpha, k, \lambda, \varphi, \tau) = \frac{\lambda\phi\tau}{\alpha\Gamma(k)} \left(\frac{x}{\alpha}\right)^{\tau k-1} \exp\left\{-\left(\frac{x}{\alpha}\right)^{\tau}\right\} \left\{\frac{1}{\Gamma(k)} \gamma\left[k, \left(\frac{x}{\alpha}\right)^{\tau}\right]\right\}^{\lambda-1} \left[1 - \left\{\frac{1}{\Gamma(k)} \gamma\left[k, \left(\frac{x}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right]^{\varphi-1}, \quad x > 0,$$

and

$$F(x) = F(x;\ \alpha, k, \lambda, \varphi, \tau) = \left[1 - \left\{\frac{1}{\Gamma(k)} \gamma\left[k, \left(\frac{x}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right]^{\varphi}, x \geqq 0$$

where $\alpha, k, \lambda, \varphi, \tau$ are all positive parameters. Taking, e.g., $h(x) \equiv 1$ and $g(x) = 1 - \left\{\frac{1}{\Gamma(k)} \gamma\left[k, \left(\frac{x}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}$, we have $\eta(x) = \frac{\varphi}{\varphi+1}\left[1 - \left\{\frac{1}{\Gamma(k)} \gamma\left[k, \left(\frac{x}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right]$ for $x > 0$.

36) Kumaraswamy Gumbel (KGu) Distribution (Cordeiro et al., [19]):

$$f(x) = f(x;\ a, b, \sigma, \mu) = \frac{ab}{\sigma} e^{-\frac{(x-\mu)}{\sigma}} \exp\left\{-ae^{-\frac{(x-\mu)}{\sigma}}\right\} \times \left[1 - \exp\left\{-ae^{-\frac{(x-\mu)}{\sigma}}\right\}\right]^{b-1}, \qquad x \in \mathbb{R},$$

and

$$F(x) = F(x;\ a, b, \sigma, \mu) = 1 - \left[1 - \exp\left\{-ae^{-\frac{(x-\mu)}{\sigma}}\right\}\right]^{b}, \qquad x \in \mathbb{R}$$

where a, b, σ (all positive) and $\mu \in \mathbb{R}$ are parameters. Taking, e.g., $h(x) = \left[1 - \exp\left\{-ae^{-\frac{(x-\mu)}{\sigma}}\right\}\right]^{1-b}$ and $g(x) = \left[1 - \exp\left\{-ae^{-\frac{(x-\mu)}{\sigma}}\right\}\right]^{2-b}$, we have $\eta(x) = \frac{1}{2}\left[1 - \exp\left\{-ae^{-\frac{(x-\mu)}{\sigma}}\right\}\right]$ for $x \in \mathbb{R}$.

37) Kumaraswamy Log Logistic (KLL) Distribution (de Santana et al., [24]):

$$f(x) = f(x;\ \alpha, \gamma, a, b) = \frac{ab\gamma}{\alpha^{a\gamma}}\, x^{a\gamma-1}\left[1 + \left(\frac{x}{\alpha}\right)^{\gamma}\right]^{-(a+1)} \times \left\{1 - \left[1 - \left(1 + \left(\frac{x}{\alpha}\right)^{\gamma}\right)^{-1}\right]^{a}\right\}^{b-1}, \quad x > 0,$$

and

$$F(x) = F(x;\ \alpha, \gamma, a, b) = 1 - \left\{1 - \left[1 - \left(1 + \left(\frac{x}{\alpha}\right)^{\gamma}\right)^{-1}\right]^{a}\right\}^{b}, \qquad x \geqq 0$$

where α, γ, a, b are all positive parameters. Taking, e.g.,

$$h(x) = x^{\gamma(1-a)}\left\{1 - \left[1 - \left(1 + \left(\frac{x}{\alpha}\right)^{\gamma}\right)^{-1}\right]^{a}\right\}^{1-b}$$

and

$$g(x) = x^{\gamma(1-a)}\left[1 + \left(\frac{x}{\alpha}\right)^{\gamma}\right]^{-1}\left\{1 - \left[1 - \left(1 + \left(\frac{x}{\alpha}\right)^{\gamma}\right)^{-1}\right]^{a}\right\}^{1-b},$$

we have $\eta(x) = \frac{a}{a+1}\left[1 + \left(\frac{x}{\alpha}\right)^{\gamma}\right]^{-1}$ for $x > 0$.

A more general case of Kumaraswamy Log Logistic distribution was introduced by Paranaíba et al. ([62]) called "Kumaraswamy Burr XII" distribution with cdf F given below

$$F(x) = F(x;\ \alpha, \gamma, k, a, b) = 1\left\{1 - \left[1 - \left(1 + \left(\frac{x}{\alpha}\right)^{\gamma}\right)^{-k}\right]^{a}\right\}^{b}, x \quad 0$$

where the extra parameter k is positive as well. For $k = 1$, we have cdf of Kumaraswamy Log Logistic distribution. The characterization of this generalized version will be similar to the above.

38) Kumaraswamy Modified Weibull (KMW) Distribution (Cordeiro et al., [15]):

$$\begin{aligned} f(x) = f(x;\ \alpha, \gamma, \lambda, a, b) \\ &= ab\alpha(\gamma + \lambda x)x^{\gamma-1}\{\exp[\lambda x - \alpha x^{\gamma}\exp(\lambda x)]\} \times [1 - \exp\{-\alpha x^{\gamma}\exp(\lambda x)\}]^{a-1} \\ &\times \{1 - [1 - \exp\{-\alpha x^{\gamma}\exp(\lambda x)\}]^{a}\}^{b-1}, \end{aligned}$$

for $x > 0$ and

$$F(x) = F(x;\ \alpha, \gamma, \lambda, a, b) = 1 - \{1 - [1 - \exp\{-\alpha x^{\gamma}\exp(\lambda x)\}]^{a}\}^{b-1}, \qquad x \geqq 0$$

where α, γ, a, b are all positive and $\lambda \geqq 0$ are parameters. Taking, e.g., $h(x) = \{\exp \alpha x^{\gamma} \exp(\lambda x)\}\{1 - [1 - \exp\{-\alpha x^{\gamma} \exp(\lambda x)\}]^{a}\}^{1-b}$ and $g(x) = h(x)[1 - \exp\{-\alpha x^{\gamma} \exp(\lambda x)\}]^{a}$, we have $\eta(x) = \frac{1}{2}\{1 + [1 - \exp\{-\alpha x^{\gamma} \exp(\lambda x)\}]^{a}\}$ for $x > 0$.

39) Kumaraswamy Pareto (KwP) Distribution (Bourguignon et al., [12]):

$$f(x) = f(x;\ \beta, k, a, b) = \frac{abk\beta^{k}}{x^{k+1}}\left[1 - \left(\frac{\beta}{x}\right)^{k}\right]^{a-1} \left\{1 - \left[1 - \left(\frac{\beta}{x}\right)^{k}\right]^{a}\right\}^{b-1}, x > \beta,$$

and

$$F(x) = F(x;\ \beta, k, a, b) = 1 - \left\{1 - \left[1 - \left(\frac{\beta}{x}\right)^{k}\right]^{a}\right\}^{b}, \qquad x \geqq \beta$$

where β, k, a, b are all positive parameters. Taking, e.g., $h(x) = \left\{1 - \left[1 - \left(\frac{\beta}{x}\right)^{k}\right]^{a}\right\}^{b-1}$ and $g(x) = h(x)\left[1 - \left(\frac{\beta}{x}\right)^{k}\right]^{a}$, we have $\eta(x) = \frac{1}{2}\left\{1 + \left[1 - \left(\frac{\beta}{x}\right)^{k}\right]^{a}\right\}$ for $x > \beta$.

40) Kumaraswamy Quadratic Hazard Rate (KQHR) Distribution (Elbatal and Aryal, [27]):

$$\begin{aligned} f(x) = f(x;\ \alpha, \theta, \beta, a, b) \\ &= ab(\alpha + \theta x + \beta x^{2}) e^{-\left(\alpha x + \frac{\theta}{2} x^{2} + \frac{\beta}{3} x^{3}\right)} \\ &\times \left[1 - e^{-\left(\alpha x + \frac{\theta}{2} x^{2} + \frac{\beta}{3} x^{3}\right)}\right]^{a-1} \left\{1 - \left[- e^{-\left(\alpha x + \frac{\theta}{2} x^{2} + \frac{\beta}{3} x^{3}\right)}\right]^{a}\right\}^{b-1}, \end{aligned}$$

for $x > 0$ and

$$F(x) = F(x;\ \alpha, \theta, \beta, a, b) = 1 - \left\{1 - \left[1 - e^{-\left(\alpha x + \frac{\theta}{2} x^{2} + \frac{\beta}{3} x^{3}\right)}\right]^{a}\right\}^{b}, \qquad x \geqq 0$$

where $\alpha, \theta, \beta, a, b$ are all positive parameters. Taking, e.g., $h(x) = \left\{1 - \left[1 - e^{-\left(\alpha x + \frac{\theta}{2} x^{2} + \frac{\beta}{3} x^{3}\right)}\right]^{a}\right\}^{1-b}$ and $g(x) = h(x)\left[1 - e^{-\left(\alpha x + \frac{\theta}{2} x^{2} + \frac{\beta}{3} x^{3}\right)}\right]^{a}$, we have $\eta(x) = \frac{1}{2}\left\{1 + \left[1 - e^{-\left(\alpha x + \frac{\theta}{2} x^{2} + \frac{\beta}{3} x^{3}\right)}\right]^{a}\right\}$ for $x > 0$.

Special cases of KQHR mentioned in Elbatal et al. (see [27]):

Quadratic Hazard Rate; Kumaraswamy Linear failure rate; Kumaraswamy Rayleigh; Kumaraswamy exponential; Linear failure rate; Rayleigh and Exponential.

41) Log β-Birnbaum-Saunders (LβBS) Distribution (Ortega et al., [60]):

$$f(x) = f(x;\ a,b,\alpha,\mu,\sigma)$$
$$= \frac{2\cosh\left(\frac{x-\mu}{\sigma}\right)}{\sqrt{2\pi}B(a,b)\alpha}\exp\left\{-\frac{2}{\alpha^2}\left(\sinh\left(\frac{x-\mu}{\sigma}\right)\right)^2\right\}$$
$$\times\left[\Phi\left(\frac{2}{\alpha}\sinh\left(\frac{x-\mu}{\sigma}\right)\right)\right]^{a-1}\left[1-\Phi\left(\frac{2}{\alpha}\sinh\left(\frac{x-\mu}{\sigma}\right)\right)\right]^{b-1},$$

$x \in \mathbb{R}$ and

$$F(x) = F(x;\ a,b,\alpha,\mu,\sigma) = I_{\Phi\left(\frac{2}{\alpha}\sinh\left(\frac{x-\mu}{\sigma}\right)\right)}(a,b), \qquad x \in \mathbb{R}$$

where a,b,α,σ are all positive and $\mu \in \mathbb{R}$ are parameters and Φ is the cdf of the standard normal distribution. Taking, e.g., $h(x) = \left[1 - \Phi\left(\frac{2}{\alpha}\sinh\left(\frac{x-\mu}{\sigma}\right)\right)\right]^{1-b}$ and $g(x) = h(x)\left[\Phi\left(\frac{2}{\alpha}\sinh\left(\frac{x-\mu}{\sigma}\right)\right)\right]^{a}$, we have $\eta(x) = \frac{1}{2}\left\{1 + \left[\Phi\left(\frac{2}{\alpha}\sinh\left(\frac{x-\mu}{\sigma}\right)\right)\right]^{a}\right\}$ for $x \in \mathbb{R}$.

42) Log Beta Weibull (LBW) Distribution (Ortega et al., [61]):

$$f(x) = f(x;\ a,b,\sigma,\mu) = \frac{1}{\sigma B(a,b)}\exp\left\{\left(\frac{x-\mu}{\sigma}\right) - b\exp\left(\frac{x-\mu}{\sigma}\right)\right\} \times \left\{-\exp\left[-\exp\left(\frac{x-\mu}{\sigma}\right)\right]\right\}^{a-1},$$

for $x \in \mathbb{R}$ and

$$F(x) = F(x;\ a,b,\sigma,\mu) = I_{\left\{1-\exp\left[-\exp\left(\frac{x-\mu}{\sigma}\right)\right]\right\}}(a,b), \qquad x \in \mathbb{R}$$

where a,b,σ are all positive and $\mu \in \mathbb{R}$ are parameters. Taking, e.g., $h(x) = \left\{1 - \exp\left[-\exp\left(\frac{x-\mu}{\sigma}\right)\right]\right\}^{1-a}$ and $g(x) = h(x)\exp\left(\frac{x-\mu}{\sigma}\right)$, we have $\eta(x) = \frac{1}{b} + \exp\left(\frac{x-\mu}{\sigma}\right)$ for $x \in \mathbb{R}$.

43) Log-Burr XII (LBXII) Distribution (Silva et al., [70]):

$$f(x) = f(x;\ k,\sigma,\mu) = \frac{k}{\sigma}\exp\left(\frac{x-\mu}{\sigma}\right)\left[1 + \exp\left(\frac{x-\mu}{\sigma}\right)\right]^{k-1},$$

$x \in \mathbb{R}$ and

$$F(x) = F(x;\ k,\sigma,\mu) = 1 - \left[1 + \exp\left(\frac{x-\mu}{\sigma}\right)\right]^{-k}, \qquad x \in \mathbb{R}$$

where k,σ (both positive) and $\mu \in \mathbb{R}$ are parameters. Taking, e.g., $h(x) = \left[1 + \exp\left(\frac{x-\mu}{\sigma}\right)\right]^{-1}$ and $g(x) = 1$, we have $\eta(x) = \frac{k+1}{k}\left[1 + \exp\left(\frac{x-\mu}{\sigma}\right)\right]$ for $x \in \mathbb{R}$.

44) Log-Dagum (LDa) Distribution (Domma and Perri, [26]):

$$f(x) = f(x;\ \beta,\lambda,\delta) = \beta\lambda\delta e^{-\delta x}\left(1 + \lambda e^{-\delta x}\right)^{-\beta-1}, \qquad x \in \mathbb{R}$$

and

$$F(x) = F(x;\ \beta, \lambda, \delta) = \left(1 + \lambda e^{-\delta x}\right)^{-\beta}, \qquad x \in \mathbb{R}$$

where β, λ, δ are all positive parameters. Taking, e.g., $h(x) \equiv 1$ and $g(x) = \left(1 + \lambda e^{-\delta x}\right)^{-\beta}$, we have $\eta(x) = \frac{1}{2}\left\{1 + \left(1 + \lambda e^{-\delta x}\right)^{-\beta}\right\}$ for $x \in \mathbb{R}$.

45) Log-Exponentiated Weibull (LEW) Distribution (Hashimoto et al., [40]):

$$f(x) = f(x;\ \lambda, \sigma, \mu) = \frac{\lambda}{\sigma}\left[1 - \exp\left\{-e^{\frac{(x-\mu)}{\sigma}}\right\}\right]^{\lambda-1} \exp\left[\frac{(x-\mu)}{\sigma} - e^{\frac{(x-\mu)}{\sigma}}\right],$$

$x \in \mathbb{R}$ and

$$F(x) = F(x;\ \lambda, \sigma, \mu) = \left[1 - \exp\left\{-e^{\frac{(x-\mu)}{\sigma}}\right\}\right]^{\lambda}, \qquad x \in \mathbb{R}$$

where λ, σ (both positive) and $\mu \in \mathbb{R}$ are parameters. Taking, e.g., $h(x) = \left[1 - \exp\left\{-e^{\frac{(x-\mu)}{\sigma}}\right\}\right]^{1-\lambda}$ and $g(x) = h(x)e^{\frac{(x-\mu)}{\sigma}}$, we have $\eta(x) = 1 + e^{\frac{(x-\mu)}{\sigma}}$ for $x \in \mathbb{R}$.

46) Log Extended Weibull (LEW) Distribution (Silva et al., [71]):

$$f(x) = f(x;\ \lambda, \sigma, \mu) = \frac{\lambda}{\sigma}\exp\left(\frac{x-\mu}{\sigma}\right) \times \exp\left\{\mu + \exp\left(\frac{x-\mu}{\sigma}\right) + \lambda e^{\mu}\left[1 - \exp\left[\exp\left(\frac{x-\mu}{\sigma}\right)\right]\right]\right\},$$

$x \in \mathbb{R}$ and

$$F(x) = F(x;\ \lambda, \sigma, \mu) = 1 - \exp\left\{\lambda e^{\mu}\left[1 - \exp\left[\exp\left(\frac{x-\mu}{\sigma}\right)\right]\right]\right\}, x \in \mathbb{R}$$

where λ, σ (both positive) and $\mu \in \mathbb{R}$ are parameters. Taking, e.g., $h(x) = \exp\left\{(\lambda e^{\mu} - 3)\exp\left(\frac{x-\mu}{\sigma}\right)\right\}$ and $g(x) = \exp\left\{(\lambda e^{\mu} - 2)\exp\left(\frac{x-\mu}{\sigma}\right)\right\}$, we have $\eta(x) = 2\exp\left\{\exp\left(\frac{x-\mu}{\sigma}\right)\right\}$ for $x \in \mathbb{R}$.

47) Log-Modified Weibull (LMW) Distribution (Carrasco et al., [14]):

$$\begin{aligned} f(x) = {} & f(x;\ \alpha_1, \sigma, \mu) \\ & = \left\{\frac{1}{\sigma} + \exp(x - \mu)\right\} \times \exp\left\{\alpha_1 + \left(\frac{x-\mu}{\sigma}\right) + \exp(x - \mu) + \frac{\mu}{\sigma}\right\} \\ & \times \exp\left[-\exp\left\{\alpha_1 + \left(\frac{x-\mu}{\sigma}\right) + \exp(x - \mu) + \frac{\mu}{\sigma}\right\}\right], \end{aligned}$$

$x \in \mathbb{R}$ and

$$F(x) = F(x;\ \lambda, \sigma, \mu) = 1 - \exp\left[-\exp\left\{\alpha_1 + \left(\frac{x-\mu}{\sigma}\right) + \exp(x - \mu) + \frac{\mu}{\sigma}\right\}\right], x \in \mathbb{R}$$

where $\sigma > 0$ and $\alpha_1, \mu \in \mathbb{R}$ are parameters. Taking, e.g.,

$$h(x) = \exp\left\{-3\left(\frac{x-\mu}{\sigma}\right) - 3\exp(x - \mu)\right\}$$

and

$$g(x) = \exp\left\{-2\left(\frac{x-\mu}{\sigma}\right) - 2\exp(x - \mu)\right\},$$

we have $\eta(x) = 2\exp\left\{\left(\frac{x-\mu}{\sigma}\right) + \exp(x - \mu)\right\}$ for $x \in \mathbb{R}$.

48) Marshall-Olkin Extended Lomax (MOEL) Distribution (Ghitany et al. [30]):

$$f(x) = f(x;\ \alpha,\beta,\gamma) = \alpha\beta\gamma(1 + \beta x)^{\gamma-1}[(1 + \beta x)^{\gamma} - (1 - \alpha)]^{-2}, \qquad x > 0,$$

and

$$F(x) = F(x;\ \alpha,\beta,\gamma) = 1 - \alpha[(1 + \beta x)^{\gamma} - (1 - \alpha)]^{-1}, \qquad x \geqq 0$$

where α, β, γ are all positive parameters. Taking, e.g.,

$$h(x) = [(1 + \beta x)^{\gamma} - (1 - \alpha)]^{-2} \text{ and } g(x) = [(1 + \beta x)^{\gamma} - (1 - \alpha)]^{-1},$$

we have $\eta(x) = \frac{3}{2}(1 + \beta x)^{\gamma} - (1 - \alpha)$ for $x > 0$.

49) McDonald Weibull (McW) Distribution (Cordeiro et al., [18]):

$$\begin{aligned} f(x) = f(x;\ a,b,c,\gamma,\lambda) \\ &= \frac{c\gamma\lambda^{\gamma}}{B\left(\frac{a}{c},b\right)} x^{\gamma-1} \times \exp[-(\lambda x)^{\gamma}]\{1 - \exp[-(\lambda x)^{\gamma}]\}^{a-1} \\ &\times \left\{1 - \left[1 - \exp[-(\lambda x)^{\gamma}]\right]^{c}\right\}^{b-1}, \end{aligned}$$

for $x > 0$ and

$$F(x) = F(x;\ a,b,c,\gamma,\lambda) = I_{[1-\exp[-(\lambda x)^{\gamma}]]^{c}}\left(\frac{a}{c},b\right), \qquad x \geqq 0$$

where a, b, c, γ, λ are all positive parameters. Taking, e.g., $h(x) = \left\{1 - \left[1 - \exp[-(\lambda x)^{\gamma}]\right]^{c}\right\}^{1-b}$, and $g(x) = h(x)\{1 - \exp[-(\lambda x)^{\gamma}]\}^{a}$, we have $\eta(x) = \frac{1}{2}\left\{1 + 1\left[- \exp[-(\lambda x)^{\gamma}]\right]^{a}\right\}$ for $x > 0$.

50) New Four-Parameter Lifetime (NFPL) Distribution (Nadarajah et al., [55]):

$$\begin{aligned} f(x) = f(x;\ a,b,\alpha,p) \\ &= \alpha(a + bx)(1 - p) \times \frac{\exp\{-[ax + (b/2)x^2]\}\{1 - \exp(-[ax + (b/2)x^2])\}^{\alpha-1}}{\{1 - p[1 - \{1 - \exp(-[ax + (b/2)x^2])\}^{\alpha}]\}^2}, \end{aligned}$$

$x > 0$ and

$$F(x) = F(x;\ a,b,\alpha,p) = \frac{\{1 - (\exp - [ax + (b/2)x^2])\}^{\alpha}}{1 - p[1 - \{1 - (\exp - [ax + (b/2)x^2])\}^{\alpha}]},$$

$x \geqq 0$, where a, b, α (all positive) and $0 < p < 1$ are parameters. For simplicity, we take $\alpha = 1$. Then taking, e.g., $h(x) = \exp\left\{-\left[ax + \left(\frac{b}{2}\right)x^2\right]\right\}\left\{1 - p\left[\exp\left(-\left[ax + \left(\frac{b}{2}\right)x^2\right]\right)\right]\right\}^2$ and $g(x) = \left\{1 - p\left[\exp\left(-\left[ax + \left(\frac{b}{2}\right)x^2\right]\right)\right]\right\}^2$, we have $\eta(x) = 1 - p\left[\exp\left(-\left[ax + \left(\frac{b}{2}\right)x^2\right]\right)\right]$ for $x > 0$.

51) Perturbed Weibull (PW) Distribution (Mirhosseini and Lalehzari, [52]):

$$f(x) = f(x;\, \alpha, \beta, \lambda) = \lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}\left[(1 - \alpha) + 2\alpha e^{-\lambda x^{\beta}}\right], \qquad x > 0,$$

and

$$F(x) = F(x;\, \alpha, \beta, \lambda) = 1 + (\alpha - 1)e^{-\lambda x^{\beta}} - \alpha e^{-2\lambda x^{\beta}}, \qquad x \geqq 0$$

where β, λ (both positive) and $-1 \leqq \alpha \leqq 1$ are parameters. Taking, e.g., $\alpha \neq 0, h(x) \equiv 1$ and $g(x) = \left[(1 - \alpha) + 2\alpha e^{-\lambda x^{\beta}}\right]^{-1}$, we have $\eta(x) = \left[(1 - \alpha) + 2\alpha e - \lambda x^{\beta}\right]^{-1}$ for $x > 0$.

52) Skewed Cauchy (SC) Distribution (Behboodian et al., [7]):

$$f(x) = f(x;\, \lambda) = \frac{1}{\pi(1 + x^2)}\left(1 + \frac{\lambda x}{\sqrt{1 + (1 + \lambda^2)x^2}}\right), \qquad x \in \mathbb{R},$$

and

$$F(x) = F(x;\, \lambda) = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{x\sqrt{1 + (1 + \lambda^2)x^2} - \lambda}{\sqrt{1 + (1 + \lambda^2)x^2} - \lambda x}, \qquad x \in \mathbb{R}$$

where $\lambda > 0$ is a parameter. Taking, e.g., $h(x) = \left(1 + \frac{\lambda x}{\sqrt{1 + (1+\lambda^2)x^2}}\right)^{-1}$ and $g(x) = \arctan(x)\left(1 + \frac{\lambda x}{\sqrt{1 + (1+\lambda^2)x^2}}\right)^{-1}$, we have $\eta(x) = \frac{1}{2}\left[\frac{\pi}{2} + \arctan(x)\right]$ for $x \in \mathbb{R}$.

53) The Additive Weibull (TAW) Distribution (Lemonte et al., [47]):

$$f(x) = f(x;\, a, b, c, d) = (abx^{b-1} + cdx^{d-1})\exp(-ax^b - cx^d), \qquad x > 0,$$

and

$$F(x) = F(x;\, a, b, c, d) = 1 - \exp(-ax^b - cx^d), x \geqq 0$$

where a, b, c, d are all positive parameters. Taking, e.g., $h(x) = 1$ and $g(x) = \exp\{-2(ax^b + cx^d)\}$, we have $\eta(x) = \frac{1}{2}\exp\{-(ax^b + cx^d)\}$ for $x > 0$.

54) The Arc Tan-Exponential Type (ATET) Distribution (Phani et al., [66])

$$f(x) = f(x;\, u, v, \lambda) = \frac{cv\lambda\exp\left[-2\lambda\tan^{-1}\left(\frac{x - u}{v}\right)\right]}{(v^2 + (x - u)^2)}, \qquad x \in \mathbb{R},$$

and

$$F(x) = F(x;\ u, v, \lambda) = \frac{1 - \exp\left\{-\left[\pi\lambda + 2\lambda \tan^{-1}\left(\frac{x-u}{v}\right)\right]\right\}}{1 - \exp(-2\pi\lambda)}, \qquad x \in \mathbb{R}$$

where $u \in \mathbb{R}, v, \lambda > 0$ are parameters and $c = \frac{2\exp(-\pi\lambda)}{1-\exp(-2\pi\lambda)}$ is the normalizing constant. Taking, e.g.,

$$h(x) = \exp\left[2\lambda \tan^{-1}\left(\frac{x-u}{v}\right)\right] \text{ and } g(x) = h(x)\left(\tan^{-1}\left(\frac{x-u}{v}\right)\right),$$

we have $\eta(x) = \frac{1}{2}\left[\frac{\pi}{2} + \tan^{-1}\left(\frac{x-u}{v}\right)\right]$ for $x \in \mathbb{R}$.

55) The Gamma-Lomax (GL) Distribution (Cordeiro et al., [20]):

$$f(x) = f(x;\ \alpha, \beta, a) = \frac{\alpha\beta^{\alpha}}{\Gamma(a)(\beta + x)^{\alpha+1}} \exp\left\{-\alpha \log\left[\frac{\beta}{\beta + x}\right]\right\}^{a-1}, \qquad x > 0,$$

and

$$F(x) = F(x;\ \alpha, \beta, a) = \frac{\gamma\left(a, -\alpha \log\left[\frac{\beta}{\beta + x}\right]\right)}{\Gamma(a)}, \qquad x \geqq 0$$

where $\alpha, \beta, a > 0$ are parameters. Taking, e.g., $h(x) = g(x)(\beta + x)^{-1}$ and $g(x) = \exp\left\{-\alpha \log\left[\frac{\beta}{\beta+x}\right]\right\}^{1-a}$, we have $\eta(x) = \frac{\alpha+1}{\alpha}(\beta + x)$ for $x > 0$.

56) The Mc-Dagum (Mc-Da) Distribution (Oluyede and Rajasooriya, [59]):

This distribution is the same as (BDa) distribution in which β is replaced with $c\beta$.

57) The Weibull-G (W-G) Distribution (Bourguignon et al., [11]):

$$f(x) = f(x;\ \alpha, \beta, \xi) = \alpha\beta k(x;\ \xi) \frac{[K(x;\ \xi)]^{\beta-1}}{[\bar{K}(x;\ \xi)]^{\beta+1}} \exp\left\{-\alpha\left[\frac{K(x;\ \xi)}{\bar{K}(x;\ \xi)}\right]^{\beta}\right\},$$

$x \in \mathbb{R}$, and

$$F(x) = F(x;\ \alpha, \beta, \xi) = 1 - \exp\left\{-\alpha\left[\frac{K(x;\ \xi)}{\bar{K}(x;\ \xi)}\right]^{\beta}\right\}, \qquad x \in \mathbb{R}$$

where $\alpha, \beta > 0$ are parameters, $K(x;\ \xi)$ is a baseline cdf with corresponding pdf $k(x;\ \xi)$, which depends on a parameter vector ξ. Taking, e.g., $h(x) = \exp\left\{-\alpha\left[\frac{K(x;\ \xi)}{\bar{K}(x;\ \xi)}\right]^{\beta}\right\}$ and $g(x) = 1$, we have $\eta(x) = 2\exp\left\{\alpha\left[\frac{K(x;\ \xi)}{\bar{K}(x;\ \xi)}\right]^{\beta}\right\}$ for $x \in \mathbb{R}$.

58) Three-Parameter Lognormal (TPL) Distribution (Bilková and Malá, [8]):

$$f(x) = f(x;\ \mu, \sigma, \theta) = \frac{1}{\sqrt{2\pi}\sigma(x - \theta)} \exp\left\{-\frac{[\log(x - \theta) - \mu]^2}{2\sigma^2}\right\}, \qquad x > \theta,$$

and

$$F(x) = F(x;\ \mu, \sigma, \theta) = \begin{cases} \Phi \dfrac{\log(x-\theta) - \mu}{\sigma} & x > \theta, \\ 0 & x \geqq \theta \end{cases}$$

where $\mu, \theta \in \mathbb{R}, \sigma > 0$ are parameters and $\Phi(x)$ is the cdf of the standard normal distribution. Taking (for simplicity) $\theta = \mu = 0, \sigma = 1, h(x) \equiv 1$ and $g(x) = x$, we have $\eta(x) = e^{1/2}\left[\frac{1-\Phi(\log x - 1)}{1-\Phi(\log x)}\right]$ for $x > 0$, where Φ is the cumulative distribution function of the standard normal random variable.

59) Transmuted Additive Weibull (TAW) Distribution (Elbatal and Aryal, [27]):

$$f(x) = f(x;\ \alpha, \beta, \gamma, \theta, \lambda) = (\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1})e^{-\alpha x^{\theta} - \gamma x^{\beta}}\left(1 - \lambda + 2\lambda e^{-\alpha x^{\theta} - \gamma x^{\beta}}\right), \qquad x > 0,$$

and

$$F(x) = F(x;\ \alpha, \beta, \gamma, \theta, \lambda) = \left(1 - e^{-\alpha x^{\theta} - \gamma x^{\beta}}\right)\left(1 + \lambda e^{-\alpha x^{\theta} - \gamma x^{\beta}}\right), \qquad x \geqq 0$$

where $\alpha, \beta, \gamma, \theta$ all positive and $|\lambda| \leqq 1$ are parameters. Taking, e.g., $h(x) = \left(1 - \lambda + 2\lambda e - \alpha x^{\theta} - \gamma x^{\beta}\right)^{-1}$ and $g(x) = \left(\alpha x^{\theta} + \gamma x^{\beta}\right)\left(1 - \lambda + 2\lambda e - \alpha x^{\theta} - \gamma x^{\beta}\right)^{-1}$, we have $\eta(x) = 1 + \alpha x^{\theta} + \gamma x^{\beta}$ for $x > 0$.

Special cases of TAW as appeared in Elbatal et al. ([27]):

i) Transmuted modified Weibull (TMW) distribution with cdf,

$$F(x) = \left(1 - e^{-\alpha x - \gamma x^{\beta}}\right)\left(1 + \lambda e - \alpha x - \gamma x^{\beta}\right), \qquad x \geqq 0.$$

ii) Transmuted linear failure rate (TLFR) distribution with cdf,

$$F(x) = \left(1 - e^{-\alpha x - \gamma x^2}\right)1 + \lambda e^{-\alpha x - \gamma x^2}, \qquad x \geqq 0.$$

iii) Transmuted modified exponential (TME) distribution with cdf,

$$F(x) = (1 - e^{-\alpha x - \gamma x})(1 + \lambda e^{-\alpha x - \gamma x}), \qquad x \geqq 0.$$

iv) Additive Weibull (AW) distribution with cdf,

$$F(x) = 1 - e^{-\alpha x^{\theta} - \gamma x^{\beta}}, \qquad x \geqq 0.$$

v) Modified Weibull (MW) distribution with cdf,

$$F(x) = 1 - e^{-\alpha x - \gamma x^{\beta}}, \qquad x \geqq 0.$$

vi) Modified Rayleigh (MR) distribution with cdf,

$$F(x) = 1 - e^{-\alpha x - \gamma x^2}, \qquad x \geqq 0.$$

vii) Modified Exponential (ME) distribution with cdf,

$$F(x) = 1 - e^{-\alpha x - \gamma x}, \qquad x \geqq 0.$$

viii) Transmuted Weibull (TW) distribution with cdf,

$$F(x) = \left(1 - e^{-\gamma x^{\beta}}\right)\left(1 + \lambda e^{-\gamma x^{\beta}}\right), \qquad x \geqq 0.$$

ix) Transmuted Rayleigh (TR) distribution with cdf,

$$F(x) = \left(1 - e^{-\gamma x^2}\right)\left(1 + \lambda e^{-\gamma x^2}\right), \qquad x \geqq 0.$$

x) Transmuted Exponential (TE) distribution with cdf,

$$F(x) = (1 - e^{-\gamma x})(1 + \lambda e^{-\gamma x}), \qquad x \geqq 0.$$

xi) Weibull (W) distribution with cdf,

$$F(x) = 1 - e^{-\gamma x^{\beta}}, \qquad x \geqq 0.$$

xii) Rayleigh (R) distribution with cdf,

$$F(x) = 1 - e^{-\gamma x^2}, \qquad x \geqq 0.$$

xiii) Exponential (E) distribution with cdf,

$$F(x) = 1 - e^{-\gamma x}, \qquad x \geqq 0.$$

60) Transmuted Modified Rayleigh (TMR) Distribution (Merovci, [51]):

$$f(x) = f(x;\ \sigma, \lambda) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)\left(1 - \lambda + 2\lambda \exp_{\left(-\frac{x^2}{2\sigma^2}\right)}\right), \qquad x > 0,$$

and

$$F(x) = F(x;\ \sigma, \lambda) = \left(1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)\left(1 + \lambda \exp\left(-\frac{x^2}{2\sigma^2}\right)\right), \qquad x \geqq 0$$

where $\sigma > 0$ and $|\lambda| \leqq 1$ are parameters. Taking, e.g., $h(x) = \left(1 - \lambda + 2\lambda \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)^{-1}$ and $g(x) = x^2\left(1 - \lambda + 2\lambda \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)^{-1}$, we have $\eta(x) = x^2 + 2\sigma^2$ for $x > 0$.

61) Truncated Exponential Power (TEP) Distribution:

$$f(x) = f(x;\ \mu, r, k) = \frac{rk^{\frac{1}{r}}}{\Gamma\left(\frac{1}{r}\right)x} \exp\{-k(\mu - \log x)^r\}, 0 < x < e^{\mu},$$

and

$$F(x) = F(x;\ \mu, r, k) = \frac{1}{\Gamma\left(\frac{1}{r}\right)} \gamma^* \left[\frac{1}{r}; -k(\mu - \log x)^r\right], 0 \leqq x \leqq e^{\mu}$$

where $\mu \in \mathbb{R}, r, k$ (both positive) are parameters and

$$\gamma^*[\alpha;\ t] = \int_t^{\infty} u^{\alpha-1} e^{-u}\, du\,.$$

Taking, e.g., $h(x) = (\mu - \log x)^{r-1}$ and $g(x) = (\mu - \log x)^{r-1} e^{-k(\mu - \log x)^r}$, we have $\eta(x) = \frac{1}{2}\left[1 + e^{-k(\mu-\log x)^r}\right]$ for $0 < x \leqq e^{\mu}$. We like to mention here that the above distribution is a truncated version of the distribution introduced by Vianelli ([72, 73]).

62) Truncated-Exponential Skew-Symmetric (TESS) Distribution (Nadarajah et al., [54]; Personal Communication):

$$f(x) = f(x;\ \lambda) = \frac{\lambda}{1 - e^{-\lambda}}\, g_1(x) e^{-\lambda G_1(x)}, \qquad x \in \mathbb{R},$$

and

$$F(x) = F(x;\ \lambda) = \frac{1 - e^{-\lambda G_1(x)}}{1 - e^{-\lambda}}\,, \qquad x \in \mathbb{R}$$

where $\lambda \in \mathbb{R}$ is a parameter and $g_1(x)$ and $G_1(x)$ are, respectively, the pdf and cdf of a symmetric random variable. Taking, e.g., $h(x) = e^{\lambda G_1(x)}$ and $g(x) = G_1(x) e^{\lambda G_1(x)}$, we have $\eta(x) = \frac{1}{2}\left(1 + G_1(x)\right)$ for $x \in \mathbb{R}$.

63) Very Skewed Cauchy (VSC) Distribution (Nasseri et al., [58]):

$$f(x) = f(x;\ \lambda) = \exp\{-\lambda \arctan(x)\}, \qquad x \in \mathbb{R},$$

and

$$F(x) = F(x;\ \lambda) = \frac{C}{\lambda}\left(e^{\frac{\lambda\pi}{2}} - \exp\{-\lambda \arctan(x)\}\right), x \in R$$

where $\lambda > 0$ is a parameter and C is a normalizing constant. Taking, e.g., $h(x) = \exp\{\lambda \arctan(x)\}$ and $g(x) = \arctan(x) \exp\{\lambda \arctan(x)\}$, we have $\eta(x) = \frac{1}{2}\left[\frac{\pi}{2} + \arctan(x)\right]$ for $x \in \mathbb{R}$.

64) Weighted Generalized Beta of the Second Kind (WGBSK) Distribution (Ye et al., [74]):

$$f(x) = f(x;\ a, b, p, q, k) = \frac{a}{b^{ap+k} B\left(p + \frac{k}{a}, q - \frac{k}{a}\right)} x^{ap+k-1} \left[1 + \left(\frac{x}{b}\right)^a\right]^{-(p+q)},$$

$x > 0$, and

$$F(x) = F(x;\ a, b, p, q, k) = \int_0^x \frac{a}{b^{ap+k} B\left(p + \frac{k}{a}, q - \frac{k}{a}\right)} t^{ap+k-1} \times \left[1 + \left(\frac{t}{b}\right)^a\right]^{-(p+q)} dt\,, \qquad x \geqq 0$$

where α, b, p, q (all positive) and $-ap < k < aq$ are parameters. Taking, e.g., $h(x) = x^{a(1-p)-k}\left[1+\left(\frac{x}{b}\right)^{a}\right]^{p+q-3}$ and $g(x) = x^{a(1-p)-k}\left[1+\left(\frac{x}{b}\right)^{a}\right]^{p+q-2}$, we have $\eta(x) = 2\left[1 + \left(\frac{x}{b}\right)^{a}\right]$ for $x > 0$.

65) Weibull Pareto (WP) Distribution (Alzaatreh et al., [5]):

$$f(x) = f(x;\ \beta, c, \theta) = \frac{\beta c}{x}\beta \log\left(\frac{x}{\theta}\right)^{c-1}\exp\left\{-\left(\beta\log\left(\frac{x}{\theta}\right)\right)^{c}\right\}, \qquad x > \theta,$$

and

$$F(x) = F(x;\ \beta, c, \theta) = 1 - \exp\left\{-\left(\beta\log\left(\frac{x}{\theta}\right)\right)^{c}\right\}, \qquad x \geqq \theta$$

where β, c, θ are all positive parameters. Taking, e.g., $h(x) \equiv 1$ and $g(x) = \exp\left\{-\left(\beta\log\left(\frac{x}{\theta}\right)\right)^{c}\right\}$, we have $\eta(x) = \frac{1}{2}\exp\left\{-\left(\beta\log\left(\frac{x}{\theta}\right)\right)^{c}\right\}$ for $x > \theta$.

2.2. Characterizations based on truncated moment of certain functions of the 1st order statistic

Let $X_{1:n} \leqq X_{2:n} \leqq \ldots \leqq X_{n:n}$ be n order statistics from a continuous cdf F. We present here characterization results based on some functions of the 1st order statistic. Our characterizations will be a consequence of the following proposition, which is similar to the one appeared in our previous work (Hamedani, 2010).

PROPOSITION 2.2.1. *Let* $X : \Omega \to (a, b)$ *be a continuous random variable with* cdf F. *Let* $\psi(x)$ *and* $q(x)$ *be two differentiable functions on* (a, b) *such that* $\lim_{x\to b} -\ \psi(x)[1 - F(x)]^{n} = 0$ *and* $\int_a^b \frac{q'(t)}{[q(t)-\psi(t)]}dt = \infty$. *Then*

(2.2.1) $E[\psi(X_{1:n}) \mid X_{1:n} > t] = q(t),\ t > a,$

implies

(2.2.2) $F(x) = 1 - \exp\left\{-\int_a^x \frac{q'(t)}{n[\psi(t) - q(t)]}\,dt\right\}, a \leqq x < b.$

We list below the distributions from subsection 2.1 with their corresponding numbers and abbreviations which are characterized in this subsection. We would only mention some appropriate functions ψ and q for each case. Clearly, there are other choices for these functions as well.

11) CEL: Take, e.g.,

$$\psi(x) = 2(\log\{p + (1 - p)[G_1(x)]^{\alpha}\})^{n} \text{ and } q(x) = \frac{1}{2}\psi(x).$$

12) CR: Take, e.g.,

$$\psi(x) = \xi^{-n\gamma}(\xi^2 + x^2)^{\frac{n\gamma}{2}} \text{ and } q(x) = \frac{1}{2}\psi(x).$$

13) EEG: Take, e.g.,

$$\psi(x) = \theta^{-n}\left\{\theta - 1 + \left[1 - \left(1 - e^{-\lambda x}\right)^{\alpha}\right]^{-1}\right\}^{n} \text{ and } q(x) = \frac{1}{2}\psi(x).$$

19) EE: Take, e.g.,

$$\psi(x) = \frac{1}{2}\exp\{n(1 + \lambda x)^{\alpha}\} \text{ and } q(x) = 2\psi(x).$$

31) GWL: Take, e.g.,

$$\psi(x) = \exp\left\{nC\left(\frac{x}{\lambda}\right)^{\phi}\right\} \text{and } q(x) = \frac{1}{2}\psi(x).$$

35) KGG: Take, e.g., $\varphi \neq 1$,

$$\psi(x) = \frac{\varphi - 1}{\varphi}\left[1 - \left\{\frac{1}{\Gamma(k)}\gamma\left[k, \left(\frac{x}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right]^{n} \text{ and } q(x) = \frac{\varphi - 1}{\varphi}\psi(x).$$

36) KGu: Take, e.g., $b \neq 1$,

$$\psi(x) = \frac{b - 1}{b}\left[1 - \exp\left\{-ae^{-\frac{(x-\mu)}{\sigma}}\right\}\right]^{n} \text{ and } q(x) = \frac{b}{b - 1}\psi(x).$$

37) KLL: Take, e.g., $b \neq 1$,

$$\psi(x) = \frac{b - 1}{b}\left\{1 - \left[1 - \left(1 + \left(\frac{x}{\alpha}\right)\right)^{-1}\right]^{a}\right\}^{n} \text{ and } q(x) = \frac{b}{b - 1}\psi(x).$$

38) KMW: Take, e.g., b _= 1,

$$\psi(x) = \frac{b - 1}{b}\{1 - [1 - \exp - \alpha x^{\gamma}\exp(\lambda x)]^{a}\}^{n} \text{ and } q(x) = \frac{b}{b - 1}\psi(x).$$

39) KwP: Take, e.g.,

$$\psi(x) = \left\{1 - \left[1 - \left(\frac{\beta}{x}\right)^{k}\right]^{a}\right\}^{\frac{nb}{2}} \text{ and } q(x) = 2\psi(x).$$

46) LEW: Take, e.g.,

$$\psi(x) = \exp\left\{-n\lambda e^{\mu}\left[1 - \exp\left[\exp\left(\frac{x - \mu}{\sigma}\right)\right]\right]\right\} \text{ and } q(x) = \frac{1}{2}\psi(x).$$

47) LMW: Take, e.g.,

$$\psi(x) = \exp\left[n\exp\left\{\alpha_1 + \left(\frac{x - \mu}{\sigma}\right) + \exp(x - \mu) + \frac{\mu}{\sigma}\right\}\right]$$

and $q(x) = \frac{1}{2}\psi(x)$.

48) MOEL: Take, e.g.,

$$\psi(x) = 2[(1 + \beta x)^{\gamma} - (1 - \alpha)]^{n} \text{ and } q(x) = \frac{1}{2}\psi(x).$$

53) TAW: Take, e.g.,

$$\psi(x) = 2\exp\{n[ax^{b} + cx^{d}]\} \text{ and } q(x) = \frac{1}{2}\psi(x).$$

57) W-G: Take, e.g.,

$$\psi(x) = 2\exp\left\{\alpha n\left[\frac{K(x;\ \xi)}{\overline{K}(x;\ \xi)}\right]^{\beta}\right\} \text{ and } q(x) = \frac{1}{2}\psi(x).$$

60) TMR: Take, e.g.,

$$\psi(x) = \left\{e^{-x^2/\sigma^2}1 - \lambda + \lambda e^{-x^2/\sigma^2}\right\}^{n} \text{ and } q(x) = 2\psi(x).$$

65) WP: Take e.g.,

$$\psi(x) = \exp\left\{-n\left(\beta\log\left(\frac{x}{\theta}\right)\right)^{c}\right\} \text{ and } q(x) = 2\psi(x).$$

2.3. Characterizations based on truncated moment of certain functions of the *n*th order statistic

We present here characterization results based on some functions of the *n*th order statistics. Again, our characterizations will be a consequence of the following proposition, which is similar to the one appeared in our previous work (Hamedani, [38]).

PROPOSITION 2.3.1. *Let* $X : \Omega \to (a,b)$ *be a continuous random variable with* cdf F. *Let* $\psi_1(x)$ *and* $q_1(x)$ *be two differentiable functions on* (a,b) *such that* $\lim_{x\to a+}\psi_1(x)[F(x)]^n = 0$ *and* $\int_a^b \frac{q_1'(t)}{[\psi_1(t)-q_1(t)]}dt = \infty$. *Then*

(2.3.1) $E[\psi_1(X_{n:n}) \mid X_{n:n} < t] = q_1(t),\ t > a,$

implies

(2.3.2) $F(x) = \exp\left\{-\int_x^b \frac{q_1'(t)}{n[\psi_1(t)-q_1(t)]}\,dt\right\}, a \leqq x < b.$

Again, we list below the distributions from subsection 2.1 with their corresponding numbers and abbreviations which are characterized in this subsection. We would only mention some appropriate functions ψ and q for each case. Clearly, there are other choices for these functions as well.

1) BPa: Take, e.g.,

$$\psi_1(x) = 2\left\{ \int_{\left(\frac{x}{\theta}\right)^{-k}}^{1} \frac{1}{B(\alpha,\beta)} u^{\alpha-1}(1 - u)^{\beta-1}\, du \right\}^n \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

9) BLLog: Take, e.g.,

$$\psi_1(x) = 2\left\{ I_{\frac{x^\beta}{\alpha^\beta + x^\beta}}(a,b) \right\}^n \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

10) BP: Take, e.g.,

$$\psi_1(x) = 2\left\{ \int_0^{(\beta x)^\alpha} u^{a-1}(1 - u)^{b-1}\, du \right\}^n \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

15) EG: Take, e.g.,

$$\psi_1(x) = \frac{\beta + 1}{\beta}\{1 - \{1 - G_1(x)\}^\alpha\}^n \text{ and } q_1(x) = \frac{\beta}{\beta + 1}\psi_1(x).$$

16) EGG: Take, e.g.,

$$\psi_1(x) = \frac{\lambda + 1}{\lambda}\left\{ \gamma\left[k, \left(\frac{x}{\alpha}\right)^\beta \right] \right\}^n \text{ and } q_1(x) = \frac{\lambda}{\lambda + 1}\psi_1(x).$$

17) ELP: Take, e.g.,

$$\psi_1(x) = [1 - \exp\{-\lambda[1 - (1 + \beta x)^{-\gamma}]^\alpha\}]^n \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

20) FPGG: Take, e.g.,

$$\psi_1(x) = \frac{\gamma + 1}{\gamma}\left\{ \int_\theta^x (t - \theta)^{\alpha-1} e^{-\beta(t-\theta)}\, dt \right\}^n \text{ and } q_1(x) = \frac{\gamma}{\gamma + 1}\psi_1(x).$$

21) GP: Take, e.g.,

$$\psi_1(x) = \left\{ \gamma\left\{ \alpha,\ c^{-1}\log\left(\frac{x}{\theta}\right) \right\} \right\}^n \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

22) GEF: Take, e.g.,

$$\psi_1(x) = \left\{ \gamma\left(\alpha, -\alpha\log\left[1 - \exp\left(-\left(\frac{\sigma}{x}\right)^\lambda \right) \right] \right) \right\}^n \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

24) GW: Take, e.g.,

$$\psi_1(x) = 2\left\{\int_0^x u^{\gamma+\beta-2} e^{-(\mu u + b u^{\beta})}\, du\right\}^{nk} \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

26) GC: Take, e.g.,

$$\psi_1(x) = 2\left\{\int_0^{G(x)} u^{\alpha-1}(1 - u)^{\beta-1}\, du\right\}^{n} \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

28) GLL: Take, e.g.,

$$\psi_1(x) = \left[\frac{(\lambda x)^{\gamma}}{1 + (\lambda x)^{\gamma}}\right]^{n\alpha} \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

29) GLTI: Take, e.g.,

$$\psi_1(x) = (\alpha n - 1)e^{-x} - 1 \text{ and } q_1(x) = \alpha n e^{-x}.$$

30) GMW: Take, e.g.,

$$\psi_1(x) = [1 - \exp(- \alpha x^{\gamma} \exp(\lambda x))\}]^{n\beta} \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

32) IGIW: Take, e.g.,

$$\psi_1(x) = 2\left\{\int_0^x u^{-\alpha-\beta-2} e^{-(\mu u^{-1} + b u^{-\beta})}\, du\right\}^{nk} \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

33) IW: Take, e.g.,

$$\psi_1(x) = 2\left\{\int_0^x \alpha\beta u^{-\alpha-1} e - \alpha u^{-\beta} du\right\}^{n} \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

34) IWG: Take, e.g.,

$$\psi_1(x) = 2\left\{\int_0^x u^{\alpha-\beta-2} e^{-(\mu u + b u^{-\beta})}\, du\right\}^{nk} \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

44) LDa: Take, e.g.,

$$\psi_1(x) = 2\left\{\left[1 - \exp\left\{- e^{\frac{(x-\mu)}{\sigma}}\right\}\right]^{\lambda}\right\}^{n} \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

45) LEW: Take, e.g.,

$$\psi_1(x) = 2\left(1 + e^{-\delta x}\right)^{-n\beta} \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

51) PW: Take, e.g.,

$$\psi_1(x) = 2\left\{1 + (\alpha - 1)e^{-\lambda x^\beta} - \alpha e^{-2\lambda x^\beta}\right\}^n \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

60) TMR: Take, e.g.,

$$\psi_1(x) = 2\left\{\left(1 - \exp - \left(\frac{x^2}{2\sigma^2}\right)\right)\left(1 + \lambda \exp - \left(\frac{x^2}{2\sigma^2}\right)\right)\right\}^n$$

and $q_1(x) = \frac{1}{2}\psi_1(x)$.

62) TESS: Take, e.g.,

$$\psi_1(x) = 2\left\{1 - \exp\left(-\lambda G_1(x)\right)\right\}^n \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

64) WGBSK: Take, e.g.,

$$\psi_1(x) = 2\left\{\int_0^x t^{ap+k-1}\left(1 + \left(\frac{t}{b}\right)^a\right)^{-(p+q)} dt\right\}^n \text{ and } q_1(x) = \frac{1}{2}\psi_1(x).$$

We shall characterize distributions [52), SC] and [63), VSC] via the following Proposition.

PROPOSITION 2.3.2. *Let* $X : \Omega \to \mathbb{R}$ *be a continuous random variable with* cdf F. *Let* $\psi_2(x)$ *and* $q_2(x)$ *be two differentiable functions on* $\mathbb{R}$ *such that* $\lim_{x\to-\infty} \psi_2(x)[F(x)]^n = 0, \left(\frac{\psi_2'(t)-q_2'(t)}{q_2(t)}\right) > 0$ *and* $\int_{-\infty}^{\infty} \frac{\psi_2'(t)-q_2'(t)}{q_2(t)} dt = \infty$.*Then*

(2.3.3) $E[\psi_2(X_{n:n}) \mid X_{n:n} < t] = \psi_2(t) - q_2(t), \ t \in \mathbb{R}$,

implies

(2.3.4) $F(x) = \exp\left\{-\int_x^\infty \frac{\psi_2'(t)- q_2'(t)}{nq_2(t)} dt\right\}, x \in \mathbb{R}$.

Proof. If (2.3.3) holds, then using integration by parts on the left hand side of (2.3.3) and the assumption $\lim_{x\to-\infty} \psi_2(x)[F(x)]^n = 0$, we have

$$\int_{-\infty}^{t} \psi_2'(x)\left(F(x)\right)^n dx = q_2(t)\left(F(t)\right)^n.$$

Differentiating both sides of the above equation with respect to t, we arrive at

(2.3.5) $\frac{f(t)}{F(t)} = \frac{\psi_2'(t)- q_2'(t)}{nq_2(t)}, \ t \in \mathbb{R}$.

Now, integrating (2.3.5) from *x* to ∞, we have, in view of $\left(\frac{\psi_2'(t)- q_2'(t)}{q_2(t)}\right) > 0$ and $\int_{-\infty}^{\infty} \frac{\psi_2'(t)- q_2'(t)}{q_2(t)} dt = \infty$, a cdf F given by (2.3.4).

REMARKS 2.3.3. (*i*) Taking, e.g.,

$$\psi_2(x) = (n + 1)\arctan\left\{\frac{x\sqrt{1 + (1+\lambda^2)x^2} - \lambda}{\sqrt{1 + (1+\lambda^2)x^2} - \lambda x}\right\} \text{ and } q_2(x) = \frac{1}{n + 1}\psi_2(x)$$

in Proposition 2.3.2, we arrive at cdf of [52), SC].

(*ii*) Taking, e.g.,

$$\psi_2(x) = (n + 1)\exp\{-\lambda \arctan(x)\}$$

and

$$q_2(x) = \left(\exp\{-\lambda \arctan(x)\} - e^{\frac{\lambda\pi}{2}}\right)$$

in Proposition 2.3.2, we arrive at cdf of [63), VSC].

(*iii*) Clearly there are other functions $\psi 2$ and $q2$ which can be employed in both cases.

2.4. Characterization based on single truncated moment of certain function of the random variable

We like to point out that Propositions 2.2.1 and 2.3.1 hold true (with of course appropriate modifications) if we replace $X_{1:n}$ and $X_{n:n}$ with the base random variable X. In this subsection we employ a single function ψ of X and characterize the distribution of X in terms of the truncated moment of $\psi(X)$. The following propositions have already appeared in our previous work, so we will just state them here which can be used to characterize some of the above mentioned distributions.

PROPOSITION 2.4.1. *Let* $X : \Omega \to (a, b)$ *be a continuous random variable with* cdf F. *Let* $\psi(x)$ *be a differentiable function on* (a, b) *with* $\lim_{x\to a+} \psi(x) = 1$. *Then for* $\delta \neq 1$,

$$E[\psi(X)\,|X > x] = \delta\psi(x), \qquad x \in (a, b),$$

if and only if

$$\psi(x) = (1 - F(x))^{\frac{1}{\delta}-1}, \qquad x \in (a, b).$$

PROPOSITION 2.4.2. *Let* $X : \Omega \to (a, b)$ *be a continuous random variable with* cdf F. *Let* $\psi_1(x)$ *be a differentiable function on* (a, b) *with* $\boldsymbol{lim}_{x\to b-} \psi_1(x) = 1$. *Then for* $\delta_1 \neq 1$,

$$E[\psi_1(X)\,|X < x] = \delta_1\psi_1(x), \qquad x \in (a, b)$$

if and only if

$$\psi_1(x) = (F(x))^{\frac{1}{\delta_1}-1}, \qquad x \in (a, b).$$

3. Infinite Divisibility

Bondesson ([9]) showed that all the members of the following families

(3.1) $f(x) = Cx^{\beta-1}(1 + cx^{\alpha})^{-\gamma},\ x > 0,\ 0 < \alpha \leqq 1,$

(3.2) $f(x) = Cx^{\beta-1}\exp\{-cx^{\alpha}\},\ x > 0, 0 < |\alpha| \leqq 1,$

(3.3) $f(x) = Cx^{\beta-1}\exp\{-c_1x + c_2x^{-1}\}, x > 0, -\infty < \beta < \infty,$

(3.4) $f(x) = Cx^{-1}\exp\left\{-\frac{(\log x - \mu)^2}{2\sigma^2}\right\},\ x > 0,$

where the natural restrictions are put on the unspecified parameters, are infinitely divisible. The last one is the lognormal density.

REMARK 3.1. Bondesson ([10, Theorem 6.2.4]) pointed out that multiplying densities (3.1)–(3.4) by $C_1(\delta + x)^{-\nu}$ for $\delta > 0$ and $\nu > 0$, will result in densities which are also infinitely divisible.

We list below the distributions from subsection 2.1 with their corresponding numbers and abbreviations which either themselves or certain transformations of them can be expressed as one of the forms (3.1)–(3.4) mentioned above.

6) BGHN: For $a = b = 1$ and $0 < \alpha \leqq \frac{1}{2}$, the pdf of BGHN is of the form (3.2).

16) EGG: For $\lambda = 1$ and $0 < \beta \leqq 1$, pdf of EGG is of the form (3.2).

19) EE: Letting $Y = 1 + \lambda X$ and $0 < \alpha \leqq 1$, the pdf of Y is of the form (3.2).

20) FPGG: Letting $Y = X - \theta$ and $\gamma = 1$, the pdf of Y is actually a Gamma pdf and clearly of the form (3.2).

22) GEF: For $\lambda = 1$, the pdf of GEF has the form (3.3).

24) GW: For $\mu = 0, 0 < \beta \leqq 1$, the pdf of GW has the form (3.2).

27) GEM: For $\delta = \rho = 1$, the pdf of GEM has the form (3.3).

28) GLL: For $\gamma = 1$, the pdf of GLL has the form (3.1).

29) GLTI: Letting $Y = e^{-X}$, the pdf of Y is of the form (3.1).

31) GWL: For $0 < \phi \leqq 1$, the pdf of GWL has the form (3.2).

32) IGIW: Letting $Y = X^{-1}, \mu = 0$ and $0 < \beta \leqq 1$, the pdf of Y has the form (3.2).

33) IW: The pdf of IW is of the form (3.3).

34) IWG: For $\beta = 1$, the pdf of IWG is of the form (3.3).

35) KGG: For $\lambda = \varphi = 1$, the pdf of KGG is of the form (3.2).

37) KLL: For $b = 1$ and $0 < \gamma \leqq 1$, the pdf of KLL is of the form (3.1).

38) KMW: For $a = b = 1, \lambda = 0$ and $0 < \gamma \leqq 1$, the pdf of KMW is of the form (3.2).

44) LDa: Letting $Y = e^{-X}$ and $0 < \delta \leqq 1$, the pdf of Y is of the form (3.1).

48) MOEL: For $\alpha = 1$, the pdf of MOEL is of the form (3.1).

49) McW: For $a = b = 1$ and $0 < \gamma \leqq 1$, the pdf of McW is of the form (3.2).

51) PW: For $\alpha = 1$ and $0 < \beta \leqq 1$, the pdf of PW is of the form (3.2).

58) TPL: Letting $Y = X - \theta$, the pdf of TPL is of the form (3.4).

61) TEP: For $r = 2$, the pdf of TEP is of the form (3.4).

64) WGBSK: For $0 < a \leqq 1$, the pdf of WGBSK is of the form (3.1).

65) WP: Letting $Y = \log\left(\frac{X}{\theta}\right)$ and 0 < *c* _ 1, the pdf of *Y* is of the form (3.2).

4. Concluding Remarks

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will vitally depend on the characterizations of the selected distribution. A good number of recently introduced distributions which have important applications in many different fields have been mentioned in this work. Certain characterizations of these distributions have been established. We hope that these results will be of interest to the investigators who may believe their models have distributions mentioned here and are looking for justifying the validity of their models. It is known that determining a distribution is infinitely divisible or not via the existing representations is not easy. We have used Bondesson's classifications to show that some of the distributions taken up in this work are infinitely divisible or a transformation of them are infinitely divisible. This could be helpful to some researchers.

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