2002

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Majeed M. Hayat
Marquette University, majeed.hayat@marquette.edu

Muhammad Sajjad Abdullah
Acterna Corporation

Adel Joobeur
ASML Lithography

Bahaa E.A. Saleh
Boston University

IEEE Transactions on Image Processing, Vol. 11, No. 8 (2002) : 838-846. DOI.

Maximum-likelihood image estimation using photon-correlated beams

Majeed M. Hayat

Department of Electrical and Computer Engineering, The University of New Mexico, Albuquerque, NM 87131-1356 USA

Muhammad Sajjad Abdullah

Acterna Corporation, Indianapolis, IN 46260 USA

Adel Joobeur

ASML Lithography,Wilton, CT 06897-0877 USA

Bahaa E. A. Saleh

Department of Electrical and Computer Engineering, Boston University, Boston, MA 02215-2421 USA

Abstract:

A theory is presented addressing the fundamental limits of image estimation in a setup that uses two photon-correlated beams. These beams have the property that their photon arrivals, as a point process, are ideally synchronized in time and space. The true image represents the spatial distribution of the optical transmittance (or reflectance) of an object. In this setup, one beam is used to probe the image while the other is used as a reference providing additional information on the actual number of photons impinging on the object. This additional information is exploited to reduce the effect of quantum noise associated with the uncertainty in the number of photons per pixel. A stochastic model for the joint statistics of the two observation matrices is developed and used to obtain a local maximum-likelihood estimator of the image. The model captures the nonideal nature of the correlation between the photons of the beams by means of a simple random translation model. The mean-square error of the estimator is evaluated and compared to the corresponding conventional techniques. Conditions for the performance advantage of the proposed estimator are examined in terms of key system parameters. The theoretical predictions are demonstrated by means of simulation.

SECTION I.

Introduction

In many imaging applications, it is desirable to accurately estimate the spatial distribution of the transmittance (or reflectance) of a semitransparent object [1]–[2][3]. From an estimation-theoretic viewpoint, this problem is akin to the problem of estimating the underlying mean intensity function of point processes [2], [4]–[5][6][7][8][9][10]. The connection is that the unknown function in intensity estimation takes the form of a uniform function that is spatially modulated by the object's unknown transmittance. Physically, the

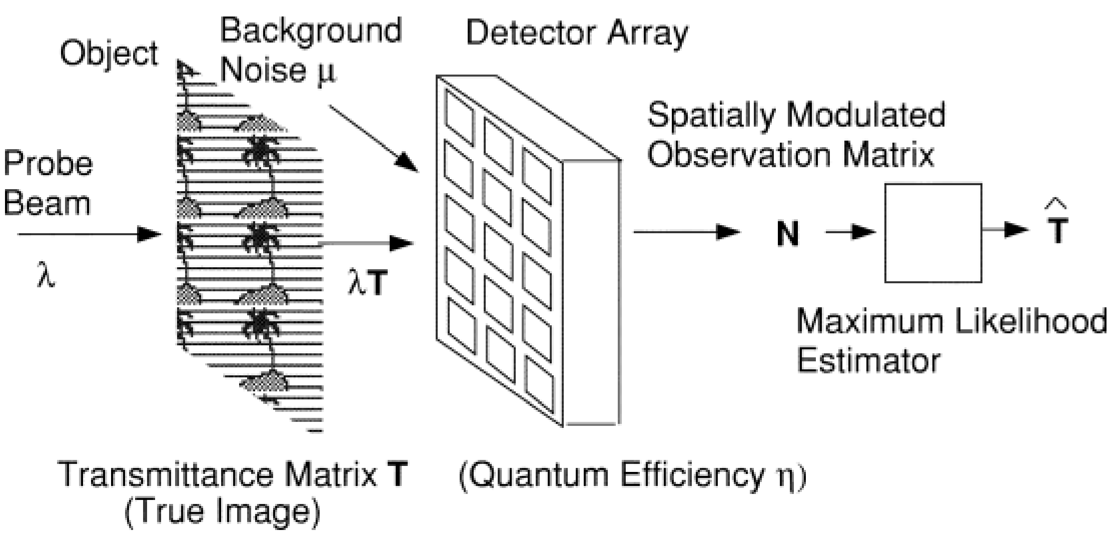

Fig. 1. Conventional maximum-likelihood image estimation setup using a single beam.

procedure for high-accuracy transmittance estimation typically involves the transmission of coherent light through the object and detecting the transmitted light by means of a detector array (e.g., a focal-plane array) operating in the photon-counting mode (see Fig. 1). Each detector in the array counts the number of photons that are transmitted through the object and detected within its active area during a specified measurement time. A detector element in the array is regarded as a pixel whose value is a noisy and spatially quantized (sampled) representation of the object's position-dependent transmittance. Under ideal conditions, the pixel value is a random variable (rv) whose mean is proportional to the average transmittance in an area covered by the detector. In this paper, we use the term “true image” to denote the two-dimensional (2-D) array of these sampled transmittance values.

Fundamentally, there are two sources of uncertainty that limit the performance of any image estimator that depends on optical measurements: 1) quantum noise and 2) photon transmission noise [1]. The performance limit governed by these two factors is referred to as the standard quantum limit (SQL) in optical measurement. The term quantum noise, or photon noise, is used to represent uncertainty resulting from the fact that the number of photons impinging on the object (per measurement time) is random (e.g., a Poisson rv if the light is in a coherent state, as the case in laser light) [11], [12]. The role of quantum noise becomes very significant in situations where the optical energy per pixel is weak (e.g., below 100 photons per pixel per measurement time) as in the case of estimating delicate biological specimens or other radiation-sensitive materials [1]. Photon transmission noise, on the other hand, is due to the fact that each photon impinging on the object is transmitted through the object with a certain probability that is equal to the transmittance of the object at that location. The very process of photon transmission through an object is therefore a random-deletion process.

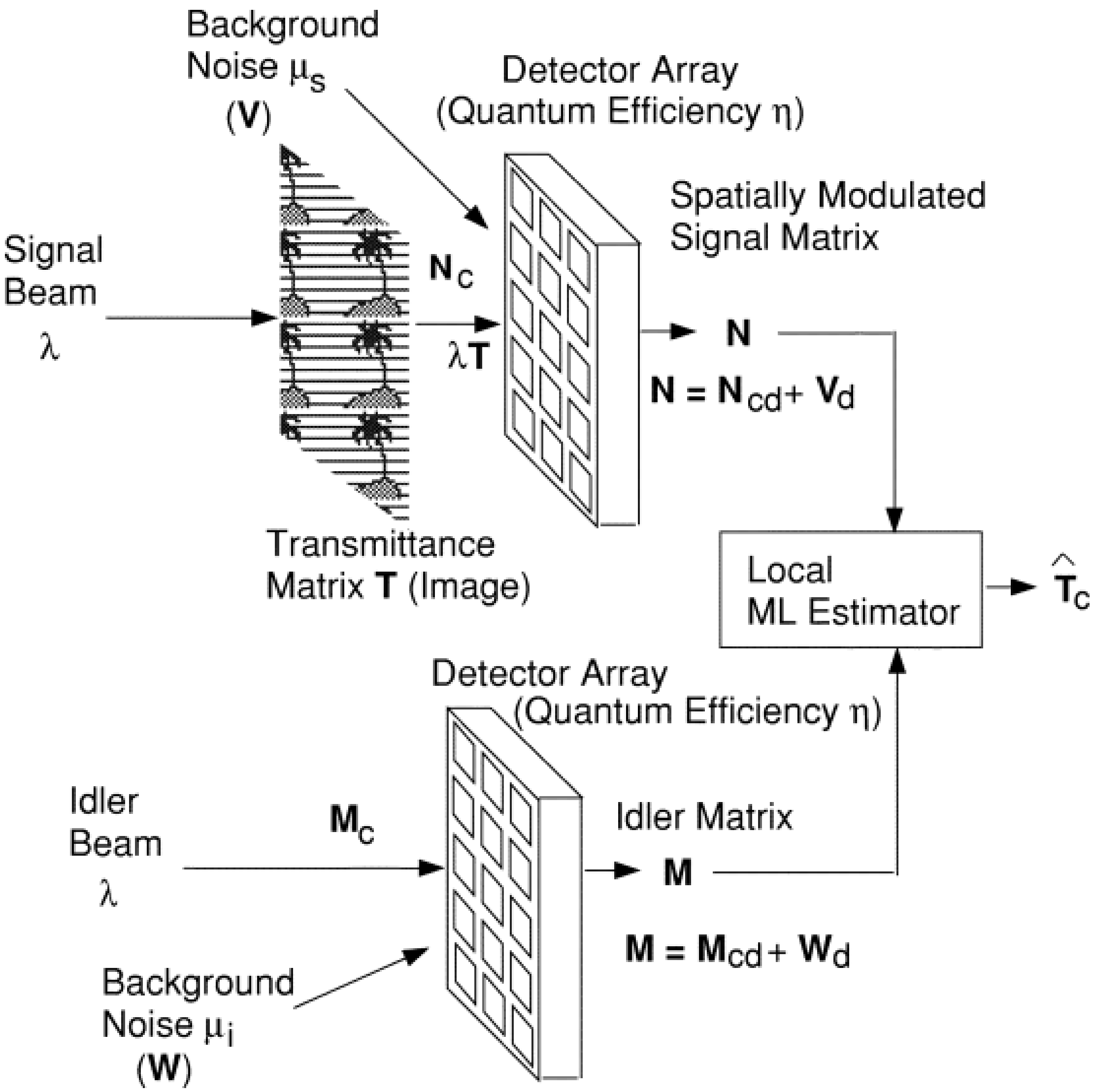

Fig. 2. Conditional maximum-likelihood image estimation setup using a pair of photon-correlated beams.

In this paper, we investigate the possibility of reducing quantum noise in image estimation by considering an alternative estimation setup, called the photon-correlated-beams setup, which uses a nonclassical light source consisting of a pair of beams whose photon streams are statistically correlated. In particular, the photons of each beam arrive in accordance with a Poisson process but the photons of the two beams are, under ideal conditions, perfectly synchronized in time and space. Photon-correlated beams can be generated, for example, by spontaneous parametric downconversion [13]–[14][15][16][17][18][19]. This type of nonclassical light has been proposed and demonstrated for use in a number of other applications including uniform transmittance (single pixel) estimation, optical communications, cryptography, and tests of the quantum theory of light [20]–[21][22][23][24][25][26][27]. For example, it was shown in [21] that a reduction up to 30% below the SQL (in the mean-square-error sense) is possible when the photon-correlated setup is used in uniform transmittance estimation. However, single-pixel estimation techniques do not suit image estimation since pixel values in the latter case are statistically correlated.

In the proposed setup, shown in Fig. 2, one of the beams, the signal, is transmitted through the object and subsequently detected by a detector array. The other beam, the idler, is not transmitted through the object but instead detected directly by another detector array and used as a reference providing information on the actual number of signal photons impinging on the object in each pixel. In this setup, the number of photons impinging on a certain image pixel and the number of photons impinging on a specified detector element in the idler detector array are highly correlated. More precisely, if a photon from the idler beam impinges on the (i,j) th pixel of the detector array, its twin signal photon will impinge on the (i,j) th pixel of the image with a certain (high) probability but may also undesirably impinge on a neighboring pixel instead. The purpose of this paper is to develop a mathematical model for this nonclassical setup and use it to obtain a maximum-likelihood (ML) estimate of the true image. The performance of such an estimator will be predicted and compared to the ML estimator for a conventional single-beam setup. Although the results are presented in the context of light sources generated by parametric downconversion, the model and the theory can be applied to other estimation setups where "correlated versions" of the underlying probing signal are available. Moreover, the theory can also be cast in the context of image fusion where two observed images (one image representing the photon spatial distribution prior to transmission through the object and the other representing the distribution of transmitted photons) are combined to generate an improved estimation of the true image.

This paper is organized as follows. In Section II, we review the conventional single-beam approach for image estimation and state the associated ML estimator. In Section III, we develop a stochastic model for the photon-correlated image estimation problem and derive a local ML estimator of the true image. In Section IV, we compare the performance of the conventional and the proposed photon-correlated estimators by means of computing the associated mean-square errors. The effects of various system parameters such as strength of correlation between photons, detector quantum efficiency (i.e., photodetection noise), optical energy, and background-noise level are investigated. Simulations of the stochastic model are also generated to support the theory.

SECTION II.

Image Estimation Using Conventional Light

For purposes of comparison, we review the theory for image estimation using the conventional single-beam setup shown in Fig. 1. The unknown transmittance is represented by an n-by-n matrix $\mathbf{T}=\{t_{ij}\}$, whose entries, t_{ij}, are real numbers in the interval [0,1] representing the object transmittance at each pixel. For example, the transmittance values 1 and 0 correspond, respectively, to full transparency and total opaqueness. The probing beam carries an average photon flux λ (photons per second per pixel), which is reduced to an average photon flux of $t_{ij}\lambda$ upon transmission through the (i,j) th pixel of the image since each photon is transmitted through the object with a probability equal to the transmittance at the pixel. The transmitted light is detected using a detector array with quantum efficiency η (i.e., each photon is detected with probability η), operating in the photon counting mode. For convenience, background stray light, dark current noise, and other sources of noise (e.g., read-out noise [28]) are collectively modeled by a noise photon flux μ (photons per second per pixel). The (i,j) th element of the detector array is a measurement of the number of photons, $N(i,j)$, detected during the measurement time T. According to the laws of photon optics for coherent light [11], $N(i,j)$ is a Poisson rv with mean value $\eta t_{ij}\lambda T+\eta\mu T$. We assume here that the noise photon statistics are also Poissonian (as an approximation) and independent of the probe-beam photons.

The conventional ML image estimation problem is described as follows. Given the measured random matrix $\mathbf{N}=\{N(i,j)\}$, and assuming that the parameters λ, μ, and η are known (from prior accurate experimental measurements of the probe beam and the detectors), determine the ML estimate of the transmittance matrix $\mathbf{T}$ of the object. To obtain this estimate, knowledge of the probability mass function (PMF) of the matrix $\mathbf{N}$ is required. Since the probe beam intensity is deterministic, the counts from different pixels are statistically independent. Hence, the joint PMF $P_{\mathbf{N}}(\mathbf{k})$, where $\mathbf{k}=\{k_{ij}\}$, can be written as the product

$$P_{\mathbf{N}}(\mathbf{k}) \triangleq \text{sf P}\{N(i,j)=k_{ij}, \text{for all } i,j\} = \prod_{i=1}^{n}\prod_{j=1}^{n} P_{N(i,j)}(k_{ij}) \tag{1}$$

where $P_{N(i,j)}(\cdot)$ is the Poissonian PMF of $N(i,j)$ given by

$$P_{N(i,j)}(k) = e^{-(\eta t_{ij}\lambda T+\eta\mu T)}\frac{(\eta t_{ij}\lambda T+\eta\mu T)^k}{k!}, k \geq 0. \tag{2}$$

The ML estimate, $\hat{\mathbf{T}}_s$, of $\mathbf{T}$ is obtained by maximizing (1) over $\mathbf{T}$. Due to the product form of $P_{\mathbf{N}}(\mathbf{k})$, this maximization can be performed on a pixel-by-pixel basis. A straightforward calculation [21] shows that, as a function of the observed pixel value $N(i,j)$, the (i,j) th entry $\hat{t}_{s,ij}$ of $\hat{\mathbf{T}}_s$ is given by

$$\hat{t}_{s,ij}(N(i,j)) = \begin{cases} 0, & \text{if } \dfrac{N(i,j)}{\eta\lambda T} \leq \dfrac{\mu}{\lambda} \\ 1, & \text{if } \dfrac{N(i,j)}{\eta\lambda T} \geq 1+\dfrac{\mu}{\lambda} \\ \dfrac{N(i,j)}{\eta\lambda T}-\dfrac{\mu}{\lambda}, & \text{otherwise.} \end{cases} \tag{3}$$

As a measure of performance, we will use the averaged mean-square error defined by

$$\varepsilon_s^2 \triangleq \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \text{sf E}\,[\{\hat{t}_{s,ij}(N(i,j)) - t_{ij}\}^2]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=0}^{\infty}\{\hat{t}_{s,ij}(k) - t_{ij}\}^2 P_{N(i,j)}(k) \tag{4}$$

which can be evaluated numerically for any fixed true image. Note that in the ideal case when μ=0 and η=1, $\varepsilon 2s$ is the error due only to the combined effect of quantum noise and the randomness in the process of photon transmission and it collectively represents the SQL in conventional image estimation. By using a photon-correlated light source, it is possible to reduce the image estimation error below the SQL to almost the random-deletion limit. This latter limit corresponds to the lowest theoretical error in image estimation (using direct detection of light) and it is achieved when the photon number (per measurement time per pixel) is deterministic. (In quantum optics, light with deterministic-photon count characteristics is referred to as maximally number-squeezed light [29].)

SECTION III.

Image Estimation Using Photon -Correlated Beams

We now consider the problem of estimating the image **T** using the photon-correlated setup shown in Fig. 2. The basic estimation problem is stated as follows. Given the observation matrices **N** and **M**, determine the ML estimate of the image **T**. Unfortunately, even with the simplest form of spatial correlation between the pixels of **M** and **N**, the conditional joint PMF of the matrix **N**, given **M**, is analytically intractable due to the fact that the correlation between the pixels of **N** propagates and spreads over the entire dimension of the matrix, which is typically large. To avoid this difficulty, we take a locally optimal estimation approach and consider instead maximizing the *scalar* conditional PMF of the pixel $N(i,j)$ given the knowledge of the immediate neighboring pixels $M(i',j')$, for $i'=i-1,\ldots,i+1$ and $j'=j-1,\ldots,j+1$. We will next develop such a local model and derive an expression for the conditional PMF of each pixel of **N** given **M**.

A. A Model for the Observed Matrices

Let $\mathbf{M}c=\{Mc(i,j)\}$ denote the n-by-n matrix representing the random number of idler-beam photons per pixel per measurement time T that are available *prior* to detection. Moreover, let the matrix $\mathbf{N}c=\{Nc(i,j)\}$ represent the number of signal photons that are transmitted through the object and are impinging on the signal-channel detector array (see Fig. 2). Each of the matrix entries $Mc(i,j)$ and $Nc(i,j)$ is modeled by a Poisson rv with mean values λT and $\lambda Ttij$, respectively, where λ represents the idler and signal photon flux (photons per second per pixel). Furthermore, since each beam is assumed to be in a coherent state, it follows that for $(i',j')\neq(i,j)$, $Mc(i,j)$ and $Mc(i',j')$ are independent rv's and so are $Nc(i,j)$ and $Nc(i',j')$. After detection (with detection quantum efficiency ηs), the signal photon matrix $\mathbf{N}c$ is reduced to the detected number of signal photons, denoted by $\mathbf{N}cd$, whose entries are Poisson rvs with a mean that is a fraction ηs of the mean of the entries of $\mathbf{N}c$. The matrix corresponding to the total number $\mathbf{N}$ of detected counts in the signal channel is given by the sum

$$\mathbf{N} = \mathbf{N}_{cd} + \mathbf{V}_{d} \tag{5}$$

where $\mathbf{V}d$ is the detected background-noise matrix whose entries are independent and identically-distributed (iid) Poisson rvs with mean $\eta s \mu s T$ (per pixel per measurement time T). Similarly, the total number $\mathbf{M}$ of detected idler photons is the sum

$$\mathbf{M} = \mathbf{M}_{cd} + \mathbf{W}_{d}$$

where $\mathbf{M}cd$ is the detected idler photon matrix and $\mathbf{W}d$ is the detected background-noise matrix for the idler channel. The entries of $\mathbf{W}d$ are iid Poisson rvs with mean $\eta i \mu s T$ and each entry of $\mathbf{M}cd$ is also a Poisson rv with mean $\eta i \mu s T$. Here, ηi is the quantum efficiency of the idler-channel detector array. In summary, the entries $N(i,j)$ and $M(i,j)$ of the observed matrices $\mathbf{N}$ and $\mathbf{M}$, respectively, are Poisson rv's with mean values $\eta_s T(t_{ij}\lambda + \mu_s)$ and $\eta_i T(\lambda + \mu_i)$.

1. Random-Translation Model Representing Photon Correlation

According to the physical assumptions on the type correlation between photons which were discussed in Section I, for each idler photon in the (i,j) th pixel of $\mathbf{M}c$, a signal photon impinges on the (i,j) th pixel of the image (with transmittance tij) with probability $1-8\beta$, where β is a known parameter, $0\leq\beta\leq1/8$. Hence, the probability that this signal-twin photon is actually present in (i,j) th pixel of $\mathbf{N}c$ is therefore $(1-8\beta)tij$. (Boundary pixels are treated in a similar way keeping in mind the appropriate neighbors.) Similarly, for each idler photon in the immediate neighbors of the (i,j) th pixel of the idler matrix $\mathbf{M}c$, the probability that its signal twin is present in the (i,j) th pixel of the signal matrix $\mathbf{N}c$ is βtij. The parameter β represents the probability that the twin signal photon appears in a specific pixel of the eight neighboring pixels of the (i,j) th pixel of $\mathbf{N}c$ and it is a measure of deviation from the ideal spatial correlation between the signal and idler photon pairs. For example, the case $\beta=0$ represent the ideal case of perfect spatial correlation.

B. A Local Conditional Model

The goal of this section is to derive an expression for the PMF of each entry of the signal observation matrix $\mathbf{N}$ given knowledge of the idler matrix $\mathbf{M}$. Since the spatial correlation between $\mathbf{N}$ and $\mathbf{M}$ is solely based the correlation between $\mathbf{M}c$ and $\mathbf{N}c$, we begin by developing a stochastic conditional specification of $\mathbf{N}c$ in terms of $\mathbf{M}c$. For brevity of notation, define the single-pixel neighborhood of a pixel (i,j) by

$$D_{ij} = \{(i',j') \colon |i-i'| \leq 1, |j-j'| \leq 1\}.$$

Now if a realization of the random matrix $\mathbf{M}c$ is given by $\{mc(i,j)\}$, then according to the single-pixel random-translation rule described in Section III-A-I, the (i,j) th pixel of the random matrix $\mathbf{N}c$ is given by

$$N_c(i,j) = B_{ij} + C_{ij}, \tag{6}$$

where B_{ij} represents the number of signal photons whose idler twin photons are in the (i,j) th pixel of the idler matrix $\mathbf{M}_c$. On the other hand, C_{ij} is the number of photons whose idler twin photons are in any of the the eight neighboring pixels of the (i,j) th pixel of the idler matrix $\mathbf{M}_c$. Note that if $\beta=0$, then C_{ij} is zero. We now describe the PMF's of B_{ij} and C_{ij}. Recall that for each idler photon in the (i,j) th pixel of $\mathbf{M}_c$, its signal twin will be in the (i,j) th pixel of $\mathbf{N}_c$ with probability $(1-8\beta)t_{ij}$. Now conditional on the fact that there are $m_c(i,j)$ photons in the (i,j) th pixel of $\mathbf{M}_c$, the number of signal photons that are in the (i,j) th pixel of $\mathbf{N}_c$ is a binomial rv with size $m_c(i,j)$ and success probability $(1-8\beta)t_{ij}$. The conditional PMF $P_{B_{ij}|\mathbf{M}_c}(k)$, defined by $\mathrm{P}\{B_{ij}=k|\mathbf{M}_c=\{m_c(i,j)\}\}$, is therefore given by

$$P_{B_{ij}|\mathbf{M}_c}(k) = b\big(k; m_c(i,j), (1-8\beta)t_{ij}\big), k \geq 0, \tag{7}$$

where for any integer K and $0\leq p\leq 1$,

$$b(k;K,p) = \begin{cases} \binom{K}{k} p^k (1-p)^{K-k}, & 0 \leq k \leq K, \\ 0, & \text{otherwise} \end{cases}$$

is a binomial PMF with size K and success probability p. As for the PMF of C_{ij}, note that each idler photon in a neighboring pixel of (i,j) (in $\mathbf{M}_c$) has a signal twin that may be present in the (i,j) th pixel of $\mathbf{N}_c$ with probability βt_{ij}. Since there is a total of

$$s(i,j) = \sum_{(i',j')\in D_{ij},(i',j')\neq(i,j)} m_c\,(i',j') \tag{8}$$

of such idler photons in all the eight neighboring pixels of the (i,j) th pixel, the total number of their signal twin photons C_{ij} is a binomial rv with size $s(i,j)$ and success probability βt_{ij}. Hence, the conditional PMF $P_{C_{ij}|\mathbf{M}_c}(k)$ is given by

$$P_{C_{ij}|\mathbf{M}_c}(k) = b\big(k, s(i,j), \beta t_{ij}\big). \tag{9}$$

To obtain the conditional PMF's of B_{ij} and C_{ij} given $\mathbf{M}$ (in place of the unobservable matrix $\mathbf{M}_c$), we replace $M_c(i',j')$ in (7) and (9) by its estimate $\hat{M}_c(M(i',j'))$ defined as the conditional mean of $M_c(i',j')$ given the knowledge of $M(i',j')$. More precisely, for $\ell\geq 0$,

$$\hat{M}_c(\ell) \triangleq \text{sf E}[M_c(i',j')|M(i',j') = \ell]. \tag{10}$$

An analytical expression for this conditional expectation can be obtained by exploiting the smoothing property of conditional expectations and Bayes' Theorem (see [21], Appendix, second to last equation). We omit the details of the calculation and state the final result as

$$\hat{M}_c(\ell) = \lambda\{(1-\eta_i)T + \frac{\ell}{\lambda+\mu_i}\}.$$

(11)

(Note that in the case when $\eta_i=1$ and $\mu_i=0$, the above conditional expectation reduces to ℓ, as expected, since in this special case $M(i,j)=M_c(i,j)$.) Hence, given that $M(i',j')=m(i',j')$ for $(i',j')\in D_{ij}$, we can use (11) to write analogs of the conditional PMF's given in (7) and (9) with the conditioning being on the knowledge of **M** instead of **M**$_c$. More precisely, we can write estimates of these PMF's as

(12)

$$P_{B_{ij}|\mathbf{M}}(k) = b\left(k; \hat{M}_c(m(i,j)), (1-8\beta)t_{ij}\right) k \geq 0$$

And

(13)

$$P_{C_{ij}|\mathbf{M}}(k) = b\left(k; \hat{s}(i,j), \beta t_{ij}\right), k \geq 0$$

Where

(14)

$$\hat{s}(i,j) = \sum_{(i',j')\in D_{ij},(i',j')\neq(i,j)} \hat{M}_c\,(m(i',j')).$$

Finally, by exploiting the mutual independence of B_{ij}, C_{ij} and the background-noise matrix **V**$_d$, we arrive at the expression for the conditional PMF of $N(i,j)$ given that $M(i,j)=m(i',j')$ for $(i',j')\in D_{ij}$:

$$P_{N(i,j)|\mathbf{M}}(k) = P_{B_{ij}|\mathbf{M}}(k) * P_{C_{ij}|\mathbf{M}}(k) * P_{V_d(i,j)}(k), k \geq 0,$$

(15)

where $P_{V_d(i,j)}(k)$ is the PMF of $V_d(i,j)$ given by

$$P_{V_d(i,j)}(k) = \frac{(\eta_s\mu_s T)^k}{k!}e^{-\eta_s\mu_s T}, k \geq 0$$

(16)

and $*$ denotes discrete convolution.

It is important to point out that the local conditional specification of the matrix **N**$_c$ given **M**$_c$ is consistent with the fact that the entries of **N**$_c$ are independent Poisson rv's, each with mean $\eta_s\lambda T t_{ij}$, as discussed in Section III-A. In fact, it can be shown using generating functions that averaging the conditional joint PMF of **N**$_c$ given **M**$_c$ (which involves the conditional pixel PMF's (7) and (9)) against the joint PMF of **M**$_c$ yields the correct joint PMF for **N**$_c$. (The key observation in the proof is that conditional on **M**$_c$, the entries of **N**$_c$ are independent.) Indeed, the observation that a random-pixel translation operation preserves the Poisson nature of a random matrix resembles the established fact that a random translation of a spatial Poisson process yields another Poisson process with a modified intensity, as proven by Snyder and Miller in [2]. In addition, as a result of the locality of the random translation model, in all the conditional PMF's developed, the conditional knowledge of **M** (or **M***c*) is equivalent to the conditional knowledge of these matrices in the neighborhood of the pixel.

1. Local Maximum-Likelihood Estimator

Suppose that the observed realizations of the signal and idler matrices are given by **N**={*n*(*i*,*j*)} and **M**={*m*(*i*,*j*)}, respectively. Then by maximizing the expression (15) over the unknown *tij*, we obtain the (*i*,*j*) th component, *t^c*,*ij*, of the local ML estimate **T**^*c*. For convenience, we summarize this procedure.

1. Calculate *s*^(*i*,*j*) using (14) and the definition of the function *Mc*^ given in (11).
2. Compute *PBij*|**M**(*k*) and *PCij*|**M**(*k*) using (12) and (13), respectively, with *tij* used as a variable.
3. Carry out the discrete convolution given in (15) to calculate *PN*(*i*,*j*)|**M**(*n*(*i*,*j*)) with *tij* used as a variable.
4. Maximize *PN*(*i*,*j*)|**M**(*n*(*i*,*j*)) over *tij* in the range [0,1]. The maximizing *tij* is the local ML estimate *t^c*,*ij*(*n*(*i*,*j*),{*m*(*i*,*j*)}).

In this paper, the maximization in the last step was done by exhaustion using a modular C++ program. In particular, for each (*i*,*j*), the arrays *PBij*|**M**(*k*) and *PCij*|**M**(*k*) were recalculated for each new value of *tij*, as *tij* was increased from zero in steps of 10−4. To make efficient use of memory, the size of the arrays (i.e., the range of *k*) was selected dynamically for each trial value of *tij*, and the arrays were terminated when negligibly small values were reached. Then, for each trial value of *tij*, (3) was executed and the error *PN*(*i*,*j*)|**M**(*n*(*i*,*j*)) was calculated and bubble sorted in search of the error-minimizing *t^c*,*ij*. The entire procedure was then repeated for every (*i*,*j*).

As in Section II, we assess the performance of the local ML estimator **T**^*c* by means of computing the averaged mean-square error

$$\begin{aligned}\varepsilon_c^2 &\triangleq \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \text{sf E}\,[\{\hat{t}_{c,ij}(N(i,j),\mathbf{M})-t_{ij}\}^2]\\ &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \text{sf E}\,[\text{sf E}[\{\hat{t}_{c,ij}(N(i,j),\mathbf{M})-t_{ij}\}^2|\mathbf{M}]]\\ &= \frac{1}{n^2}\sum_{\mathbf{q}}[\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=0}^{\infty}(\hat{t}_{c,ij}(k,\mathbf{q})-t_{ij})^2 P_{N(i,j)|\mathbf{M}}(k)]\\ &\quad\times P_{\mathbf{M}}(\mathbf{q}).\end{aligned} \tag{17}$$

(Note that the joint PMF *P***M**(**q**) is simply a product of *n*2 Poissonian PMFs.) Some examples are considered in Section IV.

Fig. 3. True image to be used in the performance analysis of the conventional and photon-correlated estimators.

SECTION IV.

Discussion

In this section, we compare the performance of the proposed local ML estimator for the photon-correlated setup to the classical single-beam ML estimator $\hat{\mathbf{T}}_s$. The average mean-square errors ε^2_s and ε^2_c, respectively given by (4) and (17), are used as measures of performance. For convenience, we also define the performance factor $\rho=\varepsilon^2_c/\varepsilon^2_s$ as a relative measure of performance advantage. For simplicity, we will assume identical signal- and idler-channel background-noise flux μ, and identical quantum efficiency η for the signal and idler detector arrays. The predicted performance advantage is shown in terms of certain key parameters including

- mean number of photons per pixel per measurement time defined by $n_p=\lambda T$, which is a measure of the optical energy used to generate the measurements;
- photon correlation parameter β;
- background-noise mean photon number $n_b=\mu T$;
- detection quantum efficiency η.

To demonstrate our results, we considered a 128 by 128 eight-bit gray-scale image shown in Fig. 3. The pixel values were normalized so that they are in the interval [0,1] (corresponding to transmittance values). This normalized image constitutes the transmittance matrix **T**.

TABLE I Comparison Between the Performance of the Conventional and the Photon-Correlated Estimates in the Absence of Background Noise (nb=0) and Under IdealDetection Conditions (η=1)

	$\beta = 0$			$\beta = 0.01$	$\beta = 0.0375$	$\beta = 0.075$
n_p	ε_s^2	ε_c^2	ρ	ρ	ρ	ρ
10	0.0272	0.0237	0.871	0.879	1.018	1.121
20	0.0137	0.0092	0.672	0.737	0.876	1.949
30	0.0092	0.0061	0.663	0.728	0.859	0.978
50	0.0055	0.0037	0.673	0.727	0.855	0.982

		β=0		β=0.01	β=0.0375	β=0.075
n_p	ε^2_s	ε^2_c	ρ	ρ	ρ	ρ
10	0.0272	0.0237	0.871	0.879	1.018	1.121
20	0.0137	0.0092	0.672	0.737	0.876	1.949
30	0.0092	0.0061	0.663	0.728	0.859	0.978
50	0.0055	0.0037	0.673	0.727	0.855	0.982

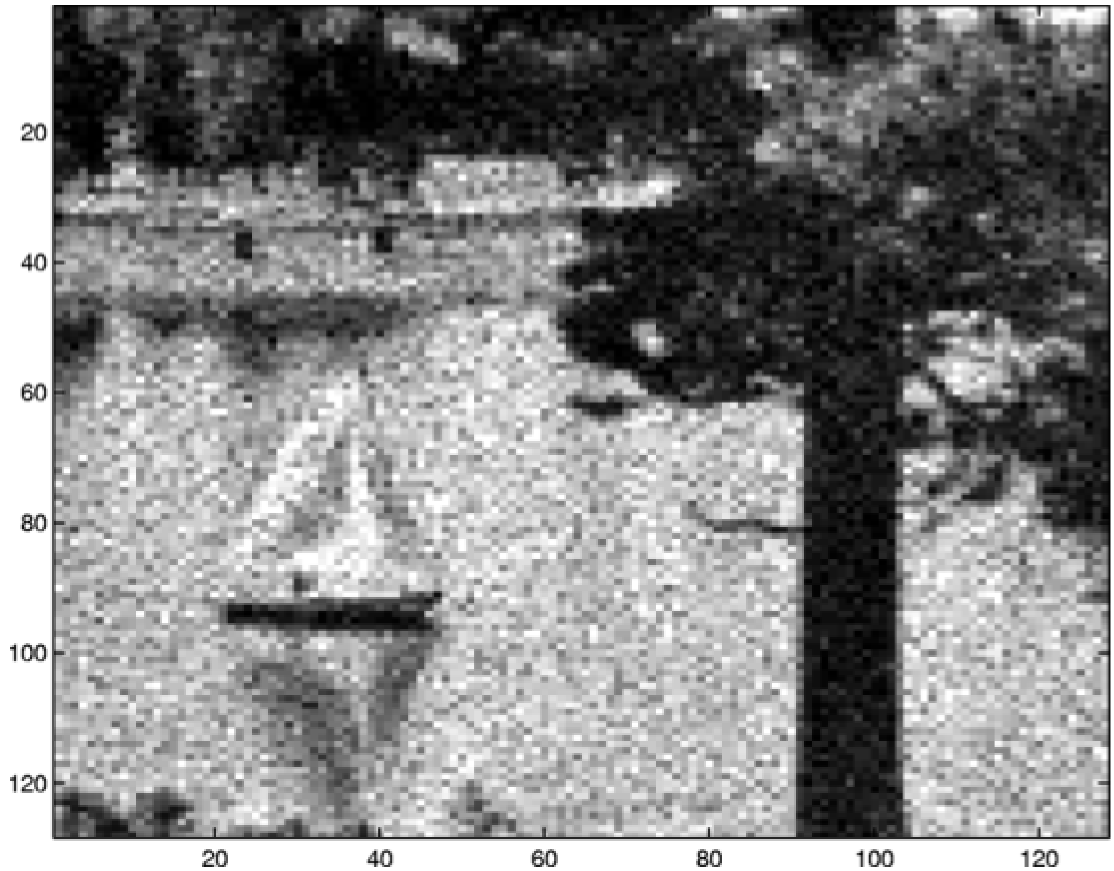

Fig. 4. ML estimate of the true image using the conventional setup. Ideal conditions of background noise (*nb*=0) and quantum efficiency (η=1) are assumed. The signal parameter *np* is 50.

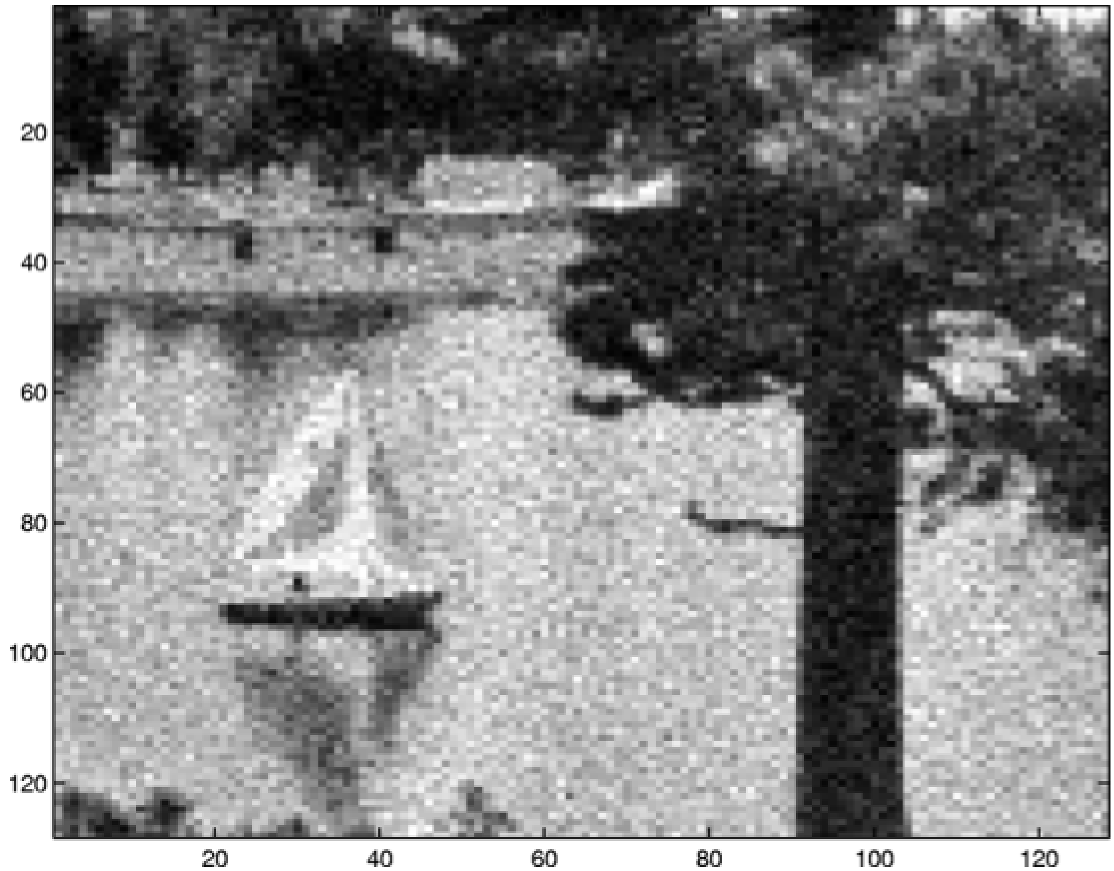

Fig. 5. Local ML estimate using photon-correlated beams. The photon correlation parameter β is 0.01. The remaining parameters are the same as those in Fig. 4.

In addition to calculating the theoretical errors, we also simulated the stochastic model for the entire process of transmission and detection of photons for both the conventional and photon-correlated setups and generated the estimated images for each estimator. We will describe the procedure for the simulation of photon-correlated setup only, simulation of the conventional setup is straightforward. We first generated a realization of a Poisson matrix representing the idler-channel photon count matrix $\mathbf{M}c$. We then created the signal-channel matrix $\mathbf{N}c$ by randomly perturbing $\mathbf{M}c$ according to the one-step random translation model described in Section III-A-I. To simulate the random deletion of photons due to transmission through the object and nonideal detection, each pixel count $Nc(i,j)$ was subsequently replaced with a realization of a binomial random variable with size $Nc(i,j)$ and success probability $tij\eta$. (Recall that the probability that a photon is transmitted through the object *and* detected by the array detector is the product $tij\eta$.) This completes the generation of a realization of the count matrix $\mathbf{N}cd$. We then added to $\mathbf{N}cd$ an independently generated Poisson noise matrix to yield the observation matrix $\mathbf{N}$. To generate a simulation of the idler observation matrix $\mathbf{M}$, we subjected the random matrix $\mathbf{M}c$ through a random deletion process (representing nonideal detection) and added to it Poisson noise matrix $\mathbf{W}d$. The estimate $\hat{\mathbf{T}}_c$ was then computed using the realizations of $\mathbf{N}$ and $\mathbf{M}$ as inputs to the photon-correlated estimators as outlined in Section III-B1. These estimates were subsequently re-scaled to form an eight-bit image.

Fig. 6. Conventional ML estimate in the presence of background noise under ideal detection conditions (η=1). The background-noise parameter *nb* is five and the signal parameter *np* is 50.

Fig. 7. Local ML estimate using photon-correlated beams in the presence of background noise when the photon correlation parameter β is 0.01. The remaining parameters are the same as those in Fig. 6.

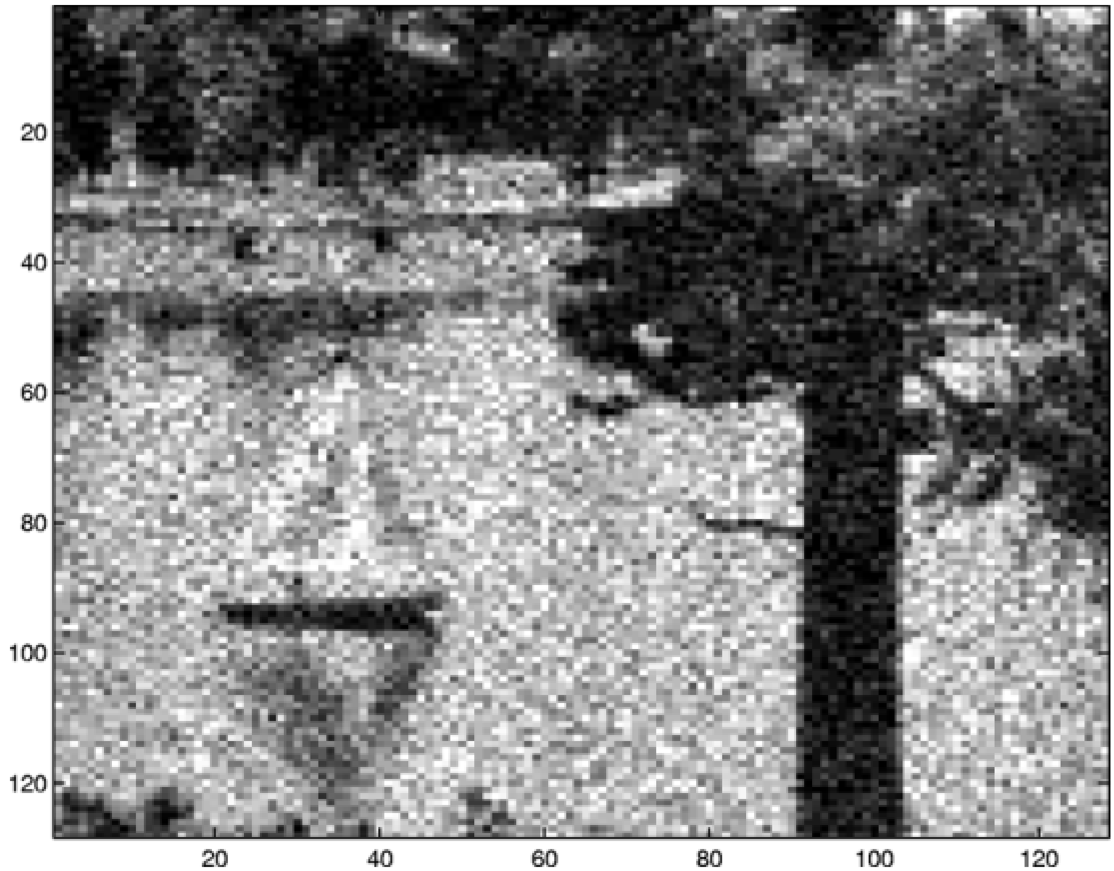

Fig. 8. Conventional ML estimate in the presence of background noise and under conditions of nonideal detection. The background-noise parameter *nb* and the signal parameter *np* are 5 and 50, respectively. The detector quantum efficiency η is 0.8.

Fig. 9. The local ML estimate using photon-correlated beams in the presence of background noise and under conditions of nonideal detection. The photon correlation parameter β is 0.01. The remaining parameters are the same as those in Fig. 8.

A. Dependence of the Performance on the Strength of Correlation Between Photons

Table I summarizes the results for the ideal case for which no background noise is present (μ=0) and the detectors are ideal (η=1). This special case serves as a benchmark for the maximum improvement possible in image estimation offered by the photon-correlated setup. In the case of perfect photon correlation (i.e., when β=0), the estimation of the unknown transmittance matrix given the observed and reference images reduces to the case of estimating the transmittance of a pixel given the signal and idler pixels. This special case had been previously investigated in [21] and the results there are consistent with the results in the "β=0" column of Table I. When β=0, the improvement factor ρ becomes approximately 0.7 as *np* increases, which corresponds to an error reduction of approximately 1.6 dB. As expected, when the correlation between the reference and signal photon counts becomes weaker, the advantage of the photon-correlated scheme becomes less significant. In fact, if the correlation parameter is β=0.075, the photon-correlated scheme is at a disadvantage for low values of *np*. The fact that ρ is greater than unity may seem contradictory at first. However, we recall that the photon-correlated estimate **T**^*c* is a *locally* optimal estimator and it is not the global ML estimator. Moreover, the optimality is in the sense of maximizing the conditional PMF, which does not necessarily yield a minimum mean-square-error estimate.

TABLE II Comparison Between the Performance of the Conventional and the Photon-Correlated Estimates in the Presence of Background Noise (*nb*=5) and Under Conditions of Ideal Detection (η=1)

	$\beta = 0$			$\beta = 0.01$	$\beta = 0.0375$	$\beta = 0.075$
n_p	ε_s^2	ε_c^2	ρ	ρ	ρ	ρ
10	0.0601	0.0707	1.176	1.181	1.119	1.126
20	0.0244	0.0241	0.989	1.009	1.052	1.135
30	0.0142	0.0122	0.856	0.901	0.978	1.081
50	0.0074	0.0058	0.787	0.828	0.940	1.340

		β=0		β=0.01	β=0.0375	β=0.075
n_p	ε^2_s	ε^2_c	ρ	ρ	ρ	ρ
10	0.0601	0.0707	1.176	1.181	1.119	1.126
20	0.0244	0.0241	0.989	1.009	1.052	1.135
30	0.0142	0.0122	0.856	0.901	0.978	1.081
50	0.0074	0.0058	0.787	0.828	0.940	1.340

TABLE III Comparison Between the Performance of the Conventional and the Photon-Correlated Estimates in the Presence of Background Noise (*nb*=5) and Under Conditions of Nonideal Detection (η=0.8)

	$\beta = 0$			$\beta = 0.01$	$\beta = 0.0375$	$\beta = 0.075$
n_p	ε_s^2	ε_c^2	ρ	ρ	ρ	ρ
10	0.0709	0.0854	1.205	1.174	1.145	1.196
20	0.03	0.0298	0.993	1.038	1.073	1.13
30	0.0176	0.0158	0.896	0.917	1.015	1.118
50	0.0092	0.0078	0.841	0.870	0.961	1.075

		β=0		β=0.01	β=0.0375	β=0.075
n_p	ε^2_s	ε^2_c	ρ	ρ	ρ	ρ
10	0.0709	0.0854	1.205	1.174	1.145	1.196
20	0.03	0.0298	0.993	1.038	1.073	1.13
30	0.0176	0.0158	0.896	0.917	1.015	1.118
50	0.0092	0.0078	0.841	0.870	0.961	1.075

In general, the improvement factor ρ decreases as a function of *np*. This behavior is also consistent with the results reported in [21] asserting that the improvement becomes more significant as *np* increases initially but then levels off at a predetermined value that depends on the true transmittance values in the image and other system parameters. Figs. 4 and 5 show realizations of the estimates of the transmittance matrix **T** using the conventional and photon-correlated schemes, respectively. The reduction in quantum noise in Fig. 5 is evident.

B. Effect of Photodetection Noise, Transmission Noise, and Background Noise
The effect of background noise on the performance advantage is seen in Table II. Although the presence of background noise reduces the performance advantage of the photon-correlated scheme for low signal and idler photon counts (e.g., *np* below 20), this effect becomes insignificant as the signal-to-noise ratio *np*/*nb* increases (e.g., when *np* is beyond 30 photons and *nb*=5). Figs. 6 and 7 show realizations of estimates of the transmittance matrix **T** using the conventional and photon-correlated setups, respectively. Finally, the effect of nonideal photodetection can be seen from Table III where the detectors' quantum efficiency is assumed as η=0.8. Although the performance improvement factor can be incrementally reduced by increasing *np*, an eventual performance advantage may not be possible at low transmittance values. This result is similar to the limitation that nonideal detection imposes on uniform transmittance estimation described in [20] and [21]. When the quantum efficiency of the detectors is not sufficiently high (e.g., η<0.5 as described in [20] and [21]), the likelihood of detecting both twin photons decreases resulting in a reduced correlation between the number of detected photons in the idler and signal beams. Figs. 8 and 9 show realizations of the estimates in this case. A moderate improvement is seen in the photon-correlated estimate.

SECTION V.

Conclusion

This paper introduces an image estimator for a class of computed-imaging problems that utilize a form of nonclassical light known as photon-correlated beams. The paper investigates the possibility of improving the performance of image estimation beyond the classical fundamental limits dictated by quantum noise. What makes the light source nonclassical is that the photon arrival times and locations of the two beams are, under ideal conditions, perfectly synchronized. One of the beams is used to probe the image and the transmitted photons are detected by means of an array detector. The other beam is detected directly and used to provide information on the number of photons impinging on each pixel of the image. Through this added information, uncertainty in the image estimation due to fluctuation in the number of photons can be reduced.

We have developed a stochastic model that locally characterizes the joint statistics of the two observation matrices and derived a local ML estimator for the true image. The theory includes the effects of background noise, photodetection noise, and the deviation from the ideal spatial synchronization (correlation) between the photons of the two beams. This latter effect is captured by a single-pixel random translation mechanism. By computing the mean-square error, it is shown that a moderate improvement in the performance is possible relative to the conventional single-beam ML estimator. The possibility of performance advantage is shown in terms of key parameters such as the mean optical energy per measurement time per pixel, detector quantum efficiency, the strength of spatial correlation between the photons of the two beams, and the level of background noise. For example, under conditions of perfect photon correlation, ideal photodetection and in the absence of background noise, a reduction in the mean-square error up to 30% below that associated with the conventional single-beam ML estimate is possible. In general, the relative improvement becomes more significant as the optical energy increases and it will eventually plateau at a level which depends on the system parameters. However, as the correlation between twin photons becomes weak, the performance of the photon-correlated estimator may become inferior to the conventional ML estimator. This is intuitively expected since when the correlation between the photons of the two beams is weak, the additional information provided by the idler beam on the signal-beam photon number becomes unreliable and will in fact obscure our knowledge of the number of photons impinging on the object. A similar performance degradation is seen when the quantum efficiency of detection is low (i.e., below 0.5). Albeit, under reasonable system parameters such as a signal-to-noise ratio of ten, detector quantum efficiency of 0.8, and an average of 50 photons per pixel per detection time, an improvement of 13% below the conventional limit is predicted under moderate levels of photon correlation.

The theory can be easily extended to suit situations where the uncertainty in the spatial correlation extends to more than one pixel in which case the single-pixel random translation model can be extended to multiple pixels.

ACKNOWLEDGMENT

The authors wish to thank Prof. R. C. Hardie for many valuable suggestions.

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