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M. H. Tahir
Islamia University, Bahawalpur

Muhammad Zubair
Government Degree College Kahrorpacca

M. Mansoor
University of Bahawalpur-Pakistan

Gauss M. Cordeiro
Universidade Federal de Pernambuco-Brazil

Morad Alizadeh
Ferdowsi University of Mashhad

See next page for additional authors

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A NEW WEIBULL-G FAMILY OF DISTRIBUTIONS

M.H.Tahir[*,†], M. Zubair[‡] , M. Mansoor[§] , Gauss M. Cordeiro[¶] , Morad Alizadeh[‖] and G. G. Hamedani[**]

Abstract

Statistical analysis of lifetime data is an important topic in reliability engineering, biomedical and social sciences and others. We introduce a new generator based on the Weibull random variable called the new Weibull-G family. We study some of its mathematical properties. Its density function can be symmetrical, left-skewed, right-skewed, bathtub and reversed-J shaped, and has increasing, decreasing, bathtub, upside-down bathtub, J, reversed-J and S shaped hazard rates. Some special models are presented. We obtain explicit expressions for the ordinary and incomplete moments, quantile and generating functions, Rényi entropy, order statistics and reliability. Three useful characterizations based on truncated moments are also proposed for the new family. The method of maximum likelihood is used to estimate the model parameters. We illustrate the importance of the family by means of two applications to real data sets.

Keywords: Generating function, hazard function, moment, reliability function, Rényi entropy, Weibull distribution.

2000 AMS Classification: 60E05; 62E10; 62N05.

1. Introduction

Broadly speaking, there has been an increased interest in defining new generators for univariate continuous families of distributions by introducing one or more additional shape parameter(s) to the baseline distribution. This induction of parameter(s) has been proved useful in exploring tail properties and also for improving the goodness-of-fit of the family under study. The well-known generators are the following: beta-G by Eugene *et al.* [18], Kumaraswamy-G (Kw-G) by Cordeiro

[*]Department of Statistics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan, Email: `mtahir.stat@gmail.com, mht@iub.edu.pk`

[†]Corresponding Author.

[‡]Department of Statistics, Government Degree College Khairpur Tamewali, Pakistan, Email: `zubair.stat@yahoo.com`

[§]Department of Statistics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan, Email: `mansoor.abbasi143@gmail.com`

[¶]Department of Statistics, Federal University of Pernambuco, 50740-540, Recife, PE, Brazil, Email: `gauss@de.ufpe.br, gausscordeiro@gmail.com`

[‖]Department of Statistics, Ferdowsi University of Mashhad, Mashhad, P.O. Box. 91775-1159, Iran, Email: `moradalizadeh78@gmail.com`

[**]Department of Mathematics, Statistics and Computer Science, Marquette University, WI 53201-1881, Milwaukee, USA, Email: `g.hamedani@mu.edu`

and de Castro [12], McDonald-G (Mc-G) by Alexander *et al.* [1], gamma-G type 1 by Zografos and Balakrishanan [29] and Amini *et al.* [7], gamma-G type 2 by Ristić and Balakrishanan [26] and Amini *et al.* [7], odd exponentiated generalized (odd exp-G) by Cordeiro *et al.* [14], transformed-transformer (T-X) (Weibull-X and gamma-X) by Alzaatreh *et al.* [4], exponentiated T-X by Alzaghal *et al.* [6], odd Weibull-G by Bourguignon *et al.* [8], exponentiated half-logistic by Cordeiro *et al.* [11], T-X{Y}-quantile based approach by Aljarrah *et al.* [3], T-R{Y} by Alzaatreh *et al.* [5], Lomax-G by Cordeiro *et al.* [15], logistic-X by Tahir *et al.* [28] and Kumaraswamy odd log-logistic-G by Alizadeh *et al.* [2].

Let $r(t)$ be the probability density function (pdf) of a random variable $T \in [a,b]$ for $-\infty < a < b < \infty$ and let $F(x)$ be the cumulative distribution function (cdf) of a random variable X such that the link function $W(\cdot) : [0,1] \longrightarrow [a,b]$ satisfies the two conditions: (i) $W(\cdot)$ is differentiable and monotonically non-decreasing, and (ii) $W(0) \to a$ and $W(1) \to b$. If the interval $[a,b]$ is half-open or open, we replace $W(0)$ and/or $W(1)$ for $\lim_{t\to 0^+} W(t) \to a$ and $\lim_{t\to 1^-} W(t) \to b$.

Recently, Alzaatreh *et al.* [4] defined the *T-X family* of distributions by

$$F(x) = \int_a^{W[G(x)]} r(t)\,dt, \tag{1.1}$$

where $W[G(x)]$ satisfies the conditions (i) and (ii). If $T \in (0,\infty)$, X is a continuous random variable and $W[G(x)] = -\log[1-G(x)]$, then the pdf corresponding to (1.1) is given by

$$f(x) = \frac{g(x)}{1-G(x)}\, r(-\log[1-G(x)]) = h_g(x)\, r[H_g(x)], \tag{1.2}$$

where $h_g(x)$ and $H_g(x)$ are the hazard and cumulative hazard functions associated to $g(x)$, respectively.

The Weibull distribution is one of the most popular and widely used model for failure time in life-testing and reliability theory. However, a drawback of this distribution as far as lifetime analysis is concerned is the monotonic behaviour of its hazard date function (hrf). In real life applications, empirical hazard rate curves often exhibit non-monotonic shapes such as a bathtub, upside-down bathtub (unimodal) and others. So, there is a genuine desire to search for some generalizations or modifications of the Weibull distribution that can provide more flexibility in lifetime modeling.

If a random variable T has the Weibull distribution with scale parameter $\alpha > 0$ and shape parameter $\beta > 0$, then its cdf and pdf are, respectively, given by

$$F_W(t) = 1 - \mathrm{e}^{-\alpha t^\beta}, \quad t > 0$$

and

$$f_W(t) = \alpha\, \beta\, t^{\beta-1}\, \mathrm{e}^{-\alpha t^\beta}, \quad t > 0. \tag{1.3}$$

In the recent literature, four Weibull based generators have appeared, namely: the beta Weibull-G by Cordeiro *et al.* [16], the Weibull-X by Alzaatreh *et al.* [4], the Weibull-G by Bourguignon *et al.* [8] and the exponentiated Weibull-X by Alzaghal *et al.* [6].

If $r(t)$ follows (1.3) and setting $W[G(x)] = -\log[1 - G(x)]$ in (1.1), Alzaatreh *et al.* [4] defined the cdf of the Weibull-X family by

$$(1.4) \qquad F(x) = \alpha\,\beta \int_0^{-\log[1-G(x)]} x^{\beta-1}\,\mathrm{e}^{-\alpha x^\beta}\,dt = 1 - \mathrm{e}^{-\alpha\{-\log[1-G(x)]\}^\beta}.$$

The pdf corresponding to (1.4) is

$$(1.5) \qquad f(x) = \alpha\,\beta\,\frac{g(x)}{1-G(x)}\,\mathrm{e}^{-\alpha(-\log[1-G(x)])^\beta}\,\{-\log[1-G(x)]\}^{\beta-1}.$$

Zagrafos and Balakrishnan [29] pioneered a versatile and flexible gamma-G class of distributions based on Stacy's generalized gamma distribution and record value theory. More recently, Bourguignon *et al.* [8] proposed the *Weibull-G* family of distributions influenced by the Zografos-Balakrishnan-G class. Bourguignon *et al.* [8] replaced the argument x by $G(x;\Theta)/\overline{G}(x;\Theta)$, where $\overline{G}(x;\Theta) = 1 - G(x;\Theta)$, and defined the cdf of their class (for $\alpha > 0$ and $\beta > 0$), say Weibull-G(α,β,Θ), by

$$(1.6) \qquad F(x;\alpha,\beta,\Theta) = \alpha\,\beta \int_0^{\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]} x^{\beta-1}\,\mathrm{e}^{-\alpha x^\beta}\,dx = 1 - \mathrm{e}^{-\alpha\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^\beta},\; x \in \Re.$$

The Weibull-G family density function becomes

$$(1.7) \qquad f(x;\alpha,\beta,\Theta) = \alpha\,\beta\,g(x;\Theta)\left[\frac{G(x;\Theta)^{\beta-1}}{\overline{G}(x;\Theta)^{\beta+1}}\right]\mathrm{e}^{-\alpha\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^\beta},$$

where $G(x;\Theta)$ and $g(x;\Theta)$ are the cdf and pdf of any baseline distribution that depend on a parameter vector Θ.

In this paper, we propose a class of distributions called the *new Weibull-G* ("NWG" for short) family, which is flexible because of the hazard rate shapes: constant, increasing, decreasing, bathtub, upside-down bathtub, J, reversed-J and S. The paper unfolds as follows. In Section 2, we define the new family. Six special models are presented in Section 3. The forms of the density and hazard rate functions are described analytically in Section 4. In Section 5, we obtain explicit expressions for the quantile function (qf), ordinary and incomplete moments, generating function and entropies. In Sections 6 and 7, we investigate the order statistics and the reliability. Section 8 refers to some characterizations of the NWG family. In Section 9, the parameters of the new family are estimated by the method of maximum likelihood. In Section 10, we illustrate its performance by means of two applications to real data sets. The paper is concluded in Section 11.

2. The new family

In equation (1.1), let $a = 0$, $r(t)$ be as in (1.3) and $W[G(x)] = -\log[G(x;\boldsymbol{\xi})]$. Then, we define the cdf of the NWG family by

$$(2.1) \qquad F(x;\alpha,\beta,\boldsymbol{\xi}) = 1 - \int_0^{-\log[G(x;\boldsymbol{\xi})]} \alpha\,\beta\,t^{\beta-1}\,\mathrm{e}^{-\alpha t^\beta}\,dt = \mathrm{e}^{-\alpha\{\,-\log[G(x;\boldsymbol{\xi})]\,\}^\beta}.$$

Hereafter, a random variable X with cdf (2.1) is denoted by $X \sim \mathrm{NWG}(\alpha,\beta,\boldsymbol{\xi})$. We can motivate equation (2.1) based on linearization of the baseline cdf $G(x;\boldsymbol{\xi})$

as follows. Let Y be a Weibull random variable with scale parameter $\alpha > 0$ and shape parameter $\beta > 0$. The extreme value random variable V can be defined as minus the log of the Weibull random variable, say $V = -\log(Y)$. It gives the limiting distribution for the smallest or largest values in samples drawn from a variety of distributions. The NWG random variable having cdf (2.1) can be derived by

$$P(X \geq x) = P\Big(Y \geq -\log[G(x;\boldsymbol{\xi})]\Big) = P\Big(V \leq -\log\{-\log[G(x;\boldsymbol{\xi})]\}\Big),$$

where $-\log\{-\log[G(x;\boldsymbol{\xi})]\}$ is a simple linearization of the baseline cdf.

Then, the pdf of X reduces to

$$f(x;\alpha,\beta,\boldsymbol{\xi}) = \alpha\beta\,\frac{g(x;\boldsymbol{\xi})}{G(x;\boldsymbol{\xi})}\,\Big\{-\log\left[G(x;\boldsymbol{\xi})\right]\Big\}^{\beta-1}\,\mathrm{e}^{-\alpha\{-\log[G(x;\boldsymbol{\xi})]\}^{\beta}}, \tag{2.2}$$

where $g(x;\boldsymbol{\xi})$ is the parent pdf. Further, we can omit sometimes the dependence on the vector $\boldsymbol{\xi}$ of the parameters and write simply $G(x) = G(x;\boldsymbol{\xi})$ and $g(x) = g(x;\boldsymbol{\xi})$. Equation (2.2) will be most tractable when the cdf $G(x)$ and pdf $g(x)$ have simple analytic expressions.

The quantile function (qf) of X is obtained by inverting (2.1). We have

$$X = Q(u) = Q_G\left(\mathrm{e}^{-t(u)};\boldsymbol{\xi}\right),$$

where $Q_G(\cdot;\cdot) = G^{-1}(\cdot;\cdot)$ is the baseline qf and $t(u) = [\log(1/u)^{1/\alpha}]^{1/\beta}$. Then, if U has a uniform distribution on $(0,1)$, $X = Q(U)$ follows the NWG$(\alpha,\beta,\boldsymbol{\xi})$ family.

Let $h(x;\boldsymbol{\xi})$ be the hrf of the parent G. The hrf $h(x;\alpha,\beta,\boldsymbol{\xi})$ of X is given by

$$h(x;\alpha,\beta,\boldsymbol{\xi}) = \frac{\alpha\beta\frac{g(x;\boldsymbol{\xi})}{G(x;\boldsymbol{\xi})}\,\{-\log\left[G(x;\boldsymbol{\xi})\right]\}^{\beta-1}\,\mathrm{e}^{-\alpha\{-\log\left[G(x;\boldsymbol{\xi})\right]\}^{\beta}}}{1-\mathrm{e}^{-\alpha\{-\log\left[G(x;\boldsymbol{\xi})\right]\}^{\beta}}}.$$

3. Special models

In this section, we provide six special models of the NWG distributions. Suppose that the parent distribution is uniform on the interval $(0,\theta)$, $\theta > 0$. We have $g(x;\theta) = 1/\theta$, $0 < x < \theta < \infty$, and $G(x;\theta) = x/\theta$, and then the Weibull-uniform (WU) cdf is given by

$$F_{WU}(x;\alpha,\beta,\theta) = \mathrm{e}^{-\alpha\left[-\log\left(\frac{x}{\theta}\right)\right]^{\beta}}, \quad 0 < x < \theta < \infty \quad \alpha,\beta,\theta > 0.$$

Now, take the parent distribution as Weibull with pdf and cdf given by $g(x) = \lambda\,\gamma\,x^{\gamma-1}\,\mathrm{e}^{-\lambda\,x^{\gamma}}$ and $G(x) = 1-\mathrm{e}^{-\lambda\,x^{\gamma}}$ for $\lambda,\gamma > 0$. Then, the Weibull-Weibull (WW) cdf becomes

$$F_{WW}(x;\alpha,\beta,\lambda,\gamma) = \mathrm{e}^{-\alpha\left[-\log\left(1-\mathrm{e}^{-\lambda\,x^{\gamma}}\right)\right]^{\beta}}, \quad x > 0, \quad \alpha,\beta,\lambda,\gamma > 0.$$

For $\gamma = 1$ and $\gamma = 2$, we obtain as special cases the Weibull-exponential (WE) and Weibull-Rayleigh (WR) distributions, respectively.

For the Weibull-logistic (WLo) distribution, we have $g(x) = \lambda\,\mathrm{e}^{-\lambda\,x}\left(1+\mathrm{e}^{-\lambda\,x}\right)^{-2}$ and $G(x) = \left(1+\mathrm{e}^{-\lambda\,x}\right)^{-1}$. Then, the WLo cdf reduces to

$$F_{WLo}(x;\alpha,\beta,\lambda) = \mathrm{e}^{-\alpha\left\{-\log\left[\left(1+\mathrm{e}^{-\lambda\,x}\right)^{-1}\right]\right\}^{\beta}}, \quad x > 0, \quad \alpha,\beta,\lambda > 0.$$

Consider the parent log-logistic distribution with parameters $s > 0$ and $c > 0$ given by $g(x; s, c) = c\, s^{-c}\, x^{c-1} \left[1 + \left(\frac{x}{s}\right)^c\right]^{-2}$ and $G(x; s, c) = 1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-1}$. Then, the Weibull-log-logistic (WLL) cdf becomes

$$F_{WLL}(x; \alpha, \beta, s, c) = \mathrm{e}^{-\alpha\left[-\log\left\{1-\left[1+\left(\frac{x}{s}\right)^c\right]^{-1}\right\}\right]^\beta}, \quad x > 0, \quad \alpha, \beta, s, c > 0.$$

We take the parent Burr XII distribution with pdf and cdf given by $g(x) = c\, k\, s^{-c}\, x^{c-1} \left[1 + (x/s)^c\right]^{-k-1}$ and $G(x) = 1 - \left[1 + (x/s)^c\right]^{-k}$. Then, the Weibull-Burr XII (WBXII) cdf reduces to

$$F_{WBXII}(x; \alpha, \beta, s, c, k) = \mathrm{e}^{-\alpha\left[-\log\left\{1-\left[1+\left(\frac{x}{s}\right)^c\right]^{-k}\right\}\right]^\beta}, \quad x > 0, \quad \alpha, \beta, s, c, k > 0.$$

For $c = 1$ and $k = 1$, we obtain as a special case the Weibull-Lomax (WLx) distribution.

Finally, if we consider the baseline normal distribution, the pdf and cdf are $g(x; \mu, \sigma) = \sigma^{-1}\, \phi\left[(x - \mu)/\sigma\right]$ and $G(x; \mu, \sigma) = \Phi\left[(x - \mu)/\sigma\right]$. Then, the Weibull-normal (WN) cdf becomes (for $x \in \Re$)

$$F_{WN}(x; \alpha, \beta, \mu, \sigma) = \mathrm{e}^{-\alpha\left[-\log\left\{\Phi\left(\frac{x-\mu}{\sigma}\right)\right\}\right]^\beta}, \quad x \in \Re, \quad \alpha, \beta, \sigma > 0, \mu \in \Re.$$

The density of the new family can be symmetrical, left-skewed, right-skewed, bathtub and reversed-J shaped, and has constant, increasing, decreasing, bathtub, upside-down bathtub, J, reversed J and S shaped hazard rates. In Figures 1 and 2, we display some plots of the pdf and hrf of (a) WU, (b) WW, (c) WLL, (d) WLo, (e) WBXII and (f) WN distributions for selected parameter values. Figure 1 indicates that the NWG family generates distributions with various shapes such as symmetrical, left-skewed, right-skewed, bathtub and reversed-J. Also, Figure 2 reveals that this family can produce flexible hazard rate shapes such as increasing, decreasing, bathtub, upside-down bathtub, J, reversed-J and S. This fact implies that the NWG family can be very useful to fit different data sets with various shapes.

4. Shapes of the pdf and hrf

The shapes of the density and hazard rate functions can be described analytically. The critical points of the NWG density are the roots of the equation:

$$\frac{g'(x)}{g(x)} + \frac{g(x)}{G(x)} + \frac{(1-\beta)g(x)}{G(x)\log\left[G(x)\right]} + \frac{\alpha\beta\, g(x)\left\{-\log\left[G(x)\right]\right\}^{\beta-1}}{G(x)} = 0. \tag{4.1}$$

The critical points of $h(x)$ are obtained from the equation

$$\begin{aligned} &\frac{g'(x)}{g(x)} + \frac{g(x)}{G(x)} + \frac{(1-\beta)g(x)}{G(x)\log\left[G(x)\right]} + \frac{\alpha\beta\, g(x)\left\{-\log\left[G(x)\right]\right\}^{\beta-1}}{G(x)} \\ &+ \frac{\alpha\beta\, g(x)\left\{-\log\left[G(x)\right]\right\}^{\beta-1}\mathrm{e}^{-\alpha\left\{-\log[\mathrm{G(x)}]\right\}^\beta}}{G(x)\left[1-\mathrm{e}^{-\alpha\left\{-\log[\mathrm{G(x)}]\right\}^\beta}\right]} = 0. \end{aligned} \tag{4.2}$$

By using most symbolic computation software platforms, we can examine equations (4.1) and (4.2) to determine the local maximums and minimums and inflexion points.

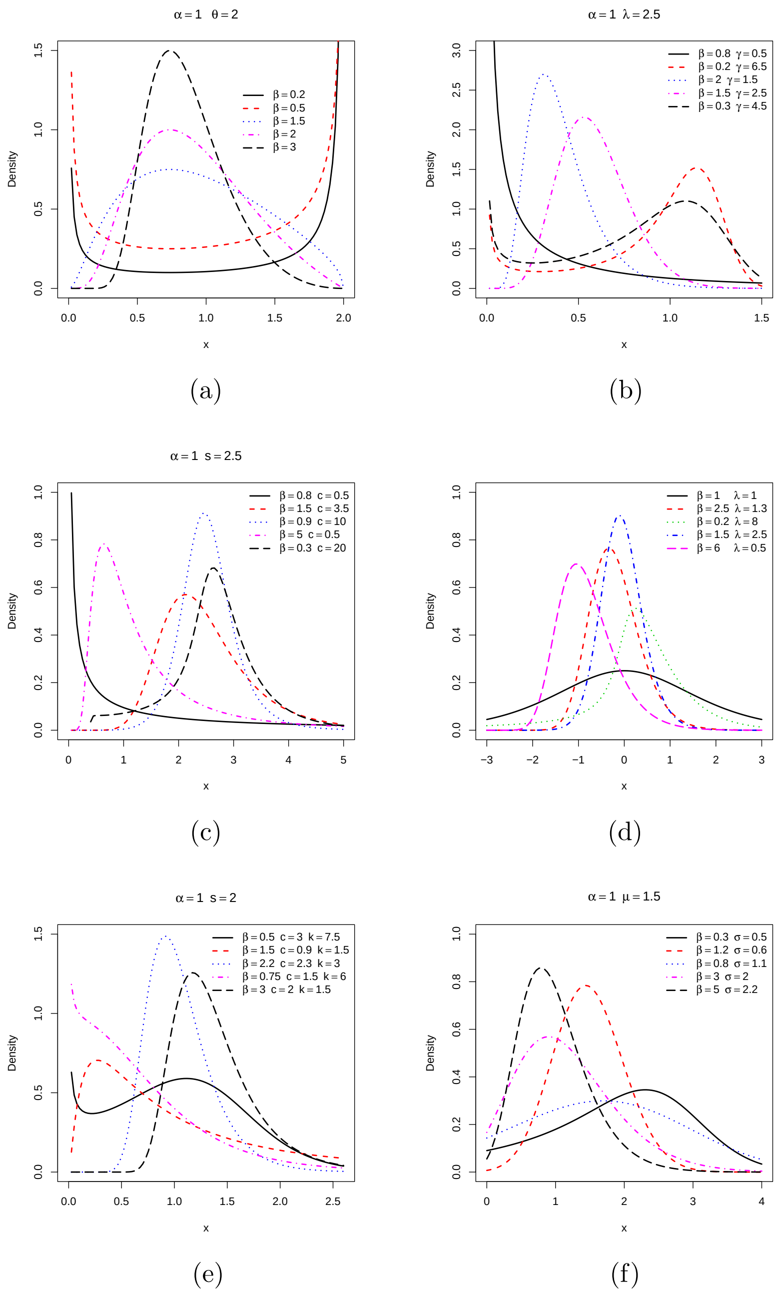


Figure 1: Plots of the (a) WU (b) WW (c) WLL (d) WLo (e) WBXII and (f) WN densities.

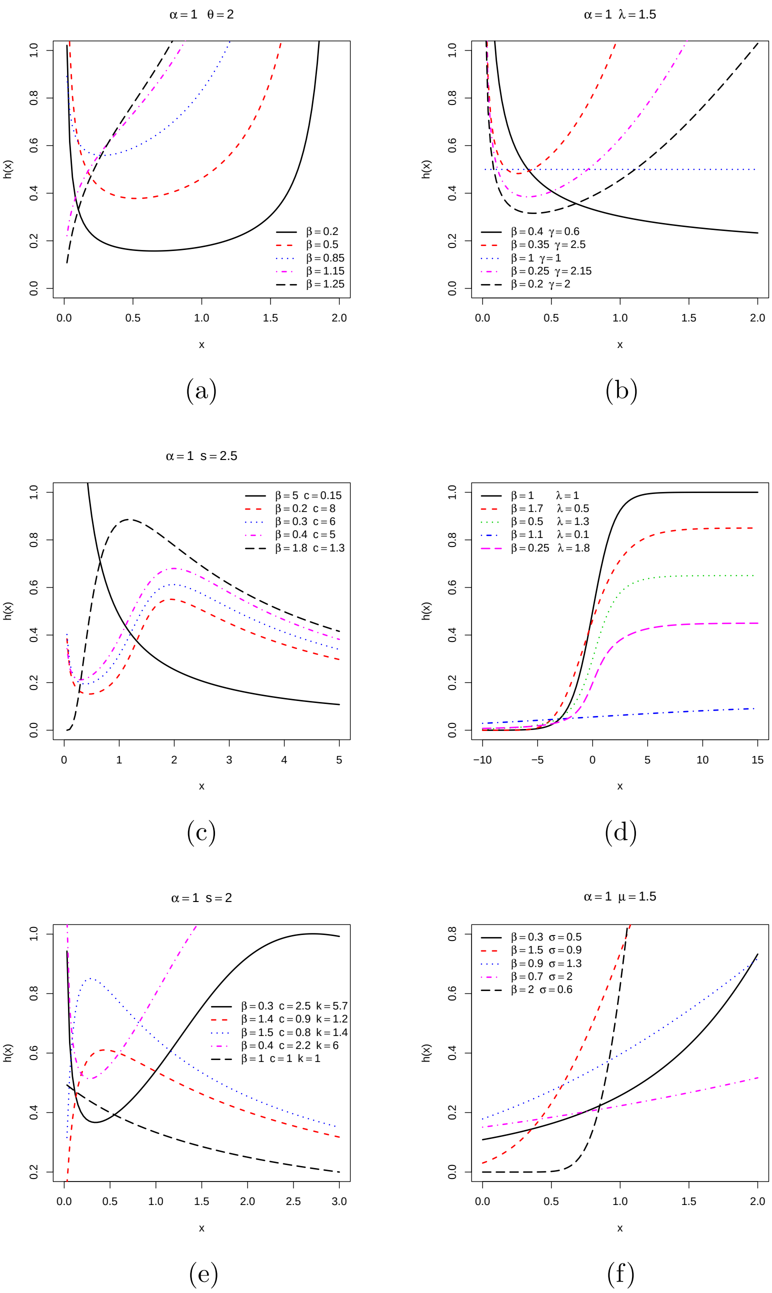


Figure 2: Plots of the (a) WU (b) WW (c) WLL (d) WLo (e) WBXII and (f) WN hazard rates.

5. Mathematical properties

The formulae derived throughout the paper can be easily handled in analytical softwares such as Maple and Mathematica which have the ability to deal with analytic expressions of formidable size and complexity. Established algebraic expansions to determine some mathematical properties of the NWG family can be more efficient than computing those directly by numerical integration of its density function, which can be prone to rounding off errors among others. The infinity limit in these sums can be substituted by a large positive integer such as 20 or 30 for most practical purposes. Here, we provide some mathematical properties of X.

5.1. Expansion for the NWG cdf. Let $A = \mathrm{e}^{-\alpha\{\,-\log[G(x;\boldsymbol{\xi})]\,\}^{\beta}}$. Then, using a power series expansion for A, we can write (2.2) as

$$F(x;\alpha,\beta,\boldsymbol{\xi}) = \sum_{i=0}^{\infty} \frac{(-1)^i\,\alpha^i}{i!}\,\{\,-\log[G(x;\boldsymbol{\xi})]\,\}^{i\,\beta}\,. \tag{5.1}$$

The following formula holds for $i \geq 1$
(`http:// functions.wolfram.com/ ElementaryFunctions/Log/06/01/04/03/`),
and then we can write

$$\begin{aligned}\{\,-\log[G(x;\boldsymbol{\xi})]\,\}^{i\,\beta} \quad &= \quad \sum_{k,l=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^{j+k+l}\,i\,\beta}{(i\beta-j)} \binom{k-i\,\beta}{k}\binom{k}{j}\binom{\beta\,i+k}{l} \\ &\quad\times \quad p_{j,k}\;[G(x;\boldsymbol{\xi})]^{l},\end{aligned}$$

where (for $j \geq 0$) $p_{j,0} = 1$ and (for $k = 1, 2, \ldots$)

$$p_{j,k} = k^{-1}\sum_{m=1}^{k} \frac{(-1)^m[m(j+1)-k]}{(m+1)}\;p_{j,k-m}.$$

By inserting the above power series in equation (5.1) gives

$$F(x;\alpha,\beta,\boldsymbol{\xi}) = \sum_{l=0}^{\infty} b_l\,G(x;\boldsymbol{\xi})^l = \sum_{l=0}^{\infty} b_l\,H_l(x;\boldsymbol{\xi}), \tag{5.2}$$

where $H_l(x;\boldsymbol{\xi}) = G(x;\boldsymbol{\xi})^l$ (for $l \geq 1$), is the exponentiated-G (exp-G) density function with power parameter l, $H_0(x;\boldsymbol{\xi}) = 1$,

$$b_l = \sum_{i,k=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^{i+j+k+l}\,i\,\beta}{i!\,(i\,\beta-j)} \binom{k-i\,\beta}{k}\binom{k}{j}\binom{i\,\beta+k}{l}\,p_{j,k}.$$

We can write the NWG family density as a mixture of exp-G densities

$$f(x;\alpha,\beta,\boldsymbol{\xi}) = \sum_{l=0}^{\infty} b_{l+1}\,h_{l+1}(x;\boldsymbol{\xi}), \tag{5.3}$$

where $h_{l+1}(x;\boldsymbol{\xi}) = (l+1)\,g(x;\boldsymbol{\xi})\,G(x;\boldsymbol{\xi})^l$ is the exp-G density function with power parameter $l+1$.

Thus, some mathematical properties of the proposed family can be derived from (5.3) and those of exp-G properties. For example, the ordinary and incomplete moments and moment generating function (mgf) of X can be obtained from those exp-G quantities. Some mathematical properties of the exp-G distributions are studied by [20, 21, 23] and others.

5.2. Moments. Let Y_l be a random variable having the exp-G density function $h_{l+1}(x)$. A first formula for the nth moment of X follows from (5.3) as

$$E(X^n) = \sum_{l=0}^{\infty} b_{l+1}\, E(Y_l^n). \tag{5.4}$$

Expressions for moments of several exp-G distributions are given by Nadarajah and Kotz [23], which can be used to obtain $E(X^n)$.

A second formula for $E(X^n)$ can be written from (5.4) in terms of the G qf as

$$E(X^n) = \sum_{l=0}^{\infty} (l+1)\, b_{l+1}\, \tau_{n,l}, \tag{5.5}$$

where $\tau_{n,l} = \int_{-\infty}^{\infty} x^n\, G(x)^l\, g(x)\, dx = \int_0^1 Q_G(u)^n\, u^l\, du$.

Cordeiro and Nadarajah [13] obtained $\tau_{n,l}$ for some well-known distributions such as normal, beta, gamma and Weibull, which can be used to determine the NWG moments.

For empirical purposes, the shapes of many distributions can be usefully described by what we call the incomplete moments. These types of moments play an important role for determining Lorenz and Bonferroni curves.

The nth incomplete moment of X is obtained as

$$m_n(y) = \sum_{l=0}^{\infty} (l+1)\, b_{l+1} \int_0^{G(y)} Q_G(u)^n\, u^l\, du. \tag{5.6}$$

The last integral can be computed for most G distributions. Equations (5.4)-(5.6) are the main results of this section.

5.3. Generating function. Let $M_X(t) = E(\mathrm{e}^{\mathrm{t\,X}})$ be the mgf of X. Then, we can write

$$M_X(t) = \sum_{l=0}^{\infty} b_{l+1}\, M_l(t), \tag{5.7}$$

where $M_l(t)$ is the mgf of Y_l. Hence, $M_X(t)$ can be determined from the exp-G generating function.

A second formula for $M_X(t)$ can be expressed as

$$M_X(t) = \sum_{l=0}^{\infty} (l+1)\, b_{l+1}\, \rho(t,l), \tag{5.8}$$

where $\rho(t,l) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{t\,x}}\, \mathrm{G(x)^l\, g(x)\, dx} = \int_0^1 \mathrm{e}^{\mathrm{t\,Q_G(u)}}\, \mathrm{u^l\, du}$.

We can obtain the mgfs of several distributions directly from equation (5.8).

5.4. Rényi entropy. Entropy has wide application in science, engineering and probability theory, and has been used in various situations as a measure of variation of the uncertainty. The entropy of a random variable X is a measure of variation of uncertainty. Here, we derive explicit expressions for the Rényi entropy [25] of the NWG family. The Shannon entropy [27] of a random variable X is defined by $E\{-\log[f(X)]\}$. It is the special case of the Rényi entropy when $\gamma \uparrow 1$.

The Rényi entropy is defined by

$$I_R(\delta) = \frac{1}{1-\delta} \log[I(\delta)],$$

where $I(\delta) = \int_{-\infty}^{\infty} f^{\delta}(x)\, dx, \delta > 0$ and $\delta \neq 1$.

Let us consider

$$f^{\delta}(x) = (\alpha\,\beta)^{\delta}\, g^{\delta}(x)\; G^{-\delta}(x)\, \{-\log[G(x)]\}^{\delta(\beta-1)}\; \mathrm{e}^{-\delta\alpha\{-\log[G(x)]\}^{\beta}}.$$

Expanding the exponential function in power series and then expanding the power of $\{-\log[G(x)]\}$ as in Section 5.1, we obtain

$$f^{\delta}(x) = (\alpha\,\beta)^{\delta} \sum_{i,k=0}^{\infty} \frac{(-1)^i\,(\alpha\,\delta)^i\,[\delta(\beta-1)+i\beta]}{i!} \binom{k-\delta(\beta-1)-i\beta}{k}$$
$$\times \sum_{j=0}^{\infty} \frac{(-1)^{j+k}\,p_{j,k}\,\binom{k}{j}}{[\delta(\beta-1)+i\beta-j]}\;\; g^{\delta}(x)\; G^{\delta}(x)\,[1-G(x)]^{k-\delta(\beta-1)-i\beta},$$

where the constants $p_{j,k}$ are given in Section 5.1.

Further, using the binomial expansion in the last equation, we can write

$$f^{\delta}(x) = \sum_{l=0}^{\infty} S_l\; g^{\delta}(x)\; G^{\delta+l}(x),$$

where

$$S_l \;=\; \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k+l}\,(\alpha\,\delta)^i\,[\delta(\beta-1)+i\beta]}{i!\,[\delta(\beta-1)+i\beta-j]}$$
$$\times\; p_{j,k-m} \binom{k}{j} \binom{k-\delta(\beta-1)-i\beta}{k} \binom{k+\delta(\beta-1)+i\beta}{l}.$$

Hence, the Rényi entropy reduces to

$$I_R(\delta) = \frac{1}{1-\delta} \log\left[\sum_{l=0}^{\infty} S_l \int_{-\infty}^{\infty} g^{\delta}(x)\, G^{\delta+l}(x)\; dx\right].$$

6. Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1, \ldots, X_n$ be observed values of a sample from the NWG family of distributions. We can write the density of the ith order statistic, say $X_{i:n}$, as

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x)\, F(x)^{j+i-1}.$$

Following similar algebraic developments of Nadarajah *et al.* [22], we can write

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} m_{r,k}\, h_{r+k+1}(x), \tag{6.1}$$

where $h_{r+k+1}(x)$ is the exp-G density function with power parameter $r+k+1$,

$$m_{r,k} = \frac{n!\,(r+1)\,(i-1)!\,b_{r+1}}{(r+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j\, f_{j+i-1,k}}{(n-i-j)!\,j!},$$

and b_k is defined in equation (5.2). Here, the quantities $f_{j+i-1,k}$ are obtained recursively by $f_{j+i-1,0} = b_0^{j+i-1}$ and (for $k \geq 1$)

$$f_{j+i-1,k} = (k\,b_0)^{-1} \sum_{m=1}^{k} [m(j+i)-k]\, b_m\, f_{j+i-1,k-m}.$$

Based on the expansion (6.1), we can obtain some mathematical properties (ordinary and incomplete moments, generating function, etc.) for the NWG order statistics from those exp-G properties.

7. Reliability

We derive the reliability $R = P(X_2 < X_1)$ when $X_1 \sim \mathrm{NWG}(\alpha_1, \beta_1, \boldsymbol{\xi_1})$ and $X_2 \sim \mathrm{NWG}(\alpha_2, \beta_2, \boldsymbol{\xi_2})$ are independent random variables with a positive support. It has many applications especially in engineering concepts. Let $f_i(x)$ and $F_i(x)$ denote the pdf and cdf of X_i for $i = 1, 2$. By using the mixture representations for $F_2(x)$ and $f_1(x)$ given in Section 5.1, we obtain

$$R = \sum_{k,s=0}^{\infty} b_k^{(1)}\; b_{s+1}^{(2)}\; R_{k,s+1},$$

where $b_l^{(1)}$ and $b_{s+1}^{(2)}$ are given in these representations and

$$R_{k,s+1} = \int_0^{\infty} H_k(x; \alpha_1, \beta_1, \boldsymbol{\xi_1})\; h_{s+1}(x; \alpha_2, \beta_2, \boldsymbol{\xi_2})\, dx.$$

If $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, then

$$R = \sum_{k,s=0}^{\infty} \frac{(s+1)}{(s+k+1)}\, b_k^{(1)}\, b_{s+1}^{(2)}.$$

Finally, if $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $\boldsymbol{\xi_1} = \boldsymbol{\xi_2}$, then $R = 1/2$ as expected.

8. Characterizations of the NWG family

Various characterizations of distributions have been established in many different directions. In this section, three characterizations of the NWG family are presented based on: (i) a simple relationship between two truncated moments; (ii) a single function of the random variable, and (iii) the hazard function.

8.1. Characterization based on truncated moments. Here, we present a characterization of the NWG family in terms of a simple relationship between two truncated moments. The characterization results employ an interesting result due to Glänzel [19] (Theorem 1, below). It has the advantage that the cdf F is not required to have a closed-form and is given in terms of an integral whose integrand depends on the solution of a first order differential equation, which can serve as a bridge between probability and differential equation.

8.1. Theorem. *Let $(\Omega, \Sigma, \mathbf{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $a < b$ ($a = -\infty, b = \infty$ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with distribution function $F(x)$ and let q_1 and q_2 be two real functions defined on H such that*

$$\mathbb{E}\left[q_1(X) \mid X \geq x\right] = \mathbb{E}\left[q_2(X) \mid X \geq x\right] \eta(x), \quad x \in H,$$

is defined with some real function η. Assume that q_1, $q_2 \in C^1(H)$, $\eta \in C^2(H)$ and $G(x)$ is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $q_2\eta = q_1$ has no real solution in the interior of H. Then G is uniquely determined by the functions q_1, q_2 and η, particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)\, q_2(u) - q_1(u)} \right| \mathrm{e}^{-s(u)}\, du,$$

where the function s is a solution of the differential equation $s' = \frac{\eta'\, q_2}{\eta\, q_2 - q_1}$ and C is a constant, chosen to make $\int_H dF = 1$.

8.2. Proposition. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_2(x) = \mathrm{e}^{\alpha\{-\log[G(x;\xi)]\}^{\beta}}$ and $q_1(x) = q_2(x)\{-\log[G(x;\xi)]\}$ for $x > 0$. The pdf of X is (2.2) if and only if the function η defined in Theorem 1 has the form*

$$\eta(x) = \frac{\beta}{\beta+1}\{-\log[G(x;\xi)]\}, \quad x > 0.$$

Proof. Let X have density (2.2), then

$$\left[1 - F(x)\right] \mathbb{E}\left[q_2(X) \mid X \geq x\right] = \alpha\{-\log[G(x;\xi)]\}^{\beta}, \quad x > 0,$$

$$\left[1 - F(x)\right] \mathbb{E}\left[q_1(X) \mid X \geq x\right] = \frac{\alpha\beta}{\beta+1}\left\{-\log[G(x;\xi)]\right\}^{\beta+1}, \quad x > 0,$$

and then

$$\eta(x)\, q_2(x) - q_1(x) = -\frac{1}{\beta+1} q_2(x) \Big\{ -\log[G(x;\xi)] \Big\} < 0 \quad \text{for } x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_2(x)}{\eta(x)\, q_2(x) - q_1(x)} = \beta\left[\frac{g(x;\xi)}{G(x;\xi)}\right]\Big\{ -\log[G(x;\xi)] \Big\}^{-1}, \quad x > 0,$$

and hence

$$s(x) = -\beta \log\Big(\{-\log[G(x;\xi)]\} \Big), \quad x > 0,$$

or

$$\mathrm{e}^{-s(x)} = \Big\{ -\log[G(x;\xi)] \Big\}^{\beta}, \quad x > 0.$$

Now, in view of Theorem 1, X has density function (2.2). □

8.3. Corollary. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_2(x)$ be as in Proposition 1. The pdf of X is (2.2) if and only if there exist functions q_1 and η defined in Theorem 1 satisfying the differential equation*

$$\frac{\eta'(x)\, q_2(x)}{\eta(x)\, q_2(x) - q_1(x)} = \beta\Big[\frac{g\,(x;\xi)}{G\,(x;\xi)}\Big] \,\big\{ -\log\left[G\,(x;\xi)\right] \big\}^{-1}, \quad x > 0.$$

Remark 1. **(a)** The general solution of the differential equation in Corollary 1 is obtained as follows:

$$\eta'(x) - \beta\Big[\frac{g\,(x;\xi)}{G\,(x;\xi)}\Big] \,\{-\log\left[G\,(x;\xi)\right]\}^{-1}$$
$$= -\beta\, q_1(x) \,\left[\tfrac{g(x;\xi)}{G(x;\xi)}\right] \Big\{ -\log\left[G\,(x;\xi)\right] \Big\}^{-1} \left[q_2(x)\right]^{-1},$$

or

$$\frac{d}{dx}\left[\{-\log\left[G\,(x;\xi)\right]\}^{\beta}\, \eta(x)\right]$$
$$= -\beta\, q_1(x) \,\left[\tfrac{g(x;\xi)}{G(x;\xi)}\right] \,\big\{ -\log\left[G\,(x;\xi)\right] \big\}^{\beta-1} \left[q_2(x)\right]^{-1}.$$

From the above equation, we obtain

$$\begin{aligned}\eta(x) \;&=\; \big\{ -\log\left[G\,(x;\xi)\right] \big\}^{-\beta} \\ &\times \left[-\int \beta\, q_1(x) \,\left[\frac{g\,(x;\xi)}{G\,(x;\xi)}\right] \Big\{ -\log\left[G\,(x;\xi)\right] \Big\}^{\beta-1} \left[q_2(x)\right]^{-1} dx + D\right],\end{aligned}$$

where D is a constant. One set of appropriate functions is given in Proposition 1 with $D = 0$.

(b) Clearly there are other triplets of functions (q_2, q_1, η) satisfying the conditions of Theorem 1. We present one such triplet in Proposition 1.

8.2. Characterization based on single function of the random variable. Here, we employ a single function ψ of X and state characterization results in terms of $\psi\,(X)$.

8.4. Proposition. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable with cdf $F(x)$. Let $\psi\,(x)$ be a differentiable function on $(0, \infty)$ with $\lim_{x\to\infty} \psi\,(x) = 1$. Then for $\delta \neq 1$,*

$$\mathbb{E}\left[\psi\,(X)\,|X < x\right] = \delta\, \psi\,(x)\,, \quad x \in (0, \infty)$$

if and only if

$$\psi\,(x) = F(x)^{\frac{1}{\delta}-1}, \quad x \in (0, \infty)\,.$$

Proof. The proof is straightforward. □

Remark 2. For $\psi\,(x) = \mathrm{e}^{-\{-\log[G(x;\xi)]\}^{\beta}}$, $x \in (0, \infty)$ and $\delta = \frac{\alpha}{\alpha+1}$, Proposition 2 provides a cdf $F(x)$ given by (2.1).

8.3. Characterizations based on the hazard function. The hrf $h_F(x)$ of a twice differentiable distribution function $F(x)$ and pdf $f(x)$ satisfies the first order differential equation

$$\frac{h'_F(x)}{h_F(x)} - h_F(x) = q(x),$$

where $q\,(y)$ is an appropriate integrable function. Although this differential equation has an obvious form since

$$\frac{f'(x)}{f(x)} = \frac{h'_F(x)}{h_F(x)} - h_F(x), \tag{8.1}$$

for many univariate continuous distributions (8.1) seems to be the only differential equation in terms of the hrf. The goal of the characterization based on the hazard function is to establish a differential equation in terms of the hrf, which has a simple form as possible and is not of the trivial form (8.1). For some general families of distributions this may not be possible.

8.5. Proposition. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The random variable X has pdf (2.2) (for $\beta = 1$) and $G(x) = (1 - \mathrm{e}^{-\lambda\, x})$ if and only if its hazard function $h_F(x)$ satisfies the differential equation*

$$\begin{aligned} h'_F(x) + \lambda \left(1 - \mathrm{e}^{-\lambda\, x}\right)^{-1} h_F(x) = {} & \frac{\alpha^2\, \lambda^2\, \mathrm{e}^{-2\lambda x}}{\left(1 - \mathrm{e}^{-\lambda\, x}\right)^2}\; \mathrm{e}^{\alpha \log\left(1 - \mathrm{e}^{-\lambda\, x}\right)} \\ & \times \left[1 - \mathrm{e}^{\alpha \log\left(1 - \mathrm{e}^{-\lambda x}\right)}\right]^{-2}, \end{aligned} \tag{8.2}$$

with initial condition $h_F(0) = 0$.

Proof. The $f(x)$ has pdf (2.2), then clearly (8.2) holds. Now, if (8.2) holds, then

$$\frac{d}{dx}\left\{\mathrm{e}^{\lambda x}\left(1 - \mathrm{e}^{-\lambda\, x}\right) h_F(x)\right\} = \alpha\, \lambda\, \frac{d}{dx}\left\{\left(1 - \mathrm{e}^{\alpha \log\left[1 - \mathrm{e}^{-\lambda\, x}\right]}\right)^{-1} + C\right\},$$

where C is an appropriate constant. Letting $C = -1$, we obtain from the above equation

$$h_F(x) = \frac{\alpha \lambda\, \mathrm{e}^{-\lambda\, x}}{(1 - e^{-\lambda x})}\left\{\mathrm{e}^{\alpha \log(1 - \mathrm{e}^{-\lambda\, x})}\left[1 - \mathrm{e}^{\alpha \log(1 - e^{-\lambda x})}\right]^{-1}\right\}.$$

Integrating both sides of the last equation from 0 to x, we arrive at

$$-\log\left[1 - F(x)\right] = -\log\left[1 - \mathrm{e}^{\alpha \log\left(1 - e^{-\lambda\, x}\right)}\right],$$

from which, we obtain

$$1 - F(x) = 1 - \mathrm{e}^{\alpha \log(1 - \mathrm{e}^{-\lambda x})}, \quad x \geq 0.$$

□

9. Estimation

We consider the estimation of the unknown parameters of the NWG family of distributions by the method of maximum likelihood. Let $x_1, \ldots, x_n$ be a sample of size n from the NWG family given by (2.2). The log-likelihood function for the vector of parameters $\boldsymbol{\Theta} = (\alpha, \beta, \boldsymbol{\xi})^\top$ can be expressed as

$$\begin{aligned}\ell(\boldsymbol{\Theta}) &= n\log\alpha + n\log\beta + \sum_{i=1}^{n}\log\,[g(x,\boldsymbol{\xi})] - \sum_{i=1}^{n}\log\,\{G(x,\boldsymbol{\xi})\} \\ &\quad +(\beta-1)\sum_{i=1}^{n}\log\,\{-\log[G(x,\boldsymbol{\xi})]\} - \alpha\sum_{i=1}^{n}\{-\log[G(x,\boldsymbol{\xi})]\}^{\beta}.\end{aligned}$$

The components of the score vector $U(\boldsymbol{\Theta})$ are given by

$$\begin{aligned}U_\alpha &= \frac{n}{\alpha} - \sum_{i=1}^{n}\{-\log[G(x,\boldsymbol{\xi})]\}^{\beta}, \\ U_\beta &= \frac{n}{\beta} - \sum_{i=1}^{n}\Big[\log\,\{-\log[G(x,\boldsymbol{\xi})]\}\Big] \\ &\quad -\alpha\sum_{i=1}^{n}\Big[\{-\log[G(x,\boldsymbol{\xi})]\}^{\beta}\,\log\,\{-\log[G(x,\boldsymbol{\xi})]\}\Big], \\ U_{\boldsymbol{\xi}_k} &= \sum_{i=1}^{n}\left[\frac{\left(\frac{\partial\, g(x,\boldsymbol{\xi})}{\partial\,\boldsymbol{\xi}_k}\right)}{g(x,\boldsymbol{\xi})}\right] - \sum_{i=1}^{n}\left[\frac{\left(\frac{\partial\, G(x,\boldsymbol{\xi})}{\partial\,\boldsymbol{\xi}_k}\right)}{G(x,\boldsymbol{\xi})}\right] \\ &\quad -(\beta-1)\sum_{i=1}^{n}\left[\frac{\left(\frac{\partial\, G(x,\boldsymbol{\xi})}{\partial\,\boldsymbol{\xi}_k}\right)}{\{-\log[G(x,\boldsymbol{\xi})]\}\;G(x,\boldsymbol{\xi})}\right] \\ &\quad +\alpha\beta\left[\sum_{i=1}^{n}\frac{\{-\log[G(x,\boldsymbol{\xi})]\}^{\beta-1}\;\left(\frac{\partial\, G(x,\boldsymbol{\xi})}{\partial\,\boldsymbol{\xi}_k}\right)}{G(x,\boldsymbol{\xi})}\right].\end{aligned}$$

Setting U_α, U_β and $U_{\boldsymbol{\xi}_k}$ equal to zero and solving numerically these equations simultaneously yields the maximum likelihood estimates (MLEs) $\widehat{\boldsymbol{\Theta}} = (\widehat{\alpha}, \widehat{\beta}, \widehat{\boldsymbol{\xi}})^\intercal$. The estimates can be obtained using the R language.

For interval estimation and hypothesis tests, we can use standard likelihood techniques based the observed information matrix, which can be obtained from the authors upon request.

10. Applications

We provide two applications to real life data sets to prove the flexibility of the Weibull-log-logistic (WLL) and Weibull-Weibull (WW) models presented in Section 3. The MLEs of the model parameters and the goodness-of-fit statistics are calculated for the WLL and WW models, and other competitive models. We compare these models with other Weibull-G models under the same baseline distribution, namely the WLL (BSC-WLL, ALF-WLL) and WW (BSC-WW, ALF-WW) models based on Bourguignon *et al.* (2014)'s generator $G(x)/[1-G(x)]$ and Alzaatreh *et al.* (2013)'s generator $-\log[1-G(x)]$. We note that the BSC-LL, ALF-LL, BSC-WW and ALF-WW models are not known in the literature. Further, we also compare the gamma exponentiated-exponential (GEE) (Ristić and Balakrishnan [26]) and exponential-exponential geometric (EEG) (Rezaei et al. [24]) models with the proposed and other competitive models. The density functions of the GEE and EEG distributions are, respectively, given by (for $x > 0$)

$$\begin{aligned} f_{GEE}(x;\lambda,\alpha,\theta) &= \frac{\alpha\,\theta}{\Gamma(\lambda)}\,\mathrm{e}^{-\theta\,x}\,\left[1-\mathrm{e}^{-\theta\,x}\right]^{\alpha-1}\,\left[-\alpha\log\left(1-\mathrm{e}^{-\theta\,x}\right)\right]^{\lambda-1}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \lambda,\alpha,\theta>0, \\ f_{EEG}(x;p,\alpha,\theta) &= \frac{\alpha\,\theta(1-p)\,\mathrm{e}^{-\theta\,x}}{\left[1-\mathrm{e}^{-\theta\,x}\right]^{\alpha-1}\left[1-p+p\,(1-\mathrm{e}^{-\theta\,x})^{\alpha}\right]^{2}}, \\ &\qquad\qquad\qquad\qquad 0<p<1, \quad \alpha,\theta>0. \end{aligned}$$

The first real data represents the breaking strength of 100 yarn reported by Duncan [17]. The second real data set corresponds to the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli reported by Bjerkedal [9].

The measures of goodness-of-fit including the log-likelihood function evaluated at the MLEs ($\hat{\ell}$), Akaike information criterion (AIC), Anderson-Darling (A^*), Cramér–von Mises (W^*) and Kolmogorovû-Smirnov (K-S), are calculated to compare the fitted models. The statistics A^* and W^* are described by Chen and Balakrishnan [10]. In general, the smaller the values of these statistics, the better the fit to the data. The required computations are carried out using the R software.

Table 1: MLEs and their standard errors (in parentheses) for the first data set.

Distribution	β	c	s	λ	α	θ	p
WLL	0.6612	25.5915	97.7523	-	-	-	-
	(0.1395)	(6.2313)	(1.0425)	-	-	-	-
BSC-WLL	4.7898	1.5601	105.0254	-	-	-	-
	(195.4617)	(63.6652)	(1.4938)	-	-	-	-
ALF-WLL	0.6528	25.9621	99.6537	-	-	-	-
	(0.1423)	(6.4490)	(1.0920)	-	-	-	-
GEE	-	-	-	20.4987	78.3734	0.0150	-
	-	-	-	(5.4222)	(11.2681)	(0.0022)	-
EEG	-	-	-	-	38.9807	0.0198	0.9974
	-	-	-	-	(5.8133)	(0.0015)	(0.0004)

Table 2: MLEs and their standard errors (in parentheses) for the second data set.

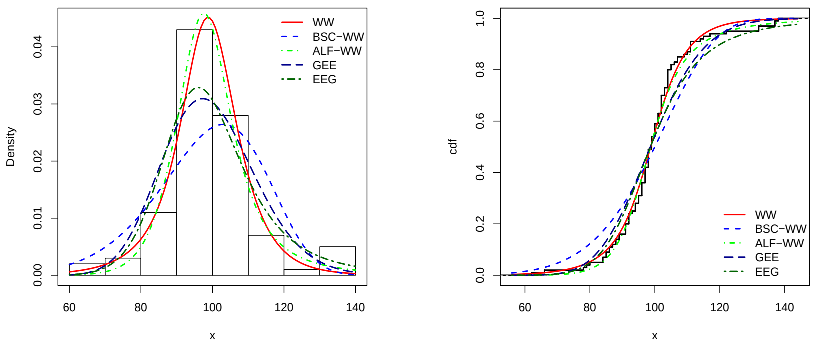
Distribution	β	γ	λ	α	θ	p
WW	2.6594	0.6933	0.0270	-	-	-
	(0.7129)	(0.1707)	(0.0193)	-	-	-
BSC-WW	11.1576	0.0881	0.4574	-	-	-
	(4.5449)	(0.0355)	(0.0770)	-	-	-
ALF-WW	1.7872	0.7795	0.0255	-	-	-
	(0.7821)	(0.3332)	(0.0400)	-	-	-
GEE	-	-	2.1138	2.6006	0.0083	-
	-	-	(1.3288)	(0.5597)	(0.0048)	-
EEG	-	-	-	2.5890	0.0004	0.9999
	-	-	-	(0.4820)	(0.0041)	(0.1036)

Table 3: The statistics $\hat{\ell}$, AIC, A^*, W^* and K-S for the fitted models to the first data set.

Distribution	$\hat{\ell}$	AIC	A^*	W^*	K-S	p-value (K-S)
WLL	-383.5896	773.1792	0.8402	0.1254	0.0805	0.5354
BSC-WLL	-404.7074	815.4147	4.7296	0.7951	0.1948	0.0010
ALF-WLL	-383.6181	773.2361	0.7432	0.1332	0.0888	0.4091
GEE	-392.7053	791.4106	2.3551	0.3976	0.1423	0.0348
EEG	-390.5435	787.0869	1.4894	0.2676	0.1442	0.0312

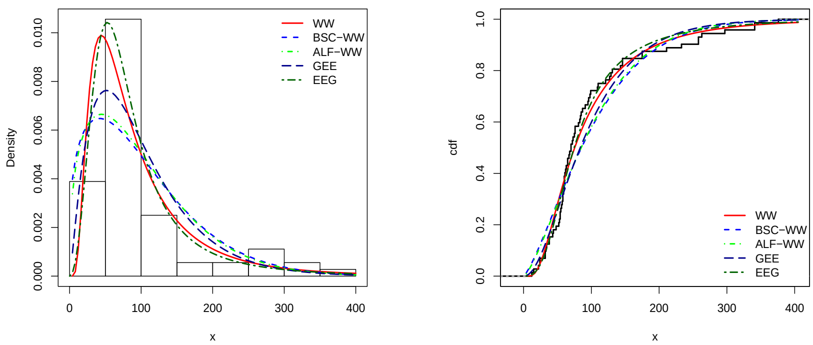
Table 4: The statistics $\hat{\ell}$, AIC, A^*, W^* and K-S for the fitted models to the second data set.

Distribution	$\hat{\ell}$	AIC	A^*	W^*	K-S	p-value (K-S)
WW	-390.2338	786.4676	0.7811	0.1427	0.1055	0.3994
BSC-WW	-397.8399	801.6797	2.4764	0.4494	0.1510	0.0749
ALF-WW	-397.1477	800.2953	2.3938	0.4348	0.1465	0.0911
GEE	-393.6235	793.2470	1.7208	0.3150	0.1347	0.1467
EEG	-389.9445	785.8890	0.5789	0.1047	0.0861	0.6282



(a) Estimated pdfs (b) Estimated cdfs

Figure 3: Plots of the estimated pdfs and cdfs of the WLL, BSC-WLL, ALF-WLL, GEE and EEG models.



(a) Estimated pdfs (b) Estimated cdfs

Figure 4: Plots of the estimated pdfs and cdfs of the WW, BSC-WW, ALF-WW, GEE and EEG models.

The MLEs and the corresponding standard errors (in parentheses) of the model parameters are given in Tables 1 and 2. The numerical values of the statistics AIC, A^*, W^* and K-S are listed in Tables 3 and 4. The histograms of the two data sets and the estimated pdfs and cdfs of the proposed and competitive models are displayed in Figures 3 and 4. Based on the figures in Tables 2 and 4, we conclude that the new WLL and WW models provide adequate fits as compared to other Weibull-G models in both applications with small values for AIC, A^*, W^* and K-S, and large p-values. In Application 1, the proposed WLL model is much better than the BSC-WLL, GEE and EEG models, and a good alternative to the ALF-WLL model. In Application 2, the proposed WW model outperforms the BSC-WEW, ALF-WW and GEE models but it is not better than EEG model. Figures 3 and 4 also support the results in Tables 2 and 4.

11. Concluding remarks

In this paper, we propose and study the *new Weibull-G* (NWG) family. We investigate some of its mathematical properties including an expansion for the density function and explicit expressions for the quantile function, ordinary and incomplete moments, generating function, entropies, reliability and order statistics. Three useful characterizations, based on truncated moments, single function of the random variable and hazard function, are formulated for the NWG family. The advantage of the characterizations given here is that the cumulative distribution is not required to have a closed-form and are given in terms of an integral whose integrand depends on the solution of a first order differential equation. They can serve as a bridge between probability and differential equation. The maximum likelihood method is employed to estimate the model parameters. We fit two special models of the new family to two real data sets to demonstrate the flexibility of the proposed family. These special models can give better fits than other competing models. We hope that the new family and its generated models will attract wider application in areas such as engineering, survival and lifetime data, hydrology, economics, among others.

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