A Comparison of Model Reduction Techniques for Multi-Zone Building Thermal Models

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ABSTRACT
A COMPARISON OF MODEL REDUCTION TECHNIQUES FOR MULTI-ZONE BUILDING THERMAL MODELS

John Burroughs
Marquette University, 2019

In this thesis model order reduction techniques are applied to a set of systems representing the thermodynamics of a multi-zone building. The models are intended to be used in a model predictive control (MPC) application, with the individual zones defined using a simplified two temperature model. There has been an increased interest in model identification and reduction for MPC applications of building models that include multi-zone and non-linear models, but the most of this work has focused on models where individual zones are represented with a higher order model. Manual pole/zero removal, dominant eigenvalue, and balanced model reduction methods are presented, along with a proposed application specific method that takes advantage of the zone model’s simplified form. The proposed method treats a set of the zones as a common airspace with comparable control and reduces the underlying resistor/capacitor (RC) network. These methods are applied to a two zone and a six zone model with various coupling configuration tested. The general form of the multi-zone model proves difficult to reduce without making modification to the original form. The effects of reducing the inputs and outputs, through methods such as using a common temperature setpoint, are presented with significant improvements to reduction capabilities. Balanced model reduction the 18th order system down to 9th and 5th based on which inputs and outputs are reduced, and the application specific methods is able to reduce the same system down to a 3rd order model when the inputs and outputs are fully reduced.
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John Burroughs

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1. INTRODUCTION

1.1 Problem Overview

As of 2018, in the United States, Heating, Ventilation and Air Conditioning (HVAC) accounts for approximately 40% of all energy consumption in residential and commercial buildings [1]. As a result, methods to reduce the energy and/or cost of operating HVAC equipment receive a great deal of interest. These methods can include improvements to the equipment, their controls, or their overall daily workflow. One method that addresses the latter two is Model Predictive Control (MPC).

MPC’s objective is to optimize a process over a horizon, through choices of controllable parameters, subject to a set of constraints. How the process propagates forward in time is based on a model, but as the model is subject to some error, the optimization routine is repeated on a regular interval with only the first time element’s control action being applied. Each consecutive run then takes in a measurement of the current state of the process as feedback. While this feedback allows for compensation of some model error, the use of an inaccurate model will limit the feedback’s effectiveness. Furthermore, as the MPC routine is required to run on some regular real time schedule, the problem and by extension model cannot become so complex that it becomes computationally intractable. For a further introduction into MPC see the tutorial by Rawlings [2].

MPC for HVAC allows for increased cost savings relative to traditional operating strategies, e.g. schedule-based setpoints, see [3]. When applying MPC to building temperature control, zones, spaces in a building controlled by a single piece of equipment, are modeled to describe how changing temperature setpoints and outdoor air conditions affect the
temperature in the zones. For applications where only a few zones are being controlled, it remains feasible to still use a complex model for each zone; however in large buildings, the number of zones can grow to the point that modeling each zone individually makes the resulting MPC optimization problem computationally intractable for use in real-time control. Thus, it becomes beneficial to reduce the model complexity provided there is limited impact on accuracy or MPC performance.

1.2 Problem Statement and Contribution to Field of Study

Recently, MPC applications for the HVAC domain have received greater interest [4] [5] [6]. While modeling the thermodynamics of zones is not a new subject [7], doing so with a control strategy like MPC in mind is a more recent development. There are three general categories the models used in MPC applications fall into: single linear zone, multi-zone, and non-linear. The identification of both linear and nonlinear models for use in MPC is explored in [8] using an optimization based fitting technique called MPC relevant identification. A nonlinear single zone model is identified using a Nonlinear Autoregressive Neural Network for use in a nonlinear MPC application in [9]. A method for fitting a multi-zone model is introduced in [10] which uses a high fidelity simulator to produce the necessary training data to fit the simplified model. Parameter fitting of the convection edges between multiple zones is explored in [11]. The use of dissimilarity-based clustering to reduce a system comprised of identical linear subsystems connected in an undirected network is applied to a set of zones in [12]. The problem definition is very similar to the multi zone model explored later in this thesis, but only explores instances where each zone has identical parameters, which is generally not the case.

Model order reduction (MOR) as a whole is a well-developed technology for linear applications, with more recent research focused on non-linear systems. Singular Value
Decomposition MOR and Input-Correlated Poor Man’s Truncated Balanced Reduction are both applied to a system with a large number of inputs and outputs, namely power grids, in [13]. These methods aim to address some of weakness of the standard projection and balanced reduction methods when dealing with larger systems. Balanced reduction and Krylov projection methods are discussed along with a method for achieving balanced reduction through Krylov projector in [14]. The interpolatory rational Krylov algorithm is presented in [15] and further applied to LED thermal models in [16]. In [17] balanced reduction is successfully applied to a low order pharmacokinetics compartment model. Proper orthogonal decomposition and greedy reduced basis methods are applied to the complex Navier-Stokes, fluid-dynamic equations in [18].

In this work several model reduction techniques are reviewed for their ability to reduce a multi-zone building thermal model for the use of MPC in HVAC applications. The four methods are transfer function pole/zero removal, dominant eigenvalue, balanced model reduction and an application specific method. The full order model is defined by coupling multiple individual zones models, where each zone model is defined using a reduced form.
2. BACKGROUND

This chapter serves as introduction to model reduction and the form of the zone model that is to be reduced. The categories standard reduction techniques are classified as are briefly discussed in section 0 and the fundamentals of building thermodynamics along with the choice of zone model is presented in 2.2. Table 1 covers the nomenclature that will be used throughout the thesis.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>∈</td>
<td>Element of</td>
</tr>
<tr>
<td>( R^n )</td>
<td>Vector Space of Real Numbers of dimension n</td>
</tr>
<tr>
<td>( C^n )</td>
<td>Vector Space of Complex Numbers of dimension n</td>
</tr>
<tr>
<td>( A^{-1} )</td>
<td>Inverse of matrix A</td>
</tr>
<tr>
<td>( A^T )</td>
<td>Transpose of matrix A</td>
</tr>
<tr>
<td>( A^{-T} )</td>
<td>Transpose of the Inverse of matrix A</td>
</tr>
<tr>
<td>( A^+ )</td>
<td>Pseudoinverse of ( A ), ( A^T(AA^T)^{-1} ) for full row rank ( A )</td>
</tr>
<tr>
<td>( A &gt; 0 )</td>
<td>A is a positive definite matrix</td>
</tr>
<tr>
<td>( | A | )</td>
<td>( l_2 )-norm of matrix A</td>
</tr>
<tr>
<td>( A^T = A^{-1} )</td>
<td>A is an Orthogonal Matrix</td>
</tr>
<tr>
<td>( I )</td>
<td>( n \times n ) identity matrix whose size is determined from context</td>
</tr>
<tr>
<td>( det(A) )</td>
<td>Determinant of matrix A</td>
</tr>
<tr>
<td>( adj(A) )</td>
<td>Adjugate of matrix A</td>
</tr>
<tr>
<td>( \dot{x} )</td>
<td>Time derivative of function ( x )</td>
</tr>
</tbody>
</table>

2.1 Model Reduction

Consider the linear state space system given as

\[
\begin{align*}
\dot{x} &= Ax + Bu \quad (1) \\
y &= Cx + Du \quad (2)
\end{align*}
\]

where \( x \in R^n \), \( u \in R^l \), and \( y \in R^m \). The goal of model reduction is to find an \( x_r \in R^r \) such that \( r < n \) and

\[
\begin{align*}
\dot{x}_r &= Fx_r + Gu \quad (3) \\
y_r &= Hx_r + Ju \quad (4)
\end{align*}
\]
while keeping $y_r$ as close to $y$ as possible.

There are a variety of MOR methods that can be classified into several broad categories: balanced truncation, Krylov subspace methods, proper orthogonal decomposition, reduced basis methods, and simplified physics methods.

Balanced truncation techniques are founded on control theory in which the controllability and observability properties are maintained. A change of variables is applied to express the system in terms of the fore-mentioned properties and is then reduced by removing state variables that are not controllable or observable [19]. This method is well understood and has known error bounds; however it is most suited for small to medium size systems [16].

Krylov subspace methods are based on matching a set of moments between the full order system and the reduced system when expressed as Laurent series [20]. This is achieved by projecting the full order system through either the input or output Krylov space. In general, the generation of the subspace's basis vectors is not numerically robust and variants of the Arnoldi and Lanczos algorithms are used [21]. These methods do not have the same level of error bounding as balanced truncations, but can handle significantly larger systems [16].

Proper orthogonal decomposition (POD) methods use data, often generated from simulations of the full order system, to determine redundancies within the data. The method is founded in statistical analysis and projects the data into a subspace spanning the eigenvectors of the sample covariance matrix. The goal is to determine a set of uncorrelated variables that maintains the variance of the original set. POD is optimal for linear systems in the sense of minimizing average squared distance between the full order and the reduced systems, but may be limited when working with real-world data [22].
Reduced basis methods are designed for parameterized differential equations that are solved for many parameters. During an offline phase, a small set of values are chosen from a large number of trial parameters using a greedy algorithm based on an a posteriori estimator. During the online phase, approximations for the differential equations are calculated and used to estimate the output [23].

When considering a reduced order model, some set of requirements should be met depending on the application [24]. Typical requirements include the following:

- The approximation error of the outputs for some or all of the inputs should be small.
- The reduced system’s order should be smaller than the original system.
- Properties, such as stability, passivity, controllability and observability, should be preserved.
- The reduction method must be computationally efficient and stable.

2.2 Zone Modeling

In this work, a zone refers to the closed loop control system containing a temperature sensor, control logic, and HVAC equipment. There are several different model types that can be used to describe a zone’s dynamics, and they fall into the general categories of a black, white, or gray box model [25], [26]. A black box model is one in which no physical knowledge of the system is known; rather a set of input-output data are fit to a chosen order and model structure. The lack of a physical knowledge is a positive in that minimal understanding is necessary to get started on the control design, but a negative in that the resulting coefficients cannot easily be modified as they lack physical meaning. Black box models generally require the most data in order to identify their coefficients due to the lack of built-in characteristics.
On the other extreme is a white box model, in which the model is built using only physical knowledge of the system. In the case of zone modeling, knowledge of the wall, door and window dimensions and materials along with the control dynamics are used to create a resistor capacitance (RC) network. This method can result in very complex but accurate models that have a well understood meaning within the building control community. In order to make this model, either a large amount of data is necessary to identify the parameters of the model or a great deal of manual effort is needed to calculate the parameters using measured dimensions and materials [27] [28].

As the name implies, a grey box model lies in between a black and white box model. The form of the model is defined in such a way that the parameters have some physical meaning, but do not necessarily map directly to any single part of the physical world. As a result, it is generally not possible to determine the parameters using measurements, but the reduced model complexity allows for parameter identification using less data than either the white or black box models would likely need. Again, an RC network is used to express the grey model, but there are several options as to what its form can be.

For either a white or grey box model using an RC network as the model form, the same base components are used to describe a thermal system in terms of electrical circuit theory. When referring to the thermal system, the term heat refers to “energy transferred across the boundary because of a temperature gradient at the boundary” [29, p. 62]. In addition, heat flow refers to energy transfer by means of heat. Lastly, the concept of thermal energy is “defined as the kinetic energy of random translational motions of atoms and molecules relative to the local frame, plus the vibrational and rotational energies of molecules” [29, p. 63].
In circuit theory, the current, $I$, through a resistor, $R_e$, is based on the voltage difference, $\Delta V$, across it along with its resistance.

$$I = \frac{\Delta V}{R_e}$$  \hspace{1cm} (5)

Comparatively, the heat flow rate, $\dot{Q}$, between two locations is dependent on the change in temperature, $\Delta T$, along with the thermal resistance, $R$.

$$\dot{Q} = \frac{\Delta T}{R}$$  \hspace{1cm} (6)

The value of the thermal resistance is dependent on the type of heat transfer: advective, conductive, convective, or radiative. Lastly, the ability of a material to store thermal energy is defined as the material’s thermal capacitance, $C$, and is computed as

$$C = \rho c V$$  \hspace{1cm} (7)

where $\rho$ is the material’s density, $c$ is the material’s specific heat, and $V$ is the material’s volume. Table 2 summarizes the equivalences between the two domains [30].

<table>
<thead>
<tr>
<th>Electrical</th>
<th>Thermal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$: Current</td>
<td>$\dot{Q}$: Heat flow rate</td>
</tr>
<tr>
<td>$V$: Voltage</td>
<td>$T$: Temperature</td>
</tr>
<tr>
<td>$R_e$: Electrical resistance</td>
<td>$R$: Thermal resistance</td>
</tr>
<tr>
<td>$C_e$: Electrical capacitance</td>
<td>$C$: Thermal capacitance</td>
</tr>
</tbody>
</table>

Table 2: Electrical circuit and thermal system equivalence

There are existing techniques to apply model reduction to detailed models, but this work explores reduction opportunities in large buildings where each zone is represented with an already simplified model. Larger buildings, such as commercial or academic, are made up of multiple zones where each zone is modeled independently with cross terms based on shared walls or open airways.
Each zone model will consist only of the indoor air temperature, which is the control variable of the HVAC equipment, and the mass temperature. Both temperatures have an associated thermal capacitance with the mass temperature expected to be much larger. The two temperatures are capable of transferring energy between the surfaces of the masses. The HVAC equipment provides heat flow to the indoor air to regulate the temperature, while heat is gained or lost to the outdoor air through windows, walls and other leaks. Other sources of heat such as the people, running electronics, and lighting in the room are collectively referred to as a load on the space. This simplified model is shown in Figure 1.

![Simplified zone model with possible energy transfer paths](image)

**Figure 1: Simplified zone model with possible energy transfer paths**

Using the circuit equivalences described earlier, the space described in Figure 1 can be expressed as the circuit seen in Figure 2.
Applying nodal analysis to the circuit at the $T_{ia}$ and $T_m$ nodes provide the following two equations

\begin{align*}
\dot{Q}_{HVAC} + \dot{Q}_{load} + \frac{1}{R_{oi}} (T_{oa} - T_{ia}) + \frac{1}{R_{mi}} (T_m - T_{ia}) &= C_{ia} \dot{T}_{ia} \\
\frac{1}{R_{mi}} (T_{ia} - T_m) &= C_m \ddot{T}_m
\end{align*}

(8) \quad (9)

Ignoring saturation and other nonlinear effects, the energy applied through the heating and cooling equipment, $\dot{Q}_{HVAC}$, can be modeled as a PI controller defined as

\begin{equation}
\dot{Q}_{HVAC} = K_p \varepsilon_{SP} + K_I \int_0^t \varepsilon_{SP}(s) \, ds \tag{10}
\end{equation}

where

\begin{equation}
\varepsilon_{SP} = T_{SP} - T_{ia} \tag{11}
\end{equation}

and $T_{SP}$ is the temperature setpoint, that is, the desired temperature of indoor air. The control error, $\varepsilon_{SP}$, is redefined in terms of its integral as

\begin{equation}
I = \int_0^t \varepsilon_{SP}(s) \, ds \tag{12}
\end{equation}

for application in a differential set of equations. By applying (10) and (12) to (8) we get...
\[ K_p \dot{I} + K_l I + \dot{Q}_{\text{Load}} + \frac{1}{R_{oi}}(T_{oa} - T_{ia}) + \frac{1}{R_{mi}}(T_m - T_{ia}) = C_{ia} \dot{T}_{ia} \]  

(13)

and applying (12) to (11) we get

\[ \dot{I} = T_{SP} - T_{ia} \]  

(14)

Equations (9), (13) and (14) can be used to define the single zone’s state space system as

\[
\begin{bmatrix}
\dot{T}_{ia} \\
\dot{T}_m \\
\dot{I}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{C_{ia}} (K_p + \frac{1}{R_{mi}} + \frac{1}{R_{oi}}) & \frac{1}{C_{ia} R_{mi}} & \frac{K_l}{C_{ia}} \\
\frac{1}{C_{ia} R_{mi}} & -\frac{1}{C_{ia} R_{mi}} & 0 \\
-1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
T_{ia} \\
T_m \\
I
\end{bmatrix} +
\begin{bmatrix}
\frac{K_p}{C_{ia}} \\
\frac{1}{C_{ia} R_{oi}} \\
1
\end{bmatrix}
\begin{bmatrix}
T_{SP} \\
T_{oa} \\
Q_{\text{Load}}
\end{bmatrix}
\]  

(15)

\[
\begin{bmatrix}
T_{ia} \\
Q_{HVAC}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
-K_p & 0 & K_l
\end{bmatrix}
\begin{bmatrix}
T_{ia} \\
T_m \\
I
\end{bmatrix} +
\begin{bmatrix}
0 & 0 & 0 \\
K_p & 0 & 0
\end{bmatrix}
\begin{bmatrix}
T_{SP} \\
T_{oa} \\
Q_{\text{Load}}
\end{bmatrix}
\]  

(16)

The multi zone model, which comprises N zones, is created by combining individual zone models and connecting their \( T_{ia} \) state variables through an additional resistor, representing the energy transfer between zones through opening and walls. Equations (15) and (16) are redefined for a specific zone, \( i \), as

\[ \dot{x}^{(i)} = A^{(i)} x^{(i)} + B^{(i)} u^{(i)} \]  

(17)

\[ y^{(i)} = C^{(i)} x^{(i)} + D^{(i)} u^{(i)} \]  

(18)

The cross term matrix, \( \beta^{(i,j)} \), for a pair of zones, \( (i,j) \), is defined as

\[
\beta^{(i,j)} =
\begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{C_{ia}} R_{\beta}^{(i,j)} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]  

(19)

where \( R_{\beta}^{(i,j)} \) is thermal resistance between zones \( i \) and \( j \). The state space system for \( N \) zones can then be defined as
\[
\begin{bmatrix}
\dot{x}^{(1)} \\
\dot{x}^{(2)} \\
\vdots \\
\dot{x}^{(N)}
\end{bmatrix} =
\begin{bmatrix}
A^{(1)} & \beta^{(1,2)} & \cdots & \beta^{(1,N)} \\
\beta^{(2,1)} & A^{(2)} & \cdots & \beta^{(2,N)} \\
\vdots & \vdots & \ddots & \vdots \\
\beta^{(N,1)} & \cdots & \cdots & A^{(N)}
\end{bmatrix}
\begin{bmatrix}
x^{(1)} \\
x^{(2)} \\
\vdots \\
x^{(N)}
\end{bmatrix} +
\begin{bmatrix}
B^{(1)} \\
B^{(2)} \\
\vdots \\
B^{(N)}
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\begin{bmatrix}
\dot{u}^{(1)} \\
\dot{u}^{(2)} \\
\vdots \\
\dot{u}^{(N)}
\end{bmatrix}
\]

\[
\begin{bmatrix}
y^{(1)} \\
y^{(2)} \\
\vdots \\
y^{(N)}
\end{bmatrix} =
\begin{bmatrix}
C^{(1)} \\
0 \\
\vdots \\
0
\end{bmatrix}
\begin{bmatrix}
x^{(1)} \\
x^{(2)} \\
\vdots \\
x^{(N)}
\end{bmatrix} +
\begin{bmatrix}
D^{(1)} \\
D^{(2)} \\
\vdots \\
D^{(N)}
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\begin{bmatrix}
\dot{u}^{(1)} \\
\dot{u}^{(2)} \\
\vdots \\
\dot{u}^{(N)}
\end{bmatrix}
\]

The inputs for \(T_{oa}\) in each \(u^{(i)}\) are the same value and can immediately reduce along with B and D as

\[
B =
\begin{bmatrix}
K_p^{(1)} & \cdots & 0 & 1 & \cdots & 0 \\
C_{ia}^{(1)} & \cdots & 0 & C_{ia}^{(1)} R_{oi}^{(1)} & \cdots & 1 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & K_p^{(N)} & \cdots & 1 & \cdots \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 1 & 0 & \cdots & 0
\end{bmatrix}
\]

\[
D =
\begin{bmatrix}
K_p^{(1)} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & K_p^{(N)} & 0 & \cdots & 0
\end{bmatrix}
\]

This multi-zone model is meant to be used in an MPC application by expressing how \(Q_{HVAC}\) can be managed through changes in the setpoint based on predictions of outdoor air temperature and \(Q_{Load}\). As mentioned earlier, for a large number of zones this model can become too complex to be optimized in a reasonable amount of time, thus the need for model reduction.
3. MODEL REDUCTION TECHNIQUES

This chapter provides an introduction to the modeling reduction techniques that are explored. Transfer function pole/zero removal, which focuses on maintaining the steady state behavior of a transfer function while non-dominant poles and zeros are eliminated is presented in section 3.1. In section 3.2, the dominant eigenvalue method, which eliminates non-dominant eigenvalues directly in the state space for is discussed. Balanced model reduction, a method founded on retaining state variables that are easily controlled and observed, is shown in section 3.3. Finally, an application specific method that takes advantage of the simplified zone model and assumes a common airspace is proposed in section 3.4.

3.1 Transfer Function Pole/Zero Removal

The order of a state space system is equivalent to the number of poles in its transfer function. By representing a system in pole/zero form and removing those poles and zeros with the least impact on the response of the system, the order of the equivalent system is reduced. Removing poles and zeros in this method also necessitates managing the steady state response, as will be explained further in this section.

The transfer function of a system is a concise way of describing the input to output relationships [31]. From state space system form, the transfer function of a system is defined as

\[ H(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D \]  

for a single-input single-output system. In general, the transfer function takes the form

\[ H(s) = \frac{b_0s^m + b_1s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n} \]  

(26)
with \( m \leq n \). The pole/zero form is useful when analyzing the response of the transfer function.

The zero/pole form is achieved by factoring the numerator and denominator.

\[
H(s) = K \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}
\]  

(27)

The poles of \( H(s) \) are the eigenvalues of \( A \), and reducing the number of poles equally reduces the number of state variables of the equivalent state space system.

The dominant poles, those which are generally retained, are those whose effect on the system last longer. The real part of each pole corresponds to an exponential envelope in the time response. If the real part of the poles are positive, the response approaches infinity as time goes to infinity and result in an unstable system, while poles with negative real values result in responses approaching the setpoint of the control. Poles with a large negative real components approach zero faster and their effect on the system is reduced sooner. As a result, poles with a smaller negative real component will have a longer effect and are considered dominant. This effect is relative to the other poles in the system, with the rule of thumb being a pole is dominant to another if its real component is approximately one order of magnitude or more smaller in value than the others [32], as demonstrated in Figure 3.
To maintain a causal system of the reduced transfer function, the number of zeros may also be reduced so that the number of finite zeros is less than or equal to the number of finite poles. While the poles and zeros determine the transient response of the system, the steady state behavior must also be taken into account. It is preferable for this value to be the same in both the original and the reduced transfer function. The DC gain of the transfer function is calculated by evaluating the response of the system when a step input is applied. Using the final value theorem [32], defined as
\[
\lim_{s \to 0} sY(s) = \lim_{s \to 0} sY(t)
\]
and substituting in the response to a step input, \(y_{step}(s) = \frac{1}{s} H(s)\), the dc gain is defined as:
\[
DC\ gain = \lim_{s \to 0} y_{step}(t) = \lim_{s \to 0} s \left[ \frac{1}{s} H(s) \right] = \lim_{s \to 0} H(s)
\]
To maintain the steady state behavior while eliminating poles and zeros, the removed values are absorbed into \( K \). While accounting for the dc gain, the reduced transfer function, \( H_r(s) \), where only the first \( r \) poles and \( j \) zeros are retained, be defined as

\[
H_r(s) = K' \frac{(s - z_1)(s - z_2) \cdots (s - z_j)}{(s - p_1)(s - p_2) \cdots (s - p_r)}
\]  

(30)

where

\[
K' = K \frac{(-z_{j+1}) \cdots (-z_m)}{(-p_{r+1}) \cdots (-p_n)}
\]  

(31)

With the reduced transfer function, the final step is to return to the state space system form. There is not a unique state space representation for a given transfer function, but the most straightforward conversions use the coefficients of the transfer function described in (26) in a standard form. One such standard form is the observable canonical form defined as

\[
\dot{x} = \begin{bmatrix}
-a_1 & 1 & 0 & \cdots & 0 \\
-a_2 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{n-1} & 0 & 0 & \cdots & 1 \\
-a_n & 0 & 0 & \cdots & 0 \\
\end{bmatrix} x + \begin{bmatrix}
b_1 - a_1 b_0 \\
b_2 - a_2 b_0 \\
\vdots \\
b_{n-1} - a_{n-1} b_0 \\
b_n - a_n b_0 \\
\end{bmatrix} u
\]

(32)

\[
y = [1 \ 0 \ \cdots \ 0 \ 0] x + [b_0] u
\]

(33)

Up until this point the system has been assumed to be single-input single-output. Equation (25) holds for the multi-input multi-output case, but results in a set of transfer functions.
\[
H(s) = \begin{bmatrix}
H_{(1,1)}(s) & H_{(1,2)}(s) & \cdots & H_{(1,l)}(s) \\
H_{(2,1)}(s) & H_{(2,2)}(s) & \cdots & H_{(2,l)}(s) \\
& & \ddots & \vdots \\
H_{(m,1)}(s) & H_{(m,2)}(s) & \cdots & H_{(m,l)}(s)
\end{bmatrix}
\]

\[
= \frac{1}{G(s)} \begin{bmatrix}
F_{(1,1)}(s) & F_{(1,2)}(s) & \cdots & F_{(1,l)}(s) \\
F_{(2,1)}(s) & F_{(2,2)}(s) & \cdots & F_{(2,l)}(s) \\
& & \ddots & \vdots \\
F_{(m,1)}(s) & F_{(m,2)}(s) & \cdots & F_{(m,l)}(s)
\end{bmatrix}
\]

(34)

To maintain a common set of dynamics, the set of transfer functions must share a common set of poles \(G(s)\), while each set of zeros \(F_{(i,j)}(s)\) can remain unique. To reduce the system, both \(G(s)\) and \(F_{(i,j)}(s)\), \(i = 1: m; j = 1: l\) need to be reduced. The resulting set of transfer functions then need to be converted to a state space system with \(r\) states. For a multi-input single-output (MISO) system the process involves adding an additional column to \(B\) and \(D\) corresponding to each additional input.

\[
B = \begin{bmatrix}
b_{1}^{(1,1)} & b_{1}^{(1,2)} & \cdots & b_{1}^{(1,l)} \\
b_{2}^{(1,1)} & b_{2}^{(1,2)} & \cdots & b_{2}^{(1,l)} \\
& & \ddots & \vdots \\
b_{n-1}^{(1,1)} & b_{n-1}^{(1,2)} & \cdots & b_{n-1}^{(1,l)} \\
b_{n}^{(1,1)} & b_{n}^{(1,2)} & \cdots & b_{n}^{(1,l)}
\end{bmatrix}
\]

(35)

\[
D = \begin{bmatrix}
b_{0}^{(1,1)} & b_{0}^{(1,2)} & \cdots & b_{0}^{(1,l)}
\end{bmatrix}
\]

(36)

where \(b_{1}^{(i,j)} \cdots b_{n}^{(i,j)}\) are the coefficients corresponding to \(F_{(i,j)}(s)\). The single-input multi-output (SIMO) case is also straightforward to define through the use of the controllable canonical form. However, there is not a standard form that can be used to define the multi-input multi-output case. Using the multi-input single output (MISO) case as a starting point and keeping the \(A\) and \(B\) matrices the \(C\) matrix takes on the form

\[
C = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & c_{1}^{(1)} & c_{2}^{(1)} & \cdots & c_{n-1}^{(1)} & c_{n}^{(1)} \\
& & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & c_{1}^{(m)} & c_{2}^{(m)} & \cdots & c_{n-1}^{(m)} & c_{n}^{(m)}
\end{bmatrix}
\]

(37)
As can be seen, the first row remains unchanged, but the following rows are not directly related to one state and the coefficients cannot be taken from the transfer functions.

Consider a case in which there exists an equivalent state space system for a multi-input multi-output (MIMO) set of transfer functions. For this statement to be valid each transfer function’s numerator and denominator must be equal to the results of (25). Breaking down (25) into these 2 parts gives

\[
\frac{C \ adj(sI - A) \ B + D \ det(sI - A)}{det(sI - A)} \tag{38}
\]

As A is the only factor in the denominator, all the transfer functions are guaranteed to have the same eigenvalues. This leaves the numerators as dependent on the rest of the state space system matrices. The values of A, B and D are all determined based on the MISO analysis of the system. This leaves the values of C to be chosen such that the coefficients for each power of s in (26) and (38) are equal. The values of C do not directly relate to each power of s, however, and instead result in a series of equations. The following is a derivation of how the values of C relate to the coefficients of the powers of s for the n dimension case, where \( b_0 \) is assumed to be 0.

The first step is to derive an expression for the adjugate of the \( n \times n \) dimension \((sI - A)\) matrix, when A is in the observable canonical form. Starting with the \( 3 \times 3 \) case, the adjugate matrix is derived as

\[
adj(sI - A) = adj \begin{bmatrix} s + a_1 & -1 & 0 \\ a_2 & s & -1 \\ a_3 & 0 & s \end{bmatrix}
\]
While some patterns are becoming visible, higher order adjugate matrices are required to fully identify the patterns. The adjugate of the $4 \times 4$ case can be found to be

$$
\begin{bmatrix}
    s^2 - a_2 s - a_3 & s^2 + a_1 s & 1 \\
    -a_2 s^2 - a_3 s - a_4 & s^3 + a_1 s^2 & s + a_1 \\
    -a_3 s^2 - a_4 s & -a_3 s - a_4 & s^3 + a_1 s^2 + a_2 s \\
    -a_4 s^2 & -a_4 s & s^3 + a_1 s^2 + a_2 s + a_3
\end{bmatrix}
$$

(40)

From here, a general form for the $n \times n$ case can be extrapolated, based on the upper and lower triangular behaviors, as follows

$$
\begin{bmatrix}
    s^{n-1} & s^{n-2} & \ldots & s & 1 \\
    -a_2 s^{n-2} - a_3 s^{n-3} - \ldots - a_n & s^{n-1} + a_1 s^{n-2} & \ldots & s^2 + a_1 s & s + a_1 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    -a_{n-1} s^{n-2} - a_n s^{n-3} & -a_{n-1} s^{n-3} - a_n s^{n-4} & \ldots & \sum_{i=0}^{n-2} a_i s^{n-1-i} & \sum_{i=0}^{n-2} a_i s^{n-2-i} \\
    -a_n s^{n-2} & -a_n s^{n-3} & \ldots & -a_n & \sum_{i=0}^{n-1} a_i s^{n-1-i}
\end{bmatrix}
$$

(41)

With the adjugate matrix defined, the entirety of the numerator can be expressed by multiplying by the row of C and column of B corresponding to input and output being fit. Again, starting with the $3 \times 3$, case the resulting coefficients for each order of $s$ are determined as
\[
\begin{bmatrix}
  c_1 & c_2 & c_3
\end{bmatrix}
\begin{bmatrix}
  s^2 & s & 1 \\
-a_2 s - a_3 & s^2 + a_1 s & s + a_1 \\
-a_3 s & -a_3 & s^2 + a_1 s + a_2
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\] (42)

\[
=[c_1 \ c_2 \ c_3]
\begin{bmatrix}
b_1(s^2) + b_2(s) + b_3(1) \\
b_1(-a_2 s - a_3) + b_2(s^2 + a_1 s) + b_3(s + a_1) \\
b_1(-a_3 s) + b_2(-a_3) + b_3(s^2 + a_1 s + a_2)
\end{bmatrix}
\]

\[
=c_1(b_1(s^2) + b_2(s) + b_3(1)) + c_2(b_1(-a_2 s - a_3) + b_2(s^2 + a_1 s) + b_3(s + a_1)) +
\]

\[
c_3(b_1(-a_3 s) + b_2(-a_3) + b_3(s^2 + a_1 s + a_2))
\]

\[
=s^2(c_1 b_1 + c_2 b_2 + c_3 b_3) + s(c_1 b_2) + c_2(a_1 b_2 - a_2 b_1 + b_3) + c_3(a_1 b_3 - a_3 b_1)
\]

\[
+(c_1 b_3) + c_2(a_1 b_3 - a_3 b_1) + c_3(a_2 b_3 - a_3 b_2)
\]

Aligning this with the coefficients \(b_1^{(i,j)} \ldots b_n^{(i,j)}\) found in \(F_{(i,j)}(s)\) where \(i \neq 1\), the problem can be framed in the form \(Ax = b\) with \(x = [c_1 \ldots c_n]^T\) and \(b = [b_1^{(i,j)} \ldots b_n^{(i,j)}]^T\).

\[
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\begin{bmatrix}
b_2 \\
b_3 \\
b_3
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\begin{bmatrix}
b_1^{(i,j)} \\
b_2^{(i,j)} \\
b_3^{(i,j)}
\end{bmatrix}
\] (44)

This process can be repeated for higher order systems until a pattern forms, see APPENDIX A for the 6th order case. The result is a general form for the \(n \times n\) case of

\[
\begin{bmatrix}
  \alpha_{(1,1)} & \ldots & \alpha_{(1,n)} \\
  \vdots & \ddots & \vdots \\
  \alpha_{(n,1)} & \ldots & \alpha_{(n,n)}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{bmatrix}
= \begin{bmatrix}
b_1^{(i,j)} \\
b_2^{(i,j)} \\
\vdots \\
b_n^{(i,j)}
\end{bmatrix}
\] (45)

where

\[
\alpha_{(k,l)} = \sum_{l=1}^{l} a_{ll-1} b_{k+l-l} - a_{k+l-ll} b_{ll-1}
\]

and \(a_0 = 1, b_0 = 0\) and \(a_k = 0\) and \(b_k = 0\) for \(k > n\).

The values of \(c\) from (45) can then be found by solving \(x = A^{-1} b\). This method works for determining a row of \(C\), (37), based on a selected \(F_{(i,j)}(s)\); however, once a row of \(C\) has been determined using the transfer function with \(i = \xi\) and \(j = \mu\), the transfer functions for \(i = \xi\)
and $j \neq \mu$ cannot be fit. As a result, there is no guarantee that the reduced system’s transfer function for $i = \xi$ and $j \neq \mu$ will equal the desired $F_{(i,j)}(s)$. In summary, even if all transfer functions have a common denominator made up of $n$ poles, it is not necessary that there exists an equivalent state space system with $n$ states.

### 3.2 Dominant Eigenvalue

The number of eigenvalues of a system, or poles the transfer function, is related to the order of the model. Dominant eigenvalue model reduction is directly based on the state-space representation of the system and focuses on maintaining only the eigenvalues which are dominant and/or critical to the system and eliminating the remainder. The method reduces the model by applying a change of variables to describe the states of the model in terms of the eigenvalues, $\lambda$, and removing the states related to the non-dominant eigenvalues. By applying a change of variables, this method naturally extends to MIMO cases. For simplicity, it is assumed that all eigenvalues of $A$ are distinct.

To start the reduction take $x_r = Kx$ where $K \in \mathbb{R}^{n \times n}$. Substituting $x = K^+ x_r$ into (1) and (2) gives

$$K^+ \dot{x}_r = AK^+ x_r + Bu \quad (47)$$

$$y_r = CK^+ x_r + Du \quad (48)$$

Multiplying (47) by $K$ gives

$$\dot{x}_r = KAK^+ x_r + KBu \quad (49)$$

Comparing (49) and (48) to the (3) and (4) respectively, the form of the reduced model is given as $F = KAK^+$, $G = KB$, $H = CK^+$, and $J = D$. The difficulty of this method lies in determining a $K$ that maintains the eigenvalues for $A$. The straightforward way of accomplishing this is to use the
eigenvalue decomposition of $A = WDW^{-1}$, where the $D$ is the diagonal matrix whose values are the eigenvalues of $A$, and the columns of $W$ are the corresponding eigenvectors $[w_1 \ w_2 \ldots \ w_n]$. Consider the case of $r = n$, in which $F = D$. To achieve this transformation one choice of $K$ is $W^{-1}$. Now, consider the case of $r < n$, in which $F = D_r$, where $D_r$ is the diagonal matrix containing the $r$ eigenvalues that are to be retained. This transformation can be accomplished by taking $K$ as the eigenvectors of $W$ that correspond to the eigenvalues to be retained [33].

This process holds for systems that include complex conjugate pairs given that the pairs are removed together and the eigenvectors are defined such that the resulting system is real. To ensure the second part, consider a real 2x2 matrix $A$, with a complex eigenvalue at $\lambda = a - bj$ and corresponding eigenvectors $v \in \mathbb{C}^n$; it can be decomposed as $A = PQP^{-1}$ for $P = \begin{bmatrix} \text{real}(v) & \text{imag}(v) \end{bmatrix}$ and $Q = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. In general, to retain the complex pair $\lambda_i$ and $\lambda_{i+1}$ the $i$ and $i + 1$ columns of $K$ are taken as $\text{real}(w_i)$ and $\text{imag}(w_i)$ respectively. The resulting system remains real with $F$ as a block diagonal matrix.

While this method maintains the dominant eigenvalues of $A$, other characteristics of the transfer functions are not necessarily maintained. This generally results in a steady state error, which is dependent on the chosen value of the eigenvectors of $A$ as follows:

$$C(-A)^{-1}B + D \neq CK^+(-KAK^+)^{-1}KB + D$$

(50)

While scaling an eigenvector results in an equivalent one, the resulting reduced model can have completely different steady state error.

This method is straightforward and easy to implement. However, there are an infinite number of reduction matrices that maintain the desired eigenvalues and do not produce the
same transfer function. As a result, this method’s applications can be limited without intervention to scale the eigenvectors.

3.2.1 Two Time Scale Property

The two time scale property refers to a system that can be expressed in terms of a slow and fast system [34].

\[
\begin{bmatrix}
\dot{x}_s \\
\dot{x}_f
\end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & A_f \end{bmatrix} \begin{bmatrix} x_s \\
{x}_f \end{bmatrix} + \begin{bmatrix} B_s \\ B_f \end{bmatrix} u
\]

\[y = \begin{bmatrix} C_s & C_f \end{bmatrix} \begin{bmatrix} x_s \\
{x}_f \end{bmatrix} + Du \quad (51)
\]

where the eigenvalues of \( A_s \) are dominant compared to \( A_f \). The system is reduced by only keeping the state variables in the slow system.

The two-time scale property can be used as a standalone model reduction technique and does not require \( A \) being a block diagonal matrix. In this case the system is transformed such that \( A \) is a diagonal matrix. To achieve this transformation, \( K \) is simply taken as a similarity transformation \( W^{-1} \). The system is then defined in terms of its fast and slow components based on the eigenvalues and desired model order and the slow system is retained. This results in a system that is similar to the previously defined dominant eigenvalue method, but with a generally better steady state error and the opportunity for manual tuning of the separation between fast and slow that is further explored in section 4.1.2.

3.3 Balanced Model Reduction

Similar to the dominant eigenvalue method, balanced model reduction is based upon applying a change of variables to the system and removing a number of the state variables. In this method, the change of variables is done in such a way to balance the system, and the states that are removed are those that are hard to control and/or observe. This method takes into
account information available in A, B, and C which lends itself to maintaining the input/output characteristics.

Consider the system

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]

A state \( x \) is controllable if there exists an input \( u \) of finite magnitude that drives any \( x_0 \) to any \( x_f \) in finite time. The system is controllable if the pair \( (A, B) \) is controllable, that is the controllability matrix, \( P_c \), defined as

\[
P_c = [B \ AB \ A^2B \ \cdots \ A^{n-1}B]
\]

is full rank. If \( A \) is also stable, then the unique solution to the Lyapunov equation

\[
AW_c + W_cA^T = -BB^T
\]

is \( W_c > 0 \) and can be expressed as

\[
W_c = \int_0^\infty e^{At}BB^Te^{A^T\tau}d\tau
\]

Equation (54) is the Controllability Grammian and the columns of \( W_c \) span the controllable subspace.

Similarly, a state \( x \) is observable if, given the input and output record for any \( t \geq 0 \), the initial state of \( x_0 \) can be determined in a finite amount of time. The conditions for observability are the dual to controllability, where the pair \( (A, C) \) is observable if the observability matrix, \( P_o \), defined as

\[
P_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}
\]
is full rank. For a stable $A$, there exists a unique $W_o > 0$ solution to the Lyapunov equation

$$A^T W_o + W_o A = -C^T C$$

whose value is the Observability Grammian and defined as

$$W_o = \int_0^\infty e^{A^T \tau} C T C e^{A\tau} d\tau$$

The columns of $W_o$ span the observable subspace, and when combined with $W_c$ contain the necessary information to determine which states are easy to control and/or observe [35] [36].

The minimum energy required to reach a state $x_0$ from the zero state, defined as

$$L_c = \min_u \frac{1}{2} \int_{-\infty}^0 \|u(\tau)\|^2 d\tau$$

subject to: $x(0) = x_0, x(-\infty) = 0$

can be shown to have a solution given as

$$L_c = \frac{1}{2} x_0^T W_c^{-1} x_0$$

Likewise, the output energy generated while observing an initial state $x_0$ with no input defined as

$$L_o = \frac{1}{2} \int_{-\infty}^0 \|y(\tau)\|^2 d\tau$$

subject to: $x(0) = x_0, u = 0$

has a solution of

$$L_o = \frac{1}{2} x_0^T W_o x_0$$

It is desirable to consider a basis in which these two concept are equal, or balanced, such that

$$W_c = W_o = \Sigma$$
where $\Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_n)$. To find the value for the transformation matrix, $T$, which converts the basis into the balanced form in (62), the transforms of $A_b = TAT^{-1}, B_b = TB$ and $C_b = CT^{-1}$ are substituted into the Lyapunov equations (53) and (56) and compared to the balanced versions. Substituting the balanced matrices into equation (53) gives

$$ (T^{-1}A_bT)W_c + W_c(T^{-1}A_bT)^T + (T^{-1}B_b)(T^{-1}B_b)^T = 0 \quad (63) $$

After distributing the transposes the equation becomes

$$ (T^{-1}A_bT)W_c + W_c(T^T A_b^T T^{-T}) + (T^{-1}B_b)B_b^T T^{-T} = 0 \quad (64) $$

which can be simplified by pre-multiplying by $T$ and post-multiplying by $T^T$ into

$$ A_b TW_c T^T + TW_c T^T A_b^T + B_b B_b^T = 0 \quad (65) $$

Comparing equation (86) to the desired balance form of

$$ A_b \Sigma + \Sigma A_b^T + B_b B_b^T = 0 \quad (66) $$

it can be seen that

$$ TW_c T^T = \Sigma \quad (67) $$

A similar process for equation (56) can be applied but instead pre-multiplying by $T^{-T}$ and post-multiplying by $T^{-1}$ as follows:

$$ (T^{-1}A_bT)^T W_o + W_o(T^{-1}A_bT) + (C_bT)^T (C_bT) = 0 $$

$$ (T^T A_b T^{-T}) W_o + W_o(T^{-1}A_b T) + (T^T C_b T) (C_bT) = 0 $$

$$ A_b^T T^{-T} W_o T^{-1} + T^{-T} W_o T^{-1} A_b + C_b^T C_b = 0 \quad (68) $$

Comparing (68) to the balanced observability Lyapunov equation

$$ A_b^T \Sigma + \Sigma A_b + C_b^T C_b = 0 \quad (69) $$

it is seen that
\[ T^{-TW_oT^{-1}} = \Sigma \]  \hspace{1cm} (70)

To solve for \( T \), the two equalities are first multiplied together:

\[ (TW_oT^T)(T^{-TW_oT^{-1}}) = \Sigma^2 \]  \hspace{1cm} (71)

After canceling out the inner \( T^T \) and \( T^{-T} \), a substitution of \( R^TR \), defined as the Cholesky decomposition of the positive definite matrix \( W_o \), is made resulting in

\[ TW_oR^TR^{-1} = \Sigma^2 \]

A multiplication of the identity matrix is injected in the form of \( I = R^{-1}R \), followed by a regrouping of the variables as follows

\[ TR^{-1}RW_oR^TR^{-1} = \Sigma^2 \]
\[ (RT^{-1})^{-1}(RW_oR^T)RT^{-1} = \Sigma^2 \]  \hspace{1cm} (72)

From equation (72) it can be seen that \( RW_oR^T \) is similar to \( \Sigma^2 \), which means \( RW_oR^T \) is positive definite and there exists an orthogonal transform, \( U \), such that \( RW_oR^T = U\Sigma^2U^{-1} \)

\[ RW_oR^T = U\Sigma^2U^{-1} \]  \hspace{1cm} (73)

Substituting (73) into (72) gives

\[ (RT^{-1})^{-1}(U\Sigma^2U^{-1})RT^{-1} = \Sigma^2 \]  \hspace{1cm} (74)

which can in turn be rewritten as

\[ V\Sigma^2V^{-1} = \Sigma^2 \]
\[ V = (RT^{-1})^{-1}U \]  \hspace{1cm} (75)
\hspace{1cm} (76)

Solving (76) for \( T \) gives

\[ T = VU^{-1}R \]  \hspace{1cm} (77)

which leaves \( V \) as an unknown. To solve for \( V \), equation (77) is substituted into (70), giving

\[ (VU^TR)^{-TW_o(VU^TR)^{-1}} = \Sigma \]  \hspace{1cm} (78)
After distributing the inverses and transposes, the Cholesky decomposition of $W_o$ is substituted in resulting in

\[(V^{-T}U^T R^{-T})R^T R (R^{-1}UV^{-1}) = \Sigma\] (79)

From here, several cancellation occur leaving only

\[V^{-T}V^{-1} = \Sigma\] (80)

As $\Sigma^2$ is diagonal, it is its own eigenvalue matrix. From the equation (75), it can be seen that $V$ is an eigenvector matrix of $\Sigma^2$, which is in turn a diagonal matrix, meaning $V^{-T} = V^{-1}$. Finally $V$ is solved for as

\[V = \Sigma^{-1/2}\] (81)

which, when substituted back into (75), results in

\[T = \Sigma^{-1/2} U^T R\] (82)

As the energy generated by observing a state is proportional to the value of $W_o$, (61), and the minimum energy required to control a state is inversely related to the value of $W_c$, (59), those states which are easy to reach and observe are those with a large corresponding value in $\Sigma$. After the similarity transformation is applied using $T$, each state variable in the resulting system

\[
\begin{align*}
\dot{x} &= A_B x + B_B u \\
y &= C_B x + D u
\end{align*}
\] (83) (84)

corresponds to a $\sigma_i$ from $\Sigma$. The reduced system is determined by keeping the states corresponding to the $r$ largest values of $\Sigma$ [37].
3.4 Application Specific Method

An application specific method is based on an understanding of the physics of the system and the application of the reduced system. In general, an application specific method will not extend to a different problem. This method is based on a simplification that treats the resistance between zones as negligible due to the application focus on total $Q_{HVAC}$ and average $T_{ia}$.

Expanding the single zone circuit, Figure 2, to the multi zone case with N zones results in network described in Figure 4. As described earlier, this model has the potential for each zone’s indoor air temperature to be correlated to any other zone through some resistance, while the mass remains independent.

Consider the case in which all values of $R_{\beta}^{(i,j)}$ are equal to 0. This scenario is the equivalent of treating the air within the zones as one large space and results in single node for indoor air temperature $T_{ia}^{(p)}$. As this node is now common to all zones, the thermal resistors with the outdoor air are in parallel along with the thermal capacitances of the indoor air. Using circuit reduction techniques, the reduced values for $R_{oi}^{(p)}$ and $C_{ia}^{(p)}$ are found as
\[ R_{oi}(\rho) = \frac{1}{\sum_{i=1}^{N} \frac{1}{R_{oi}^{(i)}}} \]  \hspace{1cm} (85)

\[ C_{ia}(\rho) = \sum_{i=1}^{N} C_{ia}^{(i)} \]  \hspace{1cm} (86)

To take this reduction one step further, the same reduction can be applied to the mass temperature. While these nodes do not have any direct connection in the original multi zone model, by treating all the indoor air temperatures as one node, the mass temperatures are now effectively connected. By treating the mass temperatures as a single node \( T_m^{(\rho)} \), the thermal resistors between the indoor air temperature and mass temperature are in parallel along with the mass’ thermal capacitors. As before, the values of \( R_{mi}(\rho) \) and \( C_m(\rho) \) are found as

\[ R_{mi}(\rho) = \frac{1}{\sum_{i=1}^{N} \frac{1}{R_{mi}^{(i)}}} \]  \hspace{1cm} (87)

\[ C_m(\rho) = \sum_{i=1}^{N} C_m^{(i)} \]  \hspace{1cm} (88)

This leaves the integral error states dependent on the number of zones being modeled. The reduction of the number of states is less straight-forward as there remains an independent setpoint for each of the original zones. The proposed approach is to use a weighted average of the setpoints when calculating the error, and sum the gain terms when used in \( F \), such that

\[ K_p(\rho) = \sum_{i=1}^{N} K_p^{(i)} \]  \hspace{1cm} (89)

\[ K_i(\rho) = \sum_{i=1}^{N} K_i^{(i)} \]  \hspace{1cm} (90)

\[ f(\rho) = \frac{1}{C_m(\rho)} \sum_{i=1}^{N} C_m^{(i)} T_{sp}^{(i)} - T_{ia}(\rho) \]  \hspace{1cm} (91)

Applying these reductions to the multi zone model results in the following form...
This method forfeits precise modeling of each zone in favor of a minimally reduced model that reasonably models the net effect of the system.
4. RESULTS

This chapter covers the results of applying the methods covered in Chapter 3 to two types of problems. First, the methods are applied to a test system to evaluate their strengths and weakness in section 4.1. In section 4.2, the multi-zone model is presented and the reduction techniques are applied.

4.1 Test System Results

To evaluate the strengths and weaknesses of the presented reduction techniques, they will first be applied to system representing a simple-machine infinite-bus power system, modeled using the Phillips-Heffron model [38]. The resulting 10th order system with 2-inputs and 2-outputs is defined as

\[
A = \begin{bmatrix}
-0.5517 & 0 & -0.3091 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1695 \\
-0.0410 & 0 & -0.0350 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 314.1593 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
9.5540 & 0 & -0.8660 & -20 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0.0421 & -0.0328 \\
-0.1962 & 10.8696 & -0.1672 & 0 & 0 & -10.8696 & 0 & 0 & 0 & 0 \\
-0.9386 & 51.9849 & -0.7999 & 0 & 0 & -41.1153 & -10.8696 & 0 & 0 & 0 \\
-0.9386 & 51.9849 & -0.7999 & 0 & 0 & -41.1153 & -10.8696 & -0.1 & 0 & 0 \\
0 & 0 & 0 & -1000 & -1000 & 0 & 0 & 1000 & -20 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0526 & -0.8211
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1000 \\
0 & 0.0926 & 0 & 0 & 0.4428 & 2.1179 & 2.1179 & 0 & 0 & 0
\end{bmatrix}^T
\]

\[
C = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.4777 & 0 & -0.0433 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

with eigenvalues of \(-18.9311 \pm 2.025i, -12.1968, -9.6484, -0.2394 \pm 3.235i, -2.1313, -0.8972 \pm 1.356i \) and \(-0.1001\). The system is observable and controllable; however, the controllability matrix is weakly full rank as one of its singular values is significantly smaller than the rest, \(6e^{-5}\) compared to the next smallest of 4 and the largest at \(1e^{14}\). While the poor controllability could be an issue in a classical control application, it implies there are states
which are not necessary to the dynamics of the system that can be removed as part of model reduction. The model is proposed in [38] and further explored in [39]. The methods explored in these papers are capable of reducing the model to a 4th order system with minimal error and further reduced down to a 3rd order system with a noticeable error that is potentially acceptable depending on the application.

Input one to the system represents the reference voltage, while input two is the mechanical torque. The two outputs are the torque angle and terminal voltage respectively, and all units are taken in per unit, p.u., form with the exception of the torque angle in units of radian, rad. The system’s response to a 0.05 step change applied to both inputs can be seen in Figure 5. After oscillating for approximately 15 seconds due to the complex poles, both outputs settle to their steady state values of ~0.02 rad and ~0.045 p.u. respectively.

Figure 5: Step response of System 1 to a 0.05 step applied to both inputs.
4.1.1 Manual Pole/Zero Removal

Of the original 10 poles roughly half can be considered dominant, leaving 4 or 5 poles available to be removed. The zeros of the original system can be seen in Table 3, where each column represents a transfer function from (34) and each entry is zero from equation (27).

<table>
<thead>
<tr>
<th>$H_{(1,1)}(s)$</th>
<th>$H_{(2,1)}(s)$</th>
<th>$H_{(1,2)}(s)$</th>
<th>$H_{(2,2)}(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20</td>
<td>-20</td>
<td>-18.995 + 2.014i</td>
<td>-30.242</td>
</tr>
<tr>
<td>-10.870 + 2.919e-07i</td>
<td>-0.1</td>
<td>-18.995 - 2.014i</td>
<td>-5.084 + 8.155i</td>
</tr>
<tr>
<td>-10.870 - 2.919e-07i</td>
<td>-10.8696</td>
<td>-1.254 + 1.088i</td>
<td>-5.084 - 8.155i</td>
</tr>
<tr>
<td>-1</td>
<td>-10.8696</td>
<td>-1.254 - 1.088i</td>
<td>-3.444 + 0.688i</td>
</tr>
<tr>
<td>-0.1</td>
<td>-1</td>
<td>-1.875</td>
<td>-3.444 - 0.688i</td>
</tr>
<tr>
<td>∞</td>
<td>0.0 + 3.488i</td>
<td>-10.910</td>
<td>-0.227</td>
</tr>
<tr>
<td>∞</td>
<td>0.0 - 3.488i</td>
<td>-10.830</td>
<td>-0.098</td>
</tr>
<tr>
<td>∞</td>
<td>∞</td>
<td>-0.100</td>
<td>-20</td>
</tr>
</tbody>
</table>

Each of the transfer functions consist of several dominant poles that can also be removed.

Consider the 7th order reduced system in which the poles at $-18.9311 \pm 2.025i$ and $-12.1968$ are removed. To maintain causality, the zero at -20 is removed from $H_{(1,1)}(s)$ and $H_{(2,1)}(s)$, and the zeros at $-18.995 \pm 2.014i$ are removed from $H_{(1,2)}(s)$. Applying the manual pole/zero removal algorithm results in the following system.

$$A = \begin{bmatrix}
-14.153 & 1 & 0 & 0 & 0 & 0 & 0 \\
-62.774 & 0 & 1 & 0 & 0 & 0 & 0 \\
-238.252 & 0 & 0 & 1 & 0 & 0 & 0 \\
-576.796 & 0 & 0 & 0 & 1 & 0 & 0 \\
-797.404 & 0 & 0 & 0 & 0 & 1 & 0 \\
-646.305 & 0 & 0 & 0 & 0 & 0 & 1 \\
-57.240 & 0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}$$
\[ B = \begin{bmatrix} 0 & 2.401 \\ 0 & 62.954 \\ -10.396 & 536.592 \\ -237.433 & 1698.126 \\ -1477.886 & 2550.556 \\ -1373.674 & 1705.077 \\ -122.825 & 146.640 \end{bmatrix} \]

\[ C = \begin{bmatrix} 1 \\ -5.552 \\ 0.525 \\ -0.037 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

A comparison of the original system and 7\textsuperscript{th} order reduced system to the same step response can be seen in Figure 6.

![Figure 6: 7th order manual pole/zero removal model step response compared to full order system.](image)

One metric for determining the reduced model’s predictive output accuracy is the mean squared error, given as

\[
\varepsilon_m = \frac{1}{N} \sum_{i=0}^{N-1} (Y_m(i) - \hat{Y}_m(i))^2
\]

(94)
for output $m$, where $N$ is the total number of samples. For the 7th order system, $\varepsilon_1 = 1.176e^{-4}$ and $\varepsilon_2 = 2.1042e^{-6}$. Repeating this process for orders 1 through 8, skipping orders which result in an unbalanced complex conjugate pair, and summing the mean squared errors of outputs one and two provides Figure 7.

![Figure 7: Total mean squared error of all outputs for each model order produced by applying manual pole/zero removal.](image)

There is a drastic increase in error when moving from 6th to 5th order models which implies 6th order is the minimal achievable order for this method. Figure 8 shows the step responses of 5th, 6th, and 7th order systems. While the steady state response is maintained for all orders, a phase shift is introduced along with an amplitude modification that becomes noticeably worse when the order is reduced to 5.
4.1.2 Dominant Eigenvalue

The same poles that were removed using the manual pole/zero method can be removed using the dominant eigenvalue method. Applying this method to generate a 7th order system by removing the \(-18.9311 \pm 2.025i\) and \(-12.1968\) eigenvalues, results in the following system.

\[
A = \begin{bmatrix}
-0.1001 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.239 & -3.235 & 0 & 0 & 0 & 0 \\
0 & 3.235 & -0.239 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.897 & -1.356 & 0 & 0 \\
0 & 0 & 0 & 1.356 & -0.897 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2.131 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -9.648
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
2.046e^{-4} & -2.561e^{-4} \\
10.745 & 15.528 \\
-3.205 & -25.486 \\
-39.359 & -24.143 \\
-79.85 & 11.13 \\
72.670 & -6.583 \\
-84.071 & 604.597
\end{bmatrix}
\]
\[ C = \begin{bmatrix} 0.6382 & -0.2885 \\ 0.2605 & -0.0075 \\ 0.212 & -0.0277 \\ -0.028 & 0.0135 \\ 0.0838 & -0.0319 \\ 0.0464 & -0.0234 \\ 1.037e^{-4} & -1.369e^{-4} \end{bmatrix}^T \]

\[ D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

The resulting step response error for this 7th order system is \( \varepsilon_1 = 4.172e^{-7} \) and \( \varepsilon_2 = 1.328e^{-5} \). The result of applying this process for the remaining possible model is seen in Figure 11.

The jump in error again happens between the 5th and 6th order systems, but the 6th order system does better than manual pole removal in terms of error. Figure 10 shows the response of the 5th, 6th, and 7th order systems. In this case, the phase and amplitude of the transient response is maintained in each of the reductions, but it is the steady state behavior that suffers. The 5th order model has a noticeable steady state error along with a phase shift, while the 6th and 7th orders do a good job of representing the full order model.
Figure 9: Total mean squared error of all outputs for each model order produced by applying dominant eigenvalue removal.

![Bar chart showing mean squared error for different model orders.]

Figure 10: Step response comparison of models created using dominant eigenvalue removal of 7th order, last valid model (6th order) and first invalid model (5th order).

![Graph showing step responses for different orders.]

After reducing the same eigenvalues but doing so using the two-time scale property, the resulting 7th order system had an updated $C$ matrix given as:
The resulting step response error for this 7\textsuperscript{th} order system is $\varepsilon_1 = 2.515e^{-7}$ and $\varepsilon_2 = 1.064e^{-5}$. Applying this process for the remaining possible model orders provides Figure 11.

Again, a 6\textsuperscript{th} order system is the cutoff for maintaining a reasonable amount of error; however, the error for the 6\textsuperscript{th} order system is several orders of magnitudes better than the manual pole removal method. Interestingly the 7\textsuperscript{th} order system manages to have a worse error than the 6\textsuperscript{th} order system. This is a surprising result as the error is expected to get worse with each additional eigenvalue removed, and it warrants further analysis.

![Figure 11: Total mean squared error of all outputs for each model order produced by applying dominant eigenvalue removal with the two time scale approach.](image)

In all of these reductions, the dominant eigenvalues are those with the smallest negative real components. Going from an 8\textsuperscript{th} order to 7\textsuperscript{th} order system eliminates the eigenvalue
at $-12.2$ and going to a $6^{th}$ order system removes $-9.65$. One theory is that the eigenvalue at $-9.65$ is worse to keep the one at $-12.12$, but by removing the eigenvalue at $-9.65$ before the other, this is shown to be only partially true as the total mean squared error is $6.04 e^{-6}$ compared to the error of the $6^{th}$ order system when both are removed of $2.2 e^{-7}$. While the two time scale property is intended to split a system into its fast and slow components and keep the slow, these results imply that focusing solely on the states with a slow time constant is insufficient, particularly due to steady state error. As the $A$ matrix is a block diagonal matrix, the relation between each input, $i$, and output, $j$, through each block can be expressed as its own partial transfer function $H_k(s)$. With each removed state/pair there is both an effect on the transient response, and the DC gain equivalent to the DC gain of the removed partial transfer function. For this system, the net effect of each partial transfer function on the DC gain is shown in Table 4.

**Table 4: Steady state effect of poles on each transfer function of academic problem it two time scale form**

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$H_{(1,1)}(s)$</th>
<th>$H_{(2,1)}(s)$</th>
<th>$H_{(1,2)}(s)$</th>
<th>$H_{(2,2)}(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.1001$</td>
<td>0.0013</td>
<td>$-5.9 e^{-4}$</td>
<td>$-0.0016$</td>
<td>$7.39 e^{-4}$</td>
</tr>
<tr>
<td>$-0.2394$</td>
<td>11.664</td>
<td>$-0.212$</td>
<td>16.857</td>
<td>$-0.306$</td>
</tr>
<tr>
<td>$\pm 3.235i$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.8972$</td>
<td>1.286</td>
<td>$-0.863$</td>
<td>0.789</td>
<td>$-0.529$</td>
</tr>
<tr>
<td>$\pm 1.356i$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-2.1313$</td>
<td>1.627</td>
<td>$-1.008$</td>
<td>$-0.1474$</td>
<td>0.0913</td>
</tr>
<tr>
<td>$-9.6484$</td>
<td>$-0.0024$</td>
<td>0.0092</td>
<td>0.017</td>
<td>$-0.0665$</td>
</tr>
<tr>
<td>$-12.1968$</td>
<td>0.0012</td>
<td>$-0.0069$</td>
<td>0.0104</td>
<td>0.0623</td>
</tr>
<tr>
<td>$\pm 2.025i$</td>
<td>$-2.31 e^{-4}$</td>
<td>0.0017</td>
<td>$1.26 e^{-5}$</td>
<td>$-9.3 e^{-5}$</td>
</tr>
</tbody>
</table>

The error decrease from a $7^{th}$ order to $6^{th}$ order model is due to the DC gains of the transfer functions based on the $-9.6484$ and $-12.1968$ eigenvalues having similar values but of the opposite sign. Specifically, the error for $H_{(2,2)}(s)$, cancels out when both of the eigenvalues are removed this way. This also explains why removing them in reverse order did
not significantly decrease the error. Based on this information, it appears that the state

corresponding to an eigenvalue of $-0.1001$, while slow, has a comparably small gain. From this
another $5^{th}$ order model is created using eigenvalues $-0.2394 \pm 3.235i$, $-0.8972 \pm 1.356i$,
and $-2.1313$ and produces a total error for both outputs of $2.19e^{-07}$ which is significantly
better than the original method.

4.1.3 Balanced Model Reduction

The first step is to balance the full system, which results in a system whose Grammians
are defined as

$$W_c = W_o = \text{diag}(12.160, 10.655, 2.487, 0.1337, 5.18e^{-3}, 1.841e^{-3}, 7.384e^{-4}, 1.113e^{-6})$$

As with the previous two methods there are 5 states whose corresponding Grammians are several
orders of magnitude larger than the rest, and 2 which are only one order of magnitude larger
then several others. Removing states whose Grammians correspond to $12.160, 10.655, 2.487$
results in the following system.

$$A = \begin{bmatrix}
-0.217 & -3.201 & -0.432 & -0.170 & -0.088 & -5.54e^{-3} & -1.247 \\
3.17 & -0.236 & -0.452 & -0.225 & -0.104 & -1.852e^{-3} & 4.398e^{-4} \\
0.275 & 0.118 & -0.849 & -1.598 & -0.528 & 0.056 & 0.029 \\
-0.167 & 0.120 & 1.585 & -0.687 & -0.750 & 0.238 & 0.106 \\
-0.038 & -0.038 & 0.634 & -0.589 & -2.105 & 1.717 & 0.719 \\
-0.025 & -0.015 & 0.323 & -0.380 & -2.283 & -15.139 & -11.970 \\
-0.031 & 0.026 & 0.090 & -0.173 & -0.445 & -2.434 & -7.664
\end{bmatrix}$$

$$B = \begin{bmatrix}
0.863 & 2.126 \\
0.321 & -2.221 \\
-2.042 & -0.237 \\
1.032 & 0.620 \\
0.746 & -0.079 \\
0.396 & -0.015 \\
0.093 & 0.140
\end{bmatrix}$$
\[
C = \begin{bmatrix}
2.292 & -0.108 \\
2.232 & -0.222 \\
1.967 & -0.597 \\
0.942 & -0.750 \\
0.440 & -0.608 \\
0.048 & 0.393 \\
0.015 & 0.167 \\
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

The resulting step response error for this 7th order system is \( \varepsilon_1 = 1.594e^{-10} \) and \( \varepsilon_2 = 4.646e^{-12} \). Applying this process for the remaining possible model orders provides Figure 12.

![Figure 12: Total mean squared error of all outputs for each model order produced by applying balanced model reduction.](image)

This method produces noticeably better results at lower orders, with the 4th order system still showing promising results. Figure 13 shows the step response of the 3rd, 4th and 7th order systems. Similar to dominant eigenvalue method, the phase and amplitude of the terminal voltage begins to degrade with a 3rd order model, while the steady state error becomes worse for both outputs. In addition to performing better in the mean squared error sense, this method
has the additional benefit of allowing any model order to be chosen as there are no complex conjugate pairs resulting in two states being removed at once.

![Balanced Model Reduction](image)

**Figure 13:** Step response comparison of models created using balanced model reduction of 7th order, last valid model (4th order) and first invalid model (3rd order).

### 4.2 Building Model Results

The first zone system to be considered is a simple 2 zone model where the zones are duplicates of each other and there is little resistance between the zones. An example of what this space might look like is seen in Figure 14. Each zone has the following parameters

\[ C_{ia} = 4179, \quad C_m = 138670, \quad R_{mi} = 0.2061, \quad R_{oi} = 0.706, \quad K_P = 2.8, \quad \text{and} \quad K_I = 0.004 \]

and the resistance between the zones \( R_{\beta}^{(1,1)} \) is 0.001.
The resulting system is defined as

\[
A = \begin{bmatrix}
-0.2414 & 0.0012 & 9.571e^{-7} & 0.2393 & 0 & 0 \\
3.499e^{-5} & -3.499e^{-5} & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0.2393 & 0 & 0 & -0.2414 & 0.0012 & 9.571e^{-7} \\
0 & 0 & 0 & 3.499e^{-5} & -3.499e^{-5} & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
6.6998e^{-4} & 0 & 3.3893e^{-4} & 2.3928e^{-4} & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 6.6998e^{-4} & 3.3893e^{-4} & 0 & 2.3928e^{-4} \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
4 & 0 & 0 & 0 & 0 & 0 \\
-2.8 & 0 & 0.004 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -2.8 & 0 & 0.004 \\
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
2.8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 2.8 & 0 & 0 & 0 \\
\end{bmatrix}
\]

and has eigenvalues of \(-0.05, -1.596e^{-3}, -5.72e^{-4}, -2.026e^{-5}, -3.668e^{-5}\), and \(-3.305e^{-5}\). This system is both observable and controllable, with less disparity of the controllability matrix's singular values than in the academic system, min of \(1e^{-8}\) versus a max of 1.

1. The first zone's two day response to a step change of \(T_{sp}^{(1)} = T_{sp}^{(2)} = 75 \ degF, T_{pa} = 3.92 \ degF\), and \(Q_{Load}^{(1)} = Q_{Load}^{(2)} = 4.52\) can be seen in Figure 15, with zone 2's response
being identical. The system is initialized to the zero state, which is practically unreasonable but simplifies management of the initial conditions when the model reduction techniques are applied. The indoor air temperature tracks to the setpoint within the first couple of hours and the $Q_{HVAC}$ has nearly settled to its steady state value of $\sim 97$ kW after the first day.

![Figure 15: Step response of zone 1 outputs in a two duplicate zone model, with $T_{sp}$ of 75, $T_{oa}$ of 3.92 and $Q_{load}$ of 4.52.](image-url)

Applying each method and following the same process used when analyzing the test system begins to show some limitations when applying model reduction to this system. The manual pole/zero removal method is unable to model the system dynamics even after reducing the order by 1 and immediately loses transient behavior which is shown in Figure 16. The dominant eigenvalue method is able to reduce the order by 1 but begins to introduce a steady state offset after that (Figure 17). The balanced model reduction method does slightly better allowing the system to be reduced to a 4$^{th}$ order system, a 33% increase, before it too breaks down, Figure 18.
Figure 16: Step response comparison of models created using manual pole/zero removal of 3$^{rd}$, 4$^{th}$ and 5$^{th}$ order models.

Figure 17: Step response comparison of models created using dominant eigenvalue/two time scale removal of 3rd, 4th and 5th order models.
The coupling between the zones allows some states to be removed, but when the resistance between the zones increases, the error introduced by reducing model order increases. Increasing $R_{\beta}^{(1,1)}$ from 0.001 to 0.01 the balanced model method is no longer able to maintain an acceptable amount of error using a 4th order system.

Lastly, the application specific method is applied to create a 3rd order model, which is the only option using this method. As the zones are duplicates of each other and the inputs to each zone are the same, there is zero error when applying this method. This demonstrates how the inputs used to test the system are too idealistic to gain a proper understanding of the reduction techniques. To address this, $T_{sp}^{(2)}$ is changed to 78 and $Q_{load}^{(1)}$ is changed to 6.74. The general model reduction techniques produce similar results as before, but the application specific method now shows noticeable error. In particular, the zones begin to fight each other as they control to different setpoints as seen in Figure 19 and Figure 20. While the individual $Q_{HVAC}$
output error is large, they are in balance, and the temperatures will consistently show similar responses compared to the full system as compared to the other reduction methods.

**Figure 19:** Zone 1's $T_{ia}$ and $Q_{hvac}$ actual application specific reduced model when system is given different setpoints.

**Figure 20:** Zone 2's $T_{ia}$ and $Q_{hvac}$ actual application specific reduced model when system is given different setpoints.
While balanced reduction method results are promising, the inability of some methods to successfully reduce the number of states by even one brings up the question of what is special about this system. From an application standpoint, there appears to be a great deal of redundancy in the system as the zones are duplicates of each other, but in reality, they are two reduced systems with distinct setpoints and loads. This system, as defined, is not well suited for model reduction. Depending on the application however, it may be necessary to further reduce the model order. To accomplish this, a change in the system’s starting definition is necessary.

There are 4 aspects of the system that are potentially unnecessary, individual $T_{SP}$, $Q_{Load}$, $Q_{HVAC}$, and $T_{ia}$. A common scenario would be one in which each zone receives the same $T_{SP}$. Less common applications but still practical are observing the total $Q_{HVAC}$, average $T_{ia}$, or averaging the inputs of $Q_{Load}$ using $C_{ia}$.

To compare these different options, only the balanced model reduction is considered at first. As this method has shown better results in the previous cases, any input/output reduction that does not show improvement with this method is likely not worth testing with other methods. Figure 21 shows the total mean squared error for model order when each of the 4 reduction options are applied individually. These results show that reducing $T_{SP}$ or $Q_{HVAC}$ can further reduce the order of the system.
Figure 21: Total mean squared error of all outputs for each model order produced by applying balanced model reduction, after combining one input or output feature.

By combining these two reductions and applying the balanced model reduction method, the system can now be successfully reduced down to a 3rd order model, Figure 22. Things become particularly interesting when an additional reduction of $T_{ia}$ is applied to this particular model. As the two zones are identical copies of each other, when these inputs and outputs are reduced, the model naturally reduces to a 3rd order system through pole zero cancelation, with the poles at $-0.05$, $-2.026e^{-5}$, and $-3.305e^{-5}$ canceling out. To verify that this is only possible due to the zones being exact copies, the same process is repeated but with one zone’s parameters updated to $C_{ia} = 10448, C_m = 343690, R_{mi} = 0.0863, R_{oi} = 0.3302, K_p = 5.6,$ and $K_l = 0.008$. As expected, there is no longer any pole/zero cancelations when the models are different even when the inputs and outputs are reduced. Figure 23 compares the error when reducing a model when only $T_{SP}$ and $Q_{Load}$ are combined versus $T_{SP}, Q_{Load}$ and $T_{ia}$. While combining 3 features does further reduce the error, it is only a small improvement over combining 2 features, so it appears it may not be worth the loss in information depending on the application.
Figure 22: Total mean squared error of all outputs for each model order produced by applying balanced model reduction, after combining $T_{SP}$ and $Q_{HVAC}$ for two duplicate zone model.

Figure 23: Total mean squared error of all outputs for each model order produced by applying balanced model reduction, after combining only $T_{SP}$ and $Q_{HVAC}$ and $T_{SP}$, $Q_{HVAC}$, and $T_{ia}$ for two different zone model.

These cases are still simple as they consist of only two zones, so the previous conclusions may be premature. As a final step to verify these methods, a 6 zone system is tested. The system uses parameters found in Table 5.
Table 5: Parameters used to create six zone model

<table>
<thead>
<tr>
<th>Zone</th>
<th>(C_{ia})</th>
<th>(C_m)</th>
<th>(R_{mi})</th>
<th>(R_{oi})</th>
<th>(K_P)</th>
<th>(K_I)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4179</td>
<td>138670</td>
<td>0.2061</td>
<td>0.706</td>
<td>2.8</td>
<td>0.004</td>
</tr>
<tr>
<td>2</td>
<td>4179</td>
<td>138670</td>
<td>0.2061</td>
<td>100</td>
<td>2.8</td>
<td>0.004</td>
</tr>
<tr>
<td>3</td>
<td>10448</td>
<td>343690</td>
<td>0.0863</td>
<td>0.3302</td>
<td>5.6</td>
<td>0.008</td>
</tr>
<tr>
<td>4</td>
<td>10448</td>
<td>343690</td>
<td>0.0863</td>
<td>0.3302</td>
<td>5.6</td>
<td>0.008</td>
</tr>
<tr>
<td>5</td>
<td>20896</td>
<td>683630</td>
<td>0.0444</td>
<td>100</td>
<td>11.2</td>
<td>0.016</td>
</tr>
<tr>
<td>6</td>
<td>20896</td>
<td>683630</td>
<td>0.0444</td>
<td>100</td>
<td>11.2</td>
<td>0.016</td>
</tr>
</tbody>
</table>

With this more complex model, reducing 3 features allows for a noticeable increase in model reduction capability with the same mean squared error, going from 9\(^{th}\) order in the 2 feature case to 5\(^{th}\) order in the 3 feature case (Figure 24).

![Figure 24: Total mean squared error of all outputs for each model order produced by applying balanced model reduction to six zone model.](image)

Finally, looking at the application specific method, the reduction is idealistic, as when the same inputs were applied to the two identical zones. This method works based on the averaging components, so when the inputs to each zone are the same and the outputs are taken as averages and totals the actions applied to the states align with the reduction applied to the
inputs/outputs; as a result, the error is always zero. Unfortunately, this is only the case when all of the inputs and outputs are collapsed, which limits its practicality to specialized applications.
5. CONCLUSION

In this thesis, several model reduction techniques are explored as they apply to multi-input multi-output thermal systems. These types of systems present a particular set of problems for classical model reduction techniques. Of the range of reduction techniques, those that were explored in this work include manual pole/zero removal, dominant eigenvalue reduction, balanced model reduction, and an application specific method. These methods were first tested using an academic problem to show their strengths and weakness, and then applied to multi zone thermodynamic model. From the test system, the balanced model reduction technique is shown to perform better in terms of predictive error and reliability. However, it was concluded that the multi zone thermodynamic model is an ill-posed problem as described in this work. The individual zones are each a minimal system and while the coupling allows for reduction of the states, a large error is produced when the inputs to each zone are not aligned. This model limitation is further expressed by the application specific method, which is designed to take full advantage of the model’s form, and yet produced noticeable error when different setpoints are applied. Modifications to the multi-zone model are explored to reframe the system in a form that is indicative of model reduction. The inputs are reduced to feed each zone with the same signal and/or the outputs are reduced to sample an aggregate version of the zones. By applying combinations of the reductions, the system becomes more receptive to the model reduction techniques.

Future topics to expand upon this thesis include additional reduction techniques and a formal definition between the reduced model characteristics and MPC behavior. Recently, a set of data-based machine learning methods have been introduced. These methods focus on fitting parameters to data to minimize some error rather than maintaining key system theoretical
characteristics of the full order system. While the reduced input/output model plus application
specific method provide a near perfect reduction, it is rigidly application specific. The machine
learning methods may alleviate the rigidity of the application method and are a potential set of
future work, along with further exploring Krylov methods. Additionally, it is of interest to
increase the fidelity of the original model when describing the multi zone system. By first
increasing the complexity and accuracy of the full order model, the model reduction methods
may be able to identify unnecessary states and redundancy that is currently hidden by the
chosen form. In addition to exploring these model reduction techniques, a study of the effect
the error has on the MPC’s ability to optimize the system would be valuable in bounding the
acceptable error. Likewise, defining the relation between the model order and optimization
complexity would provide guidance for a targeted model order.
REFERENCES


APPENDIX A

Parameter fitting matrices of a 6th order system.

\[
\begin{bmatrix}
    b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\
    b_2 & a_1 b_2 - a_2 b_1 + b_3 & a_1 b_3 - a_3 b_1 + b_4 & a_1 b_4 - a_4 b_1 + b_5 & a_1 b_5 - a_5 b_1 + b_6 & a_1 b_6 - a_6 b_1 \\
    b_3 & a_2 b_2 - a_3 b_1 + b_4 & a_2 b_3 - a_4 b_2 + a_1 b_4 - a_4 b_1 + b_5 & a_2 b_4 - a_4 b_2 + a_1 b_5 - a_5 b_1 + b_6 & a_2 b_5 - a_5 b_2 + a_1 b_6 - a_6 b_1 & a_2 b_6 - a_6 b_2 \\
    b_4 & a_3 b_2 - a_4 b_1 + b_5 & a_3 b_3 - a_4 b_2 + a_1 b_5 - a_5 b_1 + b_6 & a_3 b_4 - a_4 b_3 + a_2 b_5 - a_5 b_2 + a_1 b_6 - a_6 b_1 & a_3 b_5 - a_5 b_3 + a_2 b_6 - a_6 b_2 & a_3 b_6 - a_6 b_3 \\
    b_5 & a_4 b_2 - a_5 b_1 + b_6 & a_4 b_3 - a_5 b_2 + a_1 b_6 - a_6 b_1 & a_4 b_4 - a_5 b_3 + a_2 b_6 - a_6 b_2 & a_4 b_5 - a_5 b_4 + a_3 b_6 - a_6 b_3 & a_4 b_6 - a_6 b_4 \\
    b_6 & a_5 b_2 - a_6 b_1 & a_5 b_3 - a_6 b_2 & a_5 b_4 - a_6 b_3 & a_5 b_5 - a_6 b_4 & a_5 b_6 - a_6 b_5
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
c_5 \\
c_6
\end{bmatrix}
= \begin{bmatrix}
b_1^{(L)} \\
b_2^{(L)} \\
b_3^{(L)} \\
b_4^{(L)} \\
b_5^{(L)} \\
b_6^{(L)}
\end{bmatrix}
\]
APPENDIX B

Model reduction Matlab code

Manual Pole Zero Removal

```

    [zlr,pr,klr]= ReduceZeroPole(sys, newOrder, 1);
    
    if abs(sum(sum(imag(pr))))>1e-10
        sys_Out=[];
        return
    end

    [b11,all] = zp2tf(zlr(:,1),pr,klr(1));
    N=length(all)-1;
    for i = 2:N
        F(i-1,:)=[-a11(i) zeros(1,i-2), 1, zeros(1,N-i)];
    end
    F(N,:) = [-a11(N+1),zeros(1,N-1)];
    G=b11(1,2:end)';
    H=[1,zeros(1,N-1)];
    
    for i = 2:size(sys.C,1)
        [bli,ali] = zp2tf(zlr(:,i),pr,klr(i));
        H(i,:) = FindTransferFunctionTaps(-F(:,1),G(:,1),bli(2:end));
    end

    for i = 2:size(sys.B,2)
        [zir,pr,klr]= ReduceZeroPole(sys, newOrder, i);
        [bil,ail] = zp2tf(zir(:,1),pr,klr(1));
        G=[G bil(1,2:end)'];
    end
    
    sys_Out=ss(F,G,H,sys.D);
end

function [zr, pr, kr] = ReduceZeroPole(sys,newOrder,input)

    [z,p,k]=ss2zp(sys.A,sys.B,sys.C,sys.D,input);
    
    [~,index] = sort(real(p));
    p=flip(p(index));
    pr=p(1:newOrder);
    zeroOrder = min(newOrder-1,size(z,1));
    [~,index] = sort(real(z));
    for i = 1:size(index,2)
        z(:,i)=flip(z(index(:,i),i));
    end
```
```matlab
end
zr=z(1:zeroOrder,:);
for i=1:size(z,2)
    zz=z(zeroOrder+1:end,i);
    zz=zz(~isinf(zz));
    kr(i,1) = k(i,1)/prod(-p(newOrder+1:end)).*prod(-zz);
end
end

function [ cs ] = FindTransferFunctionTaps( as, bs, zs )
    N = length(as);
    as=[1 as(:)'; zeros(1,N)];
    bs=[0 bs(:)'; zeros(1,N)];
    for k = 1:N
        for l = 1:N
            Ac(k,l)=0;
            for ll = 1:l
                Ac(k,l)=Ac(k,l)+as(ll)*bs(k+l-ll+1)-as(k+l-ll+1)*bs(ll);
            end
        end
    end
    for i = 1:N
        Ac(i,N+1) = zs(i);
    end
    ech=rref(Ac);
    cs=ech(:,end)';
end

Dominant Eigenvalue Reduction

function [ sys_Out ] = DominantEigenValueReduction( sys, newOrder )
    [V,D]=eig(sys.A);
    [~,index] = sort(real(diag(D)));
    order=length(index);
    redIndex = index(order:-1:order-newOrder+1);
    realV=V;
    for i = 1:size(realV,1)
        if abs(sum(imag(realV(:,i))))>1e-20 &&
            abs(sum(sum(imag(realV(:,i:i+1)))))<1e-20
            realV(:,i+1)=imag(realV(:,i));
            realV(:,i)=real(realV(:,i));
        end
    end
    K=inv(realV);
    K=K(redIndex,:);
    F=K*sys.A*pinv(K);
    if abs(sum(sum(F-diag(diag(F)))))>1e-10
        sys_Out=[];
        return
    end
    G=K*sys.B;
```

Dominant Eigenvalue with Two Time Scale Reduction

```matlab
function [ sys_Out ] = TwoTimeScaleReduction( sys, newOrder )

[V,D]=eig(sys.A);

 [~,index] = sort(real(diag(D)));
 order=length(index);
 redIndex = index(order:-1:order-newOrder+1);

 realV=V;
 for i = 1:size(realV,1)
    if abs(sum(imag(realV(:,i))))>1e-20 &&
        abs(sum(sum(imag(realV(:,i:i+1)))))<1e-20
       realV(:,i+1)=imag(realV(:,i));
       realV(:,i)=real(realV(:,i));
    end
 end

 K=inv(realV);

 F=K*sys.A*inv(K);
 G=K*sys.B;
 H=sys.C*inv(K);

 F=F(redIndex,redIndex);
 eigenvalues= diag(D);
 if abs(sum(imag(eigenvalues(redIndex))))>1e-10
    sys_Out=[];
    return
 end
 G=G(redIndex,:);
 H=H(:,redIndex);

 sys_Out = ss(F,G,H,sys.D);
end
```

Balanced Model Reduction

```matlab
function [ sys_Out ] = BalancedModelReduction( sys, newOrder )

 Wc=gram(sys,'c');
 Wo=gram(sys,'o');

 U=chol(Wc)';
 [K,Sig2]=eig(U'*Wo*U);
```
Sig = sqrt(Sig2);
Tbal = sqrt(Sig)*K'*inv(U);
TbalInv = U*K*inv(sqrt(Sig));

[~, index] = sort(diag(Sig));
redIndex = index(end:-1:end-newOrder+1);

K = Tbal(redIndex, :);
F = Tbal*sys.A*inv(Tbal);
G = Tbal*sys.B;
H = sys.C*inv(Tbal);

F = F(redIndex, redIndex);
G = G(redIndex, :);
H = H(:, redIndex);

sys_Out = ss(F, G, H, sys.D);
end

Application Specific Reduction

function [ sys_Out ] = ApplicationSpecificReduction( zoneParams )
nZones = length(zoneParams);

A = zeros(3, 3);
B = zeros(3, 2*nZones+1);
C = zeros(2*nZones, 3);
D = zeros(2*nZones, 2*nZones+1);

Roi = 1/sum(1./[zoneParams(:).Roi]);
Rmi = 1/sum(1./[zoneParams(:).Rmi]);
Cia = sum([zoneParams(:).Cia]);
Cm = sum([zoneParams(:).Cm]);
Kp = sum([zoneParams(:).Kp]);
Ki = sum([zoneParams(:).Ki]);

A(1,1) = -1/Cia * (Kp + 1/Rmi + 1/Roi);
A(1,2) = 1/(Cia * Rmi);
A(1,3) = Ki/Cia;
A(2,1) = 1/(Cm * Rmi);
A(2,2) = -1/(Cm * Rmi);
A(3,1) = -1;

B(1, nZones+1) = 1/(Cia * Roi);
for i = 1:nZones
    ii = i-1;
    zp = zoneParams(i);
    B(1, ii+1) = zp.Kp/Cia;
    B(3, ii+1) = zp.Cia/Cia;
    B(1, nZones+i+ii+1) = 1/Cia;
C(ii*2+1,1)=1;
C(ii*2+2,1)=-zp*Kp;
C(ii*2+2,3)=zp*Ki;
D(ii*2+2,ii+1)=zp*Kp;
end
sys_Out = ss(A,B,C,D);
end