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Characterizations of Kumaraswamy-Laplace, McDonald Inverse Weibull and New Generalized Exponential Distributions

Gholamhossein G. Hamedani
Marquette University, gholamhoss.hamedani@marquette.edu

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Characterizations of Kumaraswamy-Laplace, McDonald Inverse Weibull and New Generalized Exponential Distributions

G.G. Hamedani
Department of Mathematics, Statistics and Computer Science
Marquette University
Milwaukee, WI 53201-1881

Abstract

Nassar (2016) considers an interesting univariate continuous distribution called Kumaraswamy-Laplace which has different forms on two subintervals. He studies certain properties and applications of this distribution. Shahbaz et al. (2016) consider another interesting distribution called McDonald Inverse Weibull distribution. They present some basic properties of their distribution and study the estimations of the parameters as well as discussing its application via an illustrative example. What is lacking in both papers, in our opinion, is the characterizations of these two interesting distributions. The present work is intended to complete, in some way, the works of Nassar and Shahbaz et al. via establishing certain characterizations of these distributions in four directions. We also introduce several New Generalized Exponential distributions and present their characterizations as well.

1. Introduction

The problem of characterizing a distribution is an important problem which can help the investigator to see if their model is the correct one. This work deals with various characterizations of Kumaraswamy-Laplace (KL) and McDonald Inverse Weibull (MIW) distributions to complement the works of Nassar (2016) and Shahbaz et al. (2016). These characterizations are presented in three directions: (i) based on the ratio of two truncated moments; (ii) in terms of the reverse hazard function and (iii) based on the conditional expectation of certain functions of the random variable. Similar characterizations as well as (iv) in terms of the hazard function, will be established for several proposed New Generalized Exponential (NGE) distributions. It should be noted that characterization (i) can be employed also when the cdf (cumulative distribution function) does not have a closed form as is the case with MIW distribution.

Nassar (2016) introduced KL distribution with cdf and pdf (probability density function) given, respectively, by

$$F(x)=F(x;\alpha,\mu,a,b)=\begin{cases} 1-\left(1-\left(\frac{1}{2}e^{\frac{x-\mu}{\alpha}}\right)^{a}\right)^{b}, & x<\mu \\ 1-\left(1-\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a}\right)^{b}, & x\geq\mu \end{cases}, \tag{1}$$

and

$$f(x)=f(x;\alpha,\mu,a,b)=\begin{cases} \dfrac{2^{-a}ab\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}\left(1-2^{-a}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}\right)^{b-1}}{\alpha}, & x<\mu \\ \dfrac{abe^{-\frac{x-\mu}{\alpha}}\left(1-\frac{1}{2}e^{\frac{x-\mu}{\alpha}}\right)\left(1-\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a}\right)^{b-1}}{2\alpha}, & x\geq\mu \end{cases}, \tag{2}$$

where α,a,b all positive and $\mu\in\mathbb{R}$ are parameters.

Shahbaz et al. (2016) proposed MIW distribution with cdf and pdf given, respectively, by

$$F(x) = F(x;\alpha,\beta,a,b,c) = \frac{ce^{-a\alpha x^{-\beta}}}{aB(ac^{-1},b)} \quad {}_2F_1\left[1-b, ac^{-1}; 1+ac^{-1}; e^{-a\alpha x^{-\beta}}\right], x \geq 0, \tag{3}$$

and

$$f(x) = f(x;\alpha,\beta,a,b,c) = \frac{\alpha\beta c x^{-(\beta+1)}}{B(ac^{-1},b)} e^{-a\alpha x^{-\beta}} \left[1-e^{-c\alpha x^{-\beta}}\right]^{b-1}, \quad x > 0, \tag{4}$$

where α, β, a, b, c are all positive parameters.

The cdf and pdf of our first NGE distribution (denoted by NGE1) are given, respectively, by

$$F(x) = F(x;\alpha,\lambda) = 1 - \exp\left\{-\lambda x\left(1-\log\left[\overline{K}(x)\right]\right)^{-\alpha}\right\}, \quad x > 0, \tag{5}$$

and

$$f(x) = f(x;\alpha,\lambda) = \lambda\left(1-\log\left[\overline{K}(x)\right]\right)^{-(\alpha+1)}\left\{1-\log\left[\overline{K}(x)\right] - \alpha x k(x)\left[\overline{K}(x)\right]^{-1}\right\} \times \exp\left\{-\lambda x\left(1-\log\left[\overline{K}(x)\right]\right)^{-\alpha}\right\}, \quad x > 0, \tag{6}$$

where $\lambda > 0, \alpha > 1$ are parameters, $K(x)$ $\left(\overline{K}(x) = 1 - K(x)\right)$ is a baseline cdf with corresponding pdf $k(x)$ such that $\lim_{x\to-\infty^+} k(x) > 0$.

Let

$$P(x) = \sum_{j=1}^{n} a_j x^j, \quad x \geq 0,$$

be a polynomial of degree n such that $a_n > 0$ and $\frac{d}{dx}P(x) > 0$ for $x > 0$.

The cdf and pdf of our second NGE distribution (denoted by NGE2) are given, respectively, by

$$F(x) = 1 - \exp\{-P(x)\}, \quad x \geq 0, \tag{7}$$

and

$$f(x) = \left(\frac{d}{dx}P(x)\right)\exp\{-P(x)\}, \quad x > 0. \tag{8}$$

Let $P(x) = \sum_{j=1}^{n} a_j x^j$, $0 \leq x \leq 1$ such that $P'(x) = \frac{d}{dx}P(x) > (x-1)^{-1}$ for $0 < x < 1$.

The cdf and pdf of our third NGE distribution (denoted by NGE3) are given, respectively, by

$$F(x) = 1 - (1-x)\exp\{-P(x)\}, \quad 0 \leq x \leq 1, \tag{9}$$

and

$$f(x) = [1 + (1-x)P'(x)]\exp\{-P(x)\}, \quad 0 < x < 1. \tag{10}$$

Let $P(x) = \sum_{j=0}^{n} a_j x^j$, $0 \leq x \leq 1$ such that $P'(x) < (1-x)^{-1}$ for $0 < x < 1$.

The cdf and pdf of our fourth NGE distribution (denoted by NGE4) are given, respectively, by

$$F(x) = 1 - c(1-x)\exp\{-P(x)\}, \quad 0 \leq x \leq 1, \tag{11}$$

and

$$f(x) = c[1 + (1-x)P'(x)]\exp\{-P(x)\}, \quad 0 < x < 1, \tag{12}$$

where $c = e^{-a_0}$.

Our fifth NGE distribution (denoted by NGE5), has cdf

$$F(x) = 1 - Q(x)\exp\{-P(x)\}, \quad x \geq 0, \tag{13}$$

with

$$Q(x) = \sum_{j=0}^{n} a_j x^j \,, \quad P(x) = \sum_{j=0}^{m} b_j x^j \,,$$

where

(a) $m \geq n, a_0 = 1, b_m > 0;$

(b) $Q(x) > 0 \; for \; x > 0;$

(c) $P(x) > \ln\{Q(x)\};$

(d) $P'(x) > \frac{Q'(x)}{Q(x)}.$

The corresponding pdf is given by

$$f(x) = \{Q(x)P'(x) - Q'(x)\}\exp\{-P(x)\}, \quad x > 0. \tag{14}$$

2. Characterizations

We present our characterizations $(i) - (iv)$ in four subsections.

2.1 Characterizations based on two truncated moments

This subsection deals with the characterizations of KL, MIW and NGE1-NGE5 distributions based on the ratio of two truncated moments. Our first characterization employs a theorem of Glänzel (1987), see Theorem 1 of Appendix A .The result, however, holds also when the interval H is not closed since the condition of Theorem 1 is on the interior of H.

Proposition 1. Let $X: \Omega \to \mathbb{R}$ be a continuous random variable and let

$$h(x) = \begin{cases} \left(1-2^{-a}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}\right)^{1-b}, & x < \mu \\ \left(1-\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a}\right)^{1-b}, & x \geq \mu \end{cases},$$

and

$$g(x) = \begin{cases} 2^{-a}\left(1-2^{-a}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}\right)^{1-b}\left(e^{\frac{x-\mu}{\alpha}}\right), & x < \mu \\ \left(1-\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a}\right)^{1-b}\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a}, & x \geq \mu \end{cases}$$

Then, the random variable X has pdf (2) if and only if the function ξ defined in Theorem 1 is of the form

$$\xi(x)=\begin{cases}\frac{1}{2}\left\{1+2^{-a}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}\right\} & , \quad x<\mu \\ \frac{1}{2}\left\{1+\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a}\right\}, & x\geq\mu\end{cases}.$$

Proof. First, we observe that the functions g, h defined above are in $C^1(H)$ and ξ is in $C^2(H)$, as required by Theorem 1 (see Appendix B). Now, suppose the random variable X has (2), then after some manipulations, we arrive at

$$\big(1-F(x)\big)E[h(x)\mid X\geq x]=\begin{cases}b\left\{1-2^{-a}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}\right\} & , \quad x<\mu \\ b\left\{1-\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a}\right\}, & x\geq\mu\end{cases},$$

and

$$\big(1-F(x)\big)E[g(x)\mid X\geq x]=\begin{cases}\frac{b}{2}\left\{1-2^{-2a}\left(e^{\frac{x-\mu}{\alpha}}\right)^{2a}\right\} & , \quad x<\mu \\ \frac{b}{2}\left\{1-\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{2a}\right\}, & x\geq\mu\end{cases},$$

and hence $\xi(x)$ has the above form. Further,

$$\xi(x)h(x)-g(x)=\begin{cases}\frac{h(x)}{2}\left\{1-2^{-a}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}\right\} & , \quad x<\mu \\ \frac{h(x)}{2}\left\{1-\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a}\right\}, & x\geq\mu\end{cases},$$

which is clearly not equal to zero for any $x<\mu$ or $x\geq\mu$.

Conversely, if ξ is of the above form, then

$$s'(x)=\frac{\xi'(x)h(x)}{\xi(x)h(x)-g(x)}=\left\{\begin{array}{ll}\dfrac{\frac{a2^{-a}}{\alpha}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}}{1-2^{-a}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}}, & x<\mu \\ \dfrac{\frac{a}{2\alpha}\left(e^{\frac{x-\mu}{\alpha}}\right)\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a-1}}{1-\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a}}, & x\geq\mu\end{array}\right\}.$$

Note that for $=\mu$, we have $s'(\mu)=s'(\mu^-)=s'(\mu^+)=\frac{a2^{-a}}{\alpha(1-2^{-a})}$. Now, according to Theorem 1, X has density (2).

Corollary 1. Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let $h(x)$ be as in Proposition 1. The random variable X has pdf (2) if and only if there exist functions g and ξ defined in Theorem 1 satisfying the following differential equation

$$\frac{\xi'(x)h(x)}{\xi(x)h(x)-g(x)}=\left\{\begin{array}{ll}\dfrac{\frac{a2^{-a}}{\alpha}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}}{1-2^{-a}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}}, & x<\mu \\ \dfrac{\frac{a}{2\alpha}\left(e^{\frac{x-\mu}{\alpha}}\right)\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a-1}}{1-\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a}}, & x\geq\mu\end{array}\right\}.$$

The general solution of the above differential equation is

$$\xi(x)=\left\{\begin{array}{ll}\left\{1-2^{-a}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}\right\}^{-1}\left[-\int\frac{a2^{-a}}{\alpha}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}\left(h(x)\right)^{-1}g(x)dx+D_1\right], & x<\mu\\ \left\{1-\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a}\right\}^{-1}\left[-\int\frac{a}{2\alpha}e^{-\frac{x-\mu}{\alpha}}\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a-1}\left(h(x)\right)^{-1}g(x)dx+D_2\right], & x\geq\mu\end{array}\right\},$$

where D_1 and D_2 are constants. We like to point out that one set of functions satisfying the above differential equation is given in Proposition 1 with $D_1=D_2=\frac{1}{2}$. Clearly, there are other triplets (h,g,ξ) which satisfy conditions of Theorem1.

Proposition 2. Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $h(x)=\left[1-e^{-c\alpha x^{-\beta}}\right]^{1-b}$ and $g(x)=h(x)e^{-a\alpha x^{-\beta}}$ for $x>0$. Then, the random variable X has pdf (4) if and only if the function ξ defined in Theorem 1 is of the form

$$\xi(x)=\frac{1}{2}\left\{1+e^{-a\alpha x^{-\beta}}\right\},\quad x>0.$$

Proof. Suppose the random variable X has (4), then

$$\left(1-F(x)\right)E[h(X)\mid X\geq x]=\frac{c}{aB(ac^{-1},b)}\left\{1-e^{-a\alpha x^{-\beta}}\right\},\qquad x>0,$$

and

$$\left(1-F(x)\right)E[g(X)\mid X\geq x]=\frac{c}{2aB(ac^{-1},b)}\left\{1-e^{-2a\alpha x^{-\beta}}\right\},\quad x>0.$$

Further,

$$\xi(x)h(x)-g(x)=\frac{h(x)}{2}\left\{1-e^{-a\alpha x^{-\beta}}\right\}>0\quad for\quad x>0.$$

Conversely, if ξ is of the above form, then

$$s'(x)=\frac{\xi'(x)h(x)}{\xi(x)h(x)-g(x)}=\frac{a\alpha\beta x^{-(\beta+1)}e^{-a\alpha x^{-\beta}}}{1-e^{-a\alpha x^{-\beta}}},\quad x>0,$$

from which we have

$$s(x)=-\log\left\{1-e^{-a\alpha x^{-\beta}}\right\},\quad x>0.$$

Now, according to Theorem 1, X has density (4).

Corollary 2. Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $h(x)$ be as in Proposition 2. The random variable X has pdf (4) if and only if there exist functions g and ξ defined in Theorem 1 satisfying the following differential equation

$$\frac{\xi'(x)h(x)}{\xi(x)h(x)-g(x)}=\frac{a\alpha\beta x^{-(\beta+1)}e^{-a\alpha x^{-\beta}}}{1-e^{-a\alpha x^{-\beta}}},\quad x>0.$$

The general solution of the above differential equation is

$$\xi(x)=\left\{1-e^{-a\alpha x^{-\beta}}\right\}^{-1}\left[-\int a\,\alpha\beta x^{-(\beta+1)}e^{-a\alpha x^{-\beta}}\left(h(x)\right)^{-1}g(x)dx+D\right],$$

where D is a constant. One set of functions satisfying the above differential equation is given in Proposition 2 with $D = 0$.

Proposition 3. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $h(x) \equiv 1$ and $g(x) = \exp\left\{-\lambda x(1 - \log\left[\overline{G}(x)\right])^{-\alpha}\right\}$ for $x > 0$. The random variable X belongs to the family (6) if and only if the function ξ defined in Theorem1 has the form

$$\xi(x) = \frac{1}{2}\left\{1 + \exp\left\{-\lambda x(1 - \log\left[\overline{K}(x)\right])^{-\alpha}\right\}\right\}, \quad x > 0.$$

Proof. Let X be a random variable with (6), then

$$\left(1 - F(x)\right)E[h(X) \mid X \geq x] = \exp\left\{-\lambda x(1 - \log\left[\overline{K}(x)\right])^{-\alpha}\right\}, \quad x > 0,$$

and

$$\left(1 - F(x)\right)E[g(X) \mid X \geq x] = \frac{1}{2}\exp\left\{-\lambda x(1 - \log\left[\overline{K}(x)\right])^{-\alpha}\right\}, \quad x > 0,$$

and finally

$$\xi(x)h(x) - g(x) = -\frac{1}{2}\exp\left\{-\lambda x(1 - \log\left[\overline{K}(x)\right])^{-\alpha}\right\} < 0 \ for \ \ x > 0.$$

Conversely, if ξ is given as above, then

$$s'(x) = \frac{\xi'(x)h(x)}{\xi(x)h(x) - g(x)} = \frac{\lambda\left\{1 - log\left[\overline{K}(x)\right] - \alpha x k(x)[\overline{K}(x)]^{-1}\right\}}{(1 - log\left[\overline{K}(x)\right])^{(\alpha+1)}}, \quad x > 0,$$

and hence

$$s(x) = \lambda x\left(\left\{1 - \log\left[\overline{K}(x)\right]\right\}\right)^{-\alpha}, \quad x > 0.$$

Now, according to Theorem 1, X has density (6).

Corollary 3. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $h(x)$ be as in Proposition 3. Then, X has pdf (6) if and only if there exist functions g and ξ defined in Theorem 1 satisfying the differential equation

$$\frac{\xi'(x)}{\xi(x) - g(x)} = \frac{\lambda\left\{1 - log\left[\overline{K}(x)\right] - \alpha x k(x)[\overline{K}(x)]^{-1}\right\}}{(1 - log\left[\overline{K}(x)\right])^{(\alpha+1)}}, \quad x > 0.$$

The general solution of the differential equation in Corollary 3 is

$$\xi(x) = \exp\left\{\lambda x(1 - \log\left[\overline{K}(x)\right])^{-\alpha}\right\}\left[\begin{array}{l} -\int \frac{\lambda\left\{1 - log\left[\overline{K}(x)\right] - \alpha x k(x)[\overline{G}(x)]^{-1}\right\}}{(1 - log\left[\overline{K}(x)\right])^{(\alpha+1)}} \times \\ \exp\left\{-\lambda x(1 - \log\left[\overline{K}(x)\right])^{-\alpha}\right\} q_2(x)dx + D \end{array}\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 3 with $D = \frac{1}{2}$.

Remark 1. A Proposition and a Corollary similar to Proposition 3 and Corollary 3 can be stated for NGE2-NGE4 distributions. The characterizations of NGE5 are more interesting which we take them up below.

Proposition 4. Let $X:\Omega \to (0,\infty)$ be a continuous random variable and let $h(x) = \{Q(x)P'(x) - Q'(x)\}^{-1}P'(x)$ and $g(x) = h(x)\exp\{-P(x)\}$ for $x > 0$. The random variable X belongs to family (14) if and only if the function ξ defined in Theorem1 has the form

$$\xi(x) = \frac{1}{2}\exp\{-P(x)\}, \quad x > 0.$$

Proof. Let X be a random variable with pdf (14), then

$$\big(1 - F(x)\big)E[h(X) \mid X \geq x] = \exp\{-P(x)\}, \quad x > 0,$$

and

$$\big(1 - F(x)\big)E[g(X) \mid X \geq x] = \frac{1}{2}\exp\{-2P(x)\}, \quad x > 0,$$

and finally

$$\xi(x)h(x) - g(x) = -\frac{1}{2}\exp\{-P(x)\} < 0 \quad for \quad x > 0.$$

Conversely, if ξ is given as above, then

$$s'(x) = \frac{\xi'(x)h(x)}{\xi(x)h(x) - g(x)} = P'(x), \quad x > 0,$$

and hence

$$s(x) = P(x), \quad x > 0.$$

Now, in view of Theorem 1, X has density (14).

Corollary 4. Let $X:\Omega \to (0,\infty)$ be a continuous random variable and let $h(x)$ be as in Proposition 4. Then, X has pdf (14) if and only if there exist functions g and ξ defined in Theorem 1 satisfying the differential equation

$$\frac{\xi'(x)h(x)}{\xi(x)h(x) - g(x)} = P'(x), \quad x > 0.$$

The general solution of the differential equation in Corollary 4 is

$$\xi(x) = \exp\{P(x)\}\left[-\int P'(x)\exp\{-P(x)\}\big(h(x)\big)^{-1}g(x)dx + D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 4 with $D = 0$.

2.2 Characterization in terms of the reverse hazard function

The reverse hazard function, r_F, of a twice differentiable distribution function, F , is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \quad x \in support\ of\ F.$$

Proposition 5. Let $X:\Omega \to \mathbb{R}$ be a continuous random variable. For $b=1$, the random variable X has pdf (2) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r_F'(x)=\begin{cases} 0 \quad , & x<\mu \\ -\frac{1}{\alpha}r_F(x)+\frac{a}{4\alpha^2}e^{-\frac{2(x-\mu)}{\alpha}}\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{-2} \quad , & x\geq\mu \end{cases},$$

with boundary condition $r_F(\mu)=\frac{a}{\alpha}$.

Proof. Straightforward and hence omitted.

Proposition 6. Let $X:\Omega \to (0,\infty)$ be a continuous random variable. For $b=1$, the random variable X has pdf (4) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r_F'(x)+(\beta+1)x^{-1}r_F(x)=0.$$

2.3 Characterization based on the conditional expectation of certain functions of the random variable

In this subsection we employ a single function ψ of X and characterize the distribution of X in terms of the truncated moment of $\psi(X)$. The following proposition has already appeared in Hamedani's previous work (2013), so we will just state it as a proposition here, which can be used to characterize KL , NGE1 distributions.

Proposition 7. Let $X:\Omega \to (d,e)$ be a continuous random variable with cdf F. Let $\psi(x)$ be a differentiable function on (d,e) with $\lim_{x\to d^+}\psi(x)=1$. Then for $\delta\neq 1$,

$$E[\psi(X) \mid X\geq x]=\delta\psi(x), x\in(d,e)$$

if and only if

$$\psi(x)=\left(1-F(x)\right)^{\frac{1}{\delta}-1}. x\in(d,e)$$

Remark 2. (A) For $(d,e)=\mathbb{R}$, $b=1$, $\psi(x)=\begin{cases} 2^{-1}e^{\frac{x-\mu}{\alpha}}, & x<\mu \\ 1-2^{-1}e^{-\frac{x-\mu}{\alpha}}, & x\geq\mu \end{cases}$ and $\delta=\frac{a}{a+1}$, Proposition 7 provides a characterization of KL distribution. (B) For $(a,b)=(0,\infty)$, $\psi(x)=\exp\left\{-x\left(1-\log\left[\overline{K}(x)\right]\right)^{-\alpha}\right\}$ and $\delta=\frac{\lambda}{\lambda+1}$, Proposition 7 provides a characterization of NGE1.

2.4 Characterization based on hazard function

It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)}=\frac{h_F'(x)}{h_F(x)}-h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterizations establish a non-trivial characterizations for (6) and (14) in terms of the hazard function which is not of the trivial form given above. Again, similar results hold for $(8), (10)$ and (12).

Proposition 8. Let $X: \Omega \rightarrow (0, \infty)$ be a continuous random variable. Then, X has pdf (6) if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) + \frac{(\alpha+1)k(x)}{\overline{K}(x)\left(1 - log\left[\overline{K}(x)\right]\right)} h_F(x)$$
$$= \lambda\left(1 - \log\left[\overline{K}(x)\right]\right)^{-(\alpha+1)} \frac{d}{dx}\left\{1 - \log\left[\overline{K}(x)\right] - \frac{\alpha x k(x)}{\overline{K}(x)}\right\},$$

with the initial condition $h_F(0) = \lambda$.

Proof. If X has pdf (6), then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left\{\left(1 - \log\left[\overline{K}(x)\right]\right)^{\alpha+1} h_F(x)\right\} = \lambda\left\{1 - \log\left[\overline{K}(x)\right] - \alpha x k(x)\left[\overline{K}(x)\right]^{-1}\right\},$$

or

$$h_F(x) = \frac{\lambda\left\{1 - log\left[\overline{K}(x)\right] - \alpha x k(x)\left[\overline{K}(x)\right]^{-1}\right\}}{\left(1 - log\left[\overline{K}(x)\right]\right)^{\alpha+1}}, \quad x > 0,$$

which is the hazard function of (6).

Proposition 9. Let $X: \Omega \rightarrow (0, \infty)$ be a continuous random variable. Then X has pdf (14) if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) - \frac{Q''(x)}{Q'(x)} h_F(x) = P''(x) - \frac{Q''(x)}{Q'(x)} P'(x) + \left(\frac{Q'(x)}{Q(x)}\right)^2,$$

with the initial condition $h_F(0) = b_1 - a_1$.

Proof. If X has pdf (14), then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left\{\left(Q'(x)\right)^{-1} h_F(x)\right\} = \frac{d}{dx}\left\{P'(x)\left(Q'(x)\right)^{-1} - \left(Q(x)\right)^{-1}\right\},$$

or

$$h_F(x) = P'(x) - \frac{Q'(x)}{Q(x)}, \quad x > 0,$$

which is the hazard function of (14).

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Appendix A

Theorem 1. Let $(\Omega, \mathcal{F}, \boldsymbol{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ ($d = -\infty$, $e = \infty$ might as well be allowed). Let $X: \Omega \to H$ be a continuous random variable with the distribution function F and let g and h be two real functions defined on H such that

$$\boldsymbol{E}[g(X) \mid X \geq x] = \boldsymbol{E}[h(X) \mid X \geq x]\xi(x), \quad x \in H,$$

is defined with some real function η. Assume that $g,\ h \in C^1(H)$, $\xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi h = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g, h and ξ , particularly

$$F(x) = \int_a^x C \left| \frac{\xi'(u)}{\xi(u)h(u) - g(u)} \right| \exp\left(-s(u)\right) \, du \ ,$$

where the function s is a solution of the differential equation $s' = \frac{\xi' \ h}{\xi \ h \ - \ g}$ and C is the normalization constant, such that $\int_H d\,F = 1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see, Glänzel 1990), in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions h_n , g_n and ξ_n $(n \in \mathbb{N})$ satisfy the conditions of Theorem 1 and let $h_n \to h$, $g_n \to g$ for some continuously differentiable real functions h and g . Let, finally, X be a random variable with distribution F . Under the condition that $h_n\,(X)$ and $g_n\,(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if ξ_n converges to ξ , where

$$\xi(x) = \frac{E[g(X) \mid X \geq x]}{E[h(X) \mid X \geq x]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions h , g and ξ , respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if $\alpha \to \infty$, as was pointed out in Glänzel and Hamedani (2001).

A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions h, g and, specially, ξ should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose ξ as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.

In some cases, one can take $h(x) \equiv 1$, as we did in this paper, which reduces the condition of Theorem 1 to $\boldsymbol{E}[g(X) \mid X \geq x] = \xi(x), \quad x \in H$. We, however, believe that employing three functions h, g and ξ will enhance the domain of applicability of Theorem 1.

Appendix B

Here we compute the derivatives of $h(x)$ and $g(x)$ given in Proposition 1 to confirm that $h, g \in C^1(\mathbb{R})$. Similarly for that of $\xi \in C^2(\mathbb{R})$.

$$h'(x) = \begin{cases} -\frac{a(1-b)2^{-a}}{\alpha}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}\left(1-2^{-a}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}\right)^{-b}, & x < \mu \\ -\frac{a(1-b)}{2\alpha}e^{-\frac{x-\mu}{\alpha}}\left(1-\frac{1}{2}e^{\frac{x-\mu}{\alpha}}\right)^{a-1}\left(1-\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a}\right)^{-b}, & x \geq \mu \end{cases}.$$

It is easy to show that the left and right derivatives of $h(x)$ are equal at $x = \mu$. We also have

$$g'(x) = \begin{cases} -\frac{2^{-a}}{\alpha}e^{\frac{x-\mu}{\alpha}}\left\{a(1-b)2^{-a}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}\left(1-2^{-a}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}\right)^{-b}-\left(1-2^{-a}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}\right)^{1-b}\right\}, & x < \mu \\ -\frac{a(1-b)}{\alpha}e^{\frac{x-\mu}{\alpha}}\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a-1}\left(1-\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a}\right)^{-b}\left\{\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a}-\left(1-\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a}\right)\right\}, & x \geq \mu \end{cases},$$

and can easily check that the left and right derivatives of $g(x)$ are equal at $x = \mu$.

For $\xi(x)$, observe that

$$\xi'(x) = \begin{cases} \frac{a2^{-a-1}}{\alpha}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}, & x < \mu \\ \frac{a}{4\alpha}e^{\frac{x-\mu}{\alpha}}\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a-1}, & x \geq \mu \end{cases},$$

and clearly the left and right derivatives of $\xi(x)$ are equal at $x = \mu$. Also, note that

$$\xi''(x) = \begin{cases} \frac{a^2 2^{-a-1}}{\alpha^2}\left(e^{\frac{x-\mu}{\alpha}}\right)^{a}, & x < \mu \\ \frac{a}{4\alpha^2}e^{\frac{x-\mu}{\alpha}}\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)^{a-2}\left\{\left(1-\frac{1}{2}e^{-\frac{x-\mu}{\alpha}}\right)+\frac{a-1}{2}\right\}, & x \geq \mu \end{cases},$$

and hence the left and right derivatives of $\xi'(x)$ are equal at $x = \mu$.