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New Classes of Univariate Continuous Exponential Power Series Distributions

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Abstract: Recently, many researchers have developed various classes of continuous probability distributions which can be generated via the generalized Pearson differential equation and other techniques. In this paper, motivated by the importance of the power series in probability theory and its applications, we derive some new classes of univariate exponential power series distributions for a real-valued continuous random variable, which we call exponential power series distributions. Various mathematical properties of the proposed classes of distributions are discussed. Based on these distributional properties, we have established some characterizations of these distributions as well. It is hoped that the findings of the paper will be useful for researchers in the fields of probability, statistics and other applied sciences.

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1. INTRODUCTION

Probability distribution of a continuous random variable plays an important role in both pure and applied sciences. Various systems of distributions have been constructed to provide approximations to a wide variety of distributions. The development of the general theories of the normal distributions began during the 18th century with the work of de Moivre in his studies of approximations to certain binomial distributions for large positive integer n. Afterward, further developments have been continued from the 19th century to date with the contributions of many well-known researchers and scientists, among them Legendre, Gauss, Laplace, Bessel, Bravais, Airy, Schols, Galton, Helmert, Tchebyshev, Edgeworth, Pearson, Markov, Lyapunov, Charlier, Gosset, Amoroso, Fisher, Burr and Johnson, are notable. For a detailed account of these, including historical and chronological development, the interested readers are referred to Patel et al. [18], Patel and Read [19] and Johnson et al. [15, 16] among others. These systems are designed with the requirements of the ease of computations and feasibility of algebraic manipulations. To meet the requirements, there must be as few parameters as possible in defining a member of the system.

As stated above, one of these systems is the Pearson system of distributions. A continuous distribution belongs to this system if for a real-valued continuous random variable X with its probability density function (*pdf*) f satisfies a differential equation of the following form

$$\frac{1}{f(x)}\frac{df(x)}{dx}=-\frac{x+a}{bx^2+cx+d},$$

where a, b, c, and d are real parameters such that f is a pdf. The shape of the pdf depends on the values of these parameters, based on which, Pearson classified the probability distributions into a number of types known as Pearson Types I – VI, see Pearson [21, 22]. Later in another paper, Pearson defined more special cases and subtypes, known as Pearson types VII - XII, see Pearson [23]. Many well-known distributions are special cases of Pearson type

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distributions which include normal and student t distributions (Pearson type VII), beta distribution (Pearson type I), gamma distribution (Pearson type III) to mention a few.

In recent years, for a real-valued continuous random variable X, many researchers have considered generalizations of the above Pearson differential equation, known as generalized Pearson system of differential equation (GPE), given by

$$\frac{1}{f(x)}\frac{df(x)}{dx}=\frac{\sum_{j=0}^{m}a_j x^j}{\sum_{j=0}^{n}b_j x^j},$$

where m and n are positive integers and the coefficients a_j's and b_j's are real numbers. The system of continuous univariate pdf's generated by GPE is called a generalized Pearson system of distributions which, as intended by Pearson, includes a vast majority of well-known continuous pdf's with proper choices of the parameters a_j's and b_j's, including those which belong to Pearson system of distributions. For a detailed account of these, including most recently developed distributions based on GPE, the interested readers are referred to Shakil et al. [24 – 27], Hamedani [12, 13], Ahsanullah et al. [2], Lee et al. [17], Shakil and Singh [28] and references therein.

The objective of this paper is to derive some new classes of univariate distributions for a real-valued continuous random variable based on the GPE involving some power series. The organization of the paper is as follows. In Section 2, we derive our new classes of univariate power series distributions. Section 3 contains some properties and characterizations of the proposed distribution. In Section 4, we provide some concluding remarks.

2. SOME NEW CLASSES OF UNIVARIATE CONTINUOUS EXPONENTIAL POWER SERIES DISTRIBUTIONS

Power series plays an important role in probability theory and its applications. For example, the class of discrete distributions such as the binomial, geometric, logarithmic, negative binomial and Poisson distributions is constructed from power series, see Patil [20]. Let X be a real-valued continuous random variable. Let $\sum_{k=0}^{\infty}c_k x^k$, $\sum_{k=0}^{\infty}a_k x^k$ and $\sum_{k=0}^{\infty}b_k x^k$ be the power series which are uniformly convergent in $(-\infty,\infty)$, where k is a non-negative integer, $c_k{}'s$, $a_k{}'s$ and $b_k{}'s$ are real numbers, $a_k \neq 0$ for at least one $k \in \mathrm{N}$ (set of natural numbers), $a_k \neq b_k$ for all k. Let $\sum_{k=0}^{n}c_k x^k$ denote the partial sum of order $n \in \mathrm{N}$ of the power series $\sum_{k=0}^{\infty}c_k x^k$. Let $\sum_{k=-n}^{\infty}c_k x^k$ denote the Laurent series. In this paper, we will consider the following cases of the GPE involving the power series:

Case (I): $\frac{1}{f(x)}\frac{df(x)}{dx}=\sum_{k=-n}^{\infty}c_k x^k$;

Case (II): $\frac{1}{f(x)}\frac{df(x)}{dx}=\sum_{k=0}^{\infty}c_k x^k$;

Case (III): $\frac{1}{f(x)}\frac{df(x)}{dx}=\sum_{k=0}^{n}c_k x^k$;

Case (IV): $\frac{1}{f(x)}\frac{df(x)}{dx}=\sum_{k=1}^{n}c_k x^{\alpha_k}$, $c_k>0, |\alpha_k|\le 1$; see Bondesson [3];

Case (V): $\frac{1}{f(x)}\frac{df(x)}{dx}=\frac{\sum_{k=0}^{n}b_k x^k}{\sum_{k=0}^{n}a_k x^k}$, $a_0 \neq 0$;

Case (VI): $\frac{1}{f(x)}\frac{df(x)}{dx}=\frac{\sum_{k=0}^{\infty}b_k x^k}{\sum_{k=0}^{\infty}a_k x^k}$, $a_0 \neq 0$.

2.1 Derivation of Proposed Classes of Distribution: In what follows, by using Case (VI), we will prove our main Theorem 2.1.1, as given below. Please note that Case (VI) embodies all Cases (II) – (III) and (V). Similar to Case (VI) (Theorem 2.1.1), we can also have the distribution based on other cases. For proving our main Theorem 2.1.1, we need first to state a Lemma (Lemma 2.1.1, below), see, for example, Bromwich [5, pages 136 – 138], Courant and John [6, page 556], Gradshteyn and Ryzhik [11, page 14], among others.

Lemma 2.1.1 Let $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$ be two power series which are uniformly convergent in $(0, \infty)$, where $a_k's$ and $b_k's$ are real numbers, $a_k \neq 0$ for at least one $k \in \mathrm{N}$, $a_k \neq b_k$ for all k. Then $\frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} = \frac{1}{a_0} \sum_{k=0}^{\infty} c_k x^k$, where

$$c_n + \frac{1}{a_0} \sum_{k=1}^{n} c_{n-k} a_k - b_n = 0, \ a_0 \neq 0$$

or

$$c_n = \frac{(-1)^n}{(a_0)^n} \begin{vmatrix} a_1 b_0 - a_0 b_1 & a_0 & 0 & \ldots & 0 \\ a_2 b_0 - a_0 b_2 & a_1 & a_0 & \ldots & 0 \\ a_3 b_0 - a_0 b_3 & a_2 & a_1 & \ldots & 0 \\ \ldots & & & & \\ \ldots & & & & \\ a_{n-1} b_0 - a_0 b_{n-1} & a_{n-2} & a_{n-3} & \ldots & a_0 \\ a_n b_0 - a_0 b_n & a_{n-1} & a_{n-2} & \ldots & a_0 \end{vmatrix}, \quad a_0 \neq 0.$$

Theorem 2.1.1 Let X be a real positive continuous random variable with *pdf* f(x).. Let $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$ be two power series which are uniformly convergent in $(0, \infty)$, where $a_k's$ and $b_k's$ are real numbers, $a_k \neq 0$ for at least one $k \in \mathrm{N}$, $a_k \neq b_k$ for all k. Let us consider GPE by taking the ratio of the power series $\frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k}$, as follows

$$\frac{1}{f(x)} \frac{df(x)}{dx} = -\frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k}, \quad a_0 \neq 0. \tag{1}$$

Then the solution to the differential equation (1) is given by

$$f(x) = C \exp\left[-\frac{1}{a_0} \sum_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1}\right], \quad x > 0, \tag{2}$$

where $a_0 \neq 0$, $c_n + \frac{1}{a_0} \sum_{k=1}^{n} c_{n-k} a_k - b_n = 0$ and

$$C = \left\{ \int_0^{\infty} \exp\left[-\frac{1}{a_0} \sum_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1}\right] dx \right\}^{-1} \tag{3}$$

denotes the normalizing constant.

Proof. From Lemma 2.1.1, we have

$$\frac{1}{f(x)}\frac{df(x)}{dx}=-\frac{\sum_{k=0}^{\infty}b_k x^k}{\sum_{k=0}^{\infty}a_k x^k}=-\frac{1}{a_0}\sum_{k=0}^{\infty}c_k x^k,$$

$$f(x)=C\exp\left[-\frac{1}{a_0}\sum_{k=0}^{\infty}c_k\frac{x^{k+1}}{k+1}\right],\quad x>0 \qquad (4)$$

where $c_n+\frac{1}{a_0}\sum_{k=1}^{n}c_{n-k}a_k-b_n=0$. Integrating (4) with respect to x, we arrive at

$$\ln(f(x))=-\frac{1}{a_0}\sum_{k=0}^{\infty}c_k x^{k+1}+A, \qquad (5)$$

where $C=e^A$ is the normalizing constant so that $\int_0^{\infty}f(x)dx=1$. Integrating both sides of (5) from 0 to ∞ and in view of the fact that $\int_0^{\infty}f(x)dx=1$, we obtain $C=\left\{\int_0^{\infty}\exp\left[-\frac{1}{a_0}\sum_{k=0}^{\infty}c_k\frac{x^{k+1}}{k+1}\right]dx\right\}^{-1}$, which completes the proof.

Theorem 2.1.2 The cumulative distribution function (cdf) of the positive continuous random variable X, with the probability density function given by (2), is

$$F_X(x)=C\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{h_{k,j}}{k+1}x^k, \qquad (6)$$

where C denotes the normalizing constant given by (3), and the coefficients $h_{k,j}$'s are given by

$$h_{0,j}=g_{0,k}$$

$$h_{k.j}=\sum_{m=0}^{k}\frac{g_{k-m,j}\,a^m}{m!\,j!},\ \text{for } m\ge 1, \qquad (7)$$

$$g_{0,j}=(d_0)^j,\ g_{n,j}=\frac{1}{na_0}\sum_{k=1}^{n}(kj-n+k)g_{n-k,j}d_k \text{ for n} \ge 1,$$

$$d_k=-\frac{c_{k+1}}{a_0(k+2)},k=0,1,2,...,a=-\frac{c_0}{a_0}.$$

Proof.

We have

$$f(x)=C\exp\left[-\frac{1}{a_0}\sum_{k=0}^{\infty}c_k\frac{x^{k+1}}{k+1}\right],\quad x>0.$$

We can write

$$f(x)=C\exp(a\,x)\exp\left(\sum_{k=0}^{\infty}d_k x^k\right), \qquad (8)$$

where

$$a=-\frac{c_0}{a_0},d_k=-\frac{c_{k+1}}{a_0(k+2)},k=0,1,2,....$$

Now

$$\exp\left(a\,x\right)=\sum_{i=0}^{\infty}\frac{(ax)^{i}}{i!}$$

and

$$\exp\left(\sum_{k=0}^{\infty}d_k x^k\right)=\sum_{0}^{\infty}\frac{1}{j!}\left(\sum_{k=0}^{\infty}d_k x^k\right)^j=\sum_{0}^{\infty}\frac{1}{j!}\left(\sum_{k=0}^{\infty}g_{k,j}x^k\right).$$

We have

$$g_{0,j}=(d_k)^j,\; g_{n,j}=\frac{1}{nd_0}\sum_{k=1}^{n}(kj-n+k)d_k g_{n-k},\; n\geq 1.$$

Thus we can write

$$f(x)=C\sum_{i=0}^{\infty}\frac{(ax)^{i}}{i!}\sum_{j=0}^{\infty}\frac{1}{j!}\sum_{k=0}^{\infty}g_{k,j}x^k$$

$$=C\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}h_{k,j}x^k\,, \tag{9}$$

where

$$h_{0,j}=g_{0,k}$$

and

$$h_{k,j}=\sum_{m=0}^{k}\frac{a^m g_{k-m,j}}{m!\,j!},k\geq 1.$$

The corresponding cdf $F_X(x)$ from (9) is given by

$$F_X(x)=C\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}h_{k,j}\frac{x^{k+1}}{k+1} \tag{10}$$

where C denotes the normalizing constant given by (3). This completes the proof of Theorem 2.1.2.

Remark 2.1.1: In the above Theorems 2.1.1 and 2.1.2, we can easily remove the restrictions on the positive real continuous random variable X, which we can take as a real continuous random variable also.

As an illustration, we consider the probability density function in (2) with $a_0=1$ and $c_k=1$ so we have

$$f_X(x)=C\exp\left[-\sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1}\right],\text{ where } C=\left[\int_0^{\infty}\exp\left[-\sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1}\right]dx\right]^{-1},$$

for which, the graph of the pdf is given in the following Figure 1. It appears from Figure 1 that the proposed distribution is skewed to the right.

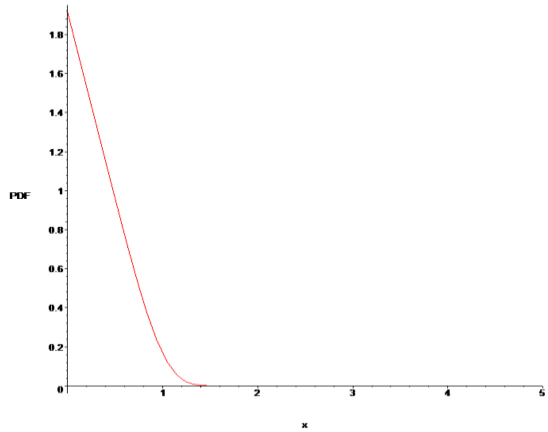

Figure 1. pdf

2.2 Some Distributional Properties of the Proposed Distribution

2.2.1 Distributional Properties: It is obvious from (2) that the mean, variance, skewness, and kurtosis of X are mathematically intractable. However, using Eq. (10), we can easily find the expressions for the survivor function, the hazard function and the cumulative hazard function on the support of X .

2.2.2 Distributional Relationships: It should be noted that, in (1), since the pairs of the series are not unique, using ratio of two different power series, the solution of differential equation (1) will result in various types of power series, and thus, we can have various exponential power series distributions for different values of coefficients, as stated in Cases (I) – (VI). For example, removing the restrictions on the positive continuous random variable X , the following are some examples of exponential power series distributions for different values of coefficients, which can easily be derived, for different values of coefficients in (1).

Example 2.2.1: $f(x) = \dfrac{e^{-m\cos x}}{I_o(m)}, \quad -\pi < x < \pi$, where $I_0(m)$ is the Bessel function of first kind and order 0 , which is the pdf of the circular distribution.

Example 2.2.2: $f(x) = e^{-x} e^{-e^{-x}}, \quad -\infty < x < \infty$, which is the pdf of the standard Gumbel distribution.

Example 2.2.3: $f(x) = \dfrac{1}{\sqrt{x(1-x)}}, \quad 0 < x < 1$, which is the pdf of the arcsine distribution.

Example 2.2.4: Taking $k = 0$ and $\frac{c_0}{a_0} = \frac{1}{\mu}$, $a_0 \neq 0$ in (2) and (3), we have $f(x) = \frac{1}{\mu} e^{-x/\mu}$, which is the pdf of an exponential distribution.

Example 2.2.5: Taking $k = 1$, $\frac{c_0}{a_0} = 2b$ and $\frac{c_1}{2a_0} = a$, $a_0 \neq 0$ in (2) and (3), and then employing Eq. 7.4.2 , page 302 Abramowitz and Stegun [1], we obtain $f(x) = \left[\frac{1}{2}\sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{a}\right) \operatorname{erf} c\left(\frac{b}{\sqrt{a}}\right)\right]^{-1} \exp\left\{-\left(ax^2 + 2b\right)\right\}$ in which $\operatorname{erf} c(\cdot)$ denotes the complementary error function.

Example 2.2.6: Taking $b = 0$, $a = \frac{1}{\sigma^2}$ in example (2.2.5), we arrive at $f(x) = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$ which is pdf of a half-normal distribution.

3. PROPERTIES AND CHARACTERIZATIONS

In this section, we provide some properties and characterizations of the proposed distribution.

3.1 Convexity and Infinite Divisibility: Here, we shall prove the convexity and infinite divisibility of the proposed distribution with the probability density function given by (2) via imposing certain restrictions on the coefficients, for which we shall first need the following Lemmas.

Lemma 3.1.1 (see Deshpande [7, page 208]): Any power series $P(x)=\sum_{k=0}^{\infty} a_k x^k$, where $a_k{}'s$ are real numbers and $a_k \neq 0$ for at least one $k \in \mathrm{N}$, with the radius of convergence R, has derivatives of all orders at each point in its domain $(-R, R)$, and for each n, the nth derivative is given via term-by-term differentiation of the $(n-1)th$ derivative. In particular, its nth derivative at 0 is given by $a_n = \dfrac{P^{(n)}(0)}{n!}$.

Lemma 3.1.2 (see Webster [30, page 197]): Let $f : \mathrm{I} \to \Re$ be a real-valued and twice differentiable function on an open interval I of the real line $\Re$. Then f is convex if and only if $f^{//}(x) \geq 0$ for all x in I.

Lemma 3.1.3 (see Webster [30, page 207]; Boyd and Vandenberghe [4, page 104]): Let $f : \mathrm{I} \to \Re$ be a real-valued function on an open interval I of the real line $\Re$. Then f is log-convex if $f(x) > 0$ for all x in I and its logarithm, $\log f : \mathrm{I} \to \Re$, is convex.

Lemma 3.1.4 (see Steutel and Harn [29, page 117]): Let F be a distribution on $\Re_+$ with $F(0)=0$, having a density f that is log-convex on $(0, \infty)$. Then F is infinitely divisible.

Theorem 3.1.1 Let X be a positive continuous random variable with the following probability density function

$$f(x) = C \exp\left[\sum_{k=0}^{\infty} \frac{c_k}{|a_0|} \frac{x^{k+1}}{k+1}\right], \quad 0 < \mathrm{x} < \mathrm{b} < \infty, \tag{11}$$

where b is such that (11) is a pdf with $a_0 \neq 0$, $c_k > 0$ for all k and

$\dfrac{1}{C} = \int_0^b \exp(\int_0^b \exp(\frac{1}{|a_0|}\sum_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1}) dx$. Then, the pdf as given in (11) is log-convex on $(0, b)$ and is infinitely divisible.

Proof. Taking natural log of (11), we have

$$\ln(f(x)) = \ln(C) + \sum_{k=0}^{\infty} \frac{c_k}{|a_0|} \frac{x^{k+1}}{k+1}, \quad 0 < \mathrm{x} < \mathrm{b}. \tag{12}$$

We have $\ln(f(x))$ is a twice differentiable function on $(0, b)$ with $\dfrac{d^2[\ln(f(x))]}{dx^2} \geq 0$. By Lemma 3.1.2, $\ln(f(x))$ is convex on $(0, b)$. It follows from Lemma 3.1.3 that the probability density function $f(x)$ as given in (11) is log-convex on $(0, b)$. Hence, by Lemma 3.1.4, it is infinitely-divisible. This completes the proof of the Theorem 3.1.1.

Remark 3.1.1 Let X be a positive continuous random variable, with the probability density function f as given by

$$f(x) = C\exp\left[-\sum_{k=1}^{n} c_k x^{\alpha_k}\right], \quad x>0,\ c_k>0.$$

Then, by Bondesson [3], the probability density function $f(x)$ is infinitely-divisible. Also, see Hamedani [14].

3.2. Characterizations: The problem of characterizing a distribution is an important problem in various fields which has recently attracted the attention of many researchers. These characterizations have been established in many different directions. This subsection deals with various characterizations of our new distribution. These characterizations are based on a simple relationship between two truncated moments. It should be mentioned that for the characterizations given here the cdf need not have a closed form, as in our case here. We believe, due to the nature of the cdf, there may not be other possible characterizations than the ones presented in this subsection. Our first characterization result borrows from a theorem due to Glänzel [8], see Theorem 3.2.1 below. We refer the interested reader to Glänzel [8] for a proof of Theorem 3.2.1. Note that the result holds also when the interval H is not closed. As shown in Glänzel [9], this characterization is stable in the sense of weak convergence.

Theorem 3.2.1 Let $(\Omega, \mathbf{F}, \mathbf{P})$ be a given probability space and let $H=[d,\ e]$ be an interval for some $d<e$, $(d=-\infty,\ e=\infty \text{ might as well be allowed})$. Let $X\ :\ \Omega \to H$ be a continuous random variable with the distribution function F and let g and h be two real functions defined on H such that

$$\mathbf{E}[g(X)\mid X\ge x] = \mathbf{E}[h(X)\mid X\ge x]\eta(x), \quad x\in H,$$

is defined with some real function η. Assume that $g, h \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $h\eta = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g, h and η, particularly

$$F(x) = \int_a^x C\left|\frac{\eta'(u)}{\eta(u)h(u)-g(u)}\right| \exp(-s(u))\, du,$$

where the function s is a solution of the differential equation $s' = \dfrac{\eta' h}{\eta h - g}$ and C is the normalization constant, such that $\int_H dF = 1$.

Remark 3.2.1 The goal is to have as simple functional form for $\eta(x)$ as possible. For the detailed explanation of this goal, please see Glänzel [10].

We now present our first characterization of (2) in terms of a simple relationship between two truncated moments. We assume a_0 and c_0 are positive constants.

Proposition 3.2.1 Let $X\ :\ \Omega \to (0,\infty)$ be a continuous random variable and let $h(x) = \exp\left\{\frac{1}{a_0}\sum_{k=1}^{\infty}\left(\frac{c_k}{k+1}\right)x^{k+1}\right\}$ and $g(x) = x\,h(x)$ for $x>0$. The random variable X belongs to the family (2) if and only if the function η defined in Theorem 3.2.1 has the form

$$\eta(x) = \frac{c_0}{a_0} + x, \quad x>0. \tag{13}$$

Proof. Let X be a random variable with density (2) , then

$$(1-F(x))E[h(X) \mid X \geq x] = \frac{C\,a_0}{c_0} e^{-\frac{c_0}{a_0}x}, \quad x > 0,$$

and

$$(1-F(x))E[g(X) \mid X \geq x] = \frac{C\,a_0}{c_0} e^{-\frac{c_0}{a_0}x}\left(\frac{c_0}{a_0} + x\right), \quad x > 0,$$

and finally,

$$\eta(x)h(x) - g(x) = \frac{c_0}{a_0}h(x) > 0 \quad for \ \ x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)h(x)}{\eta(x)h(x) - g(x)} = \frac{c_0}{a_0}, \quad x > 0,$$

and hence,

$$s(x) = \frac{c_0}{a_0}x, \quad x > 0.$$

Now, in view of Theorem 3.2.1, X has a density (2).

Corollary 3.2.1 Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h(x)$ be as in Proposition 3.2.1. The pdf of X is (2) if and only if there exist functions g and η defined in Theorem 3.2.1 satisfying the differential equation

$$\frac{\eta'(x)h(x)}{\eta(x)h(x) - g(x)} = \frac{c_0}{a_0}, \quad x > 0. \tag{14}$$

The general solution of the differential equation in (14) is

$$\eta(x) = e^{\frac{c_0}{a_0}x}\left[-\int \frac{c_0}{a_0} e^{-\frac{c_0}{a_0}x}\left(h(x)\right)^{-1} g(x)dx + D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation (14) is given in Proposition 3.2.1 with $D = 0$. However, it should also be noted that there are other triplets $(h,\ g,\ \eta)$ satisfying the conditions of Theorem 3.2.1.

4. CONCLUDING REMARKS

This paper introduces a new univariate class of continuous probability distributions based on a generalized Pearson differential equation involving the ratio of two power series, which we call an exponential power series distribution. Various mathematical properties of the proposed class of distributions are discussed. We hope that the findings of this short article will be useful in many applications.

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